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ON EQUALITY OF ABSOLUTE CENTRAL AND CLASS PRESERVING AUTOMORPHISMS OF FINITE p-GROUPS

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Abstract. Let G be a finite non-abelian p-group and $L(G)$ denotes the absolute center of G. Also, let $\mathrm{Aut}^L(G)$ and $\mathrm{Aut}_c(G)$ denote the group of all absolute central and the class preserving automorphisms of G, respectively. In this paper, we give a necessary and sufficient condition for G such that $\mathrm{Aut}_c(G) = \mathrm{Aut}^L(G)$. We also characterize all finite non-abelian p-groups of order $p^n (n \leq 5)$, for which every absolute central automorphism is class preserving.

1. Introduction

Throughout this paper, all groups mentioned are assumed to be finite and p always denotes a prime number. By $G', Z(G), \Phi(G), \mathrm{Inn}(G)$ and $\mathrm{Aut}(G)$, we denote the commutator subgroup, the center, the Frattini subgroup, the group of all inner automorphisms and the group of all automorphisms of G, respectively. For $x \in G$, x^G denotes the conjugacy class of all $x^g = g^{-1}xg$, where $g \in G$, and $[x, G]$ stands the set of all commutators of the form $[x, g] = x^{-1}g^{-1}xg$, $g \in G$. Since $x^g = x[x, g]$, for all $g \in G$, we have $x^G = x[x, G]$ and so $|x^G| = |[x, G]|$. For $x \in G$ and $\alpha \in \mathrm{Aut}(G)$, the element $[x, \alpha] = x^{-1}x^\alpha$ is called the autocommutator of x and α. Also inductively, for all $\alpha_1, \alpha_2, ..., \alpha_n \in \mathrm{Aut}(G)$, define $[x, \alpha_1, \alpha_2, ..., \alpha_n] = [[x, \alpha_1, \alpha_2, ..., \alpha_{n-1}], \alpha_n]$. An automorphism α of G is called a class preserving automorphism if $x^\alpha \in x^G$, for all

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$x \in G$. The set of all class preserving automorphisms of G, denoted by $\mathrm{Aut}_c(G)$, contains $\mathrm{Inn}(G)$. Let N be a normal subgroup of G and $\alpha \in \mathrm{Aut}(G)$. If $N^\alpha = N$ (or $Nx^\alpha = Nx$ for all $x \in G$), we shall say α normalizes N (centralizes G/N respectively). Now let M and N be normal subgroups of G. We let $\mathrm{Aut}^N(G)$ denote the group of all automorphisms α of G normalizing N and centralizing G/N (or equivalently, $[x, \alpha] \in N$ for all $x \in G$), and $C_{\mathrm{Aut}^N(G)}(M)$ the group of all automorphisms of $\mathrm{Aut}^N(G)$ centralizing M. Hegarty [5] introduced the absolute center $L(G)$ of a group G as

$$L(G) = \{g \in G \mid [g, \alpha] = 1, \forall \alpha \in \mathrm{Aut}(G)\}.$$

It is easy to check that $L(G)$ is a characteristic subgroup contained in the center of G. If we choose $N = L(G)$ or $N = Z(G)$, then $\mathrm{Aut}^N(G)$ is precisely the group of all absolute central or central automorphisms of G. In this paper, we give a necessary and sufficient condition on a finite non-abelian p-group G such that $\mathrm{Aut}^L(G) = \mathrm{Aut}_c(G)$. We also characterize all finite non-abelian p-groups G of order $p^n (n \leq 5)$, for which $\mathrm{Aut}^L(G) = \mathrm{Aut}_c(G)$.

Recall an abelian p-group A has invariants or is of type $(a_1, a_2, ..., a_k)$ if it is the direct product of cyclic subgroups of orders $p^{a_1}, p^{a_2}, ..., p^{a_k}$, where $a_1 \geq a_2 \geq ... \geq a_k > 0$.

Let G be a finite non-abelian p-group such that G/G' and $L(G)$ are of types $(a_1, a_2, ..., a_k)$ and $(b_1, b_2, ..., b_l)$. Also if $G/L(G)$ is abelian, then $G/Z(G)$ and G' are of types $(e_1, e_2, ..., e_n)$ and $(d_1, d_2, ..., d_s)$. Since $G' \leq Z(G)$, by [2, Section 25], $n \leq k$ and $e_j \leq a_j$ for each $1 \leq j \leq n$.

The above notation will be used in the rest of the paper. We state the main result in the following theorem:

Theorem. *Let G be a finite non-abelian p-group. Then the following statements are equivalent:*

(i) $\mathrm{Aut}_c(G) = \mathrm{Aut}^L(G)$, *where* $L = L(G)$;

(ii) $G' = L(G)$, $Z(G) \leq \Phi(G)$, $|\mathrm{Aut}_c(G)| = \prod_{1 \leq i \leq n} |\Omega_{e_i}(G')|$ *and one of the following conditions holds:*

  (1) $L(G) = Z(G)$ *or*

  (2) $L(G) < Z(G)$, $n = k$ *and* $b_1 = e_t$, *where t is the largest integer between 1 and n such that $a_t > e_t$.*

2. Preliminary results

A p-group G is said to be extraspecial if $G' = Z(G) = \Phi(G)$ is of order p. If α is an automorphism of G and x is an element of

G, we write x^α for the image of x under α. For a finite group G, $\exp(G), d(G)$, $\Omega_i(G)$, and $o(x)$ respectively denote the exponent of G, minimal number of generators of G, the subgroup of G generated by its elements of order dividing p^i and the order of x. We use U^n for the direct product of n-copies of a group U, C_n for the cyclic group of order n where $n \geq 1$, as usual D_8, respectively Q_8, for the dihedral group, resp. the quaternion group, of order 8. Also the minimal non-abelian p-groups $M_p(n,m)$ and $M_p(n,m,1)$ of order p^{n+m} and p^{n+m+1} as defined respectively by

$$\langle x, y \mid x^{p^n} = y^{p^m} = 1, x^y = x^{1+p^{n-1}} \rangle,$$

where $n \geq 2$, $m \geq 1$ and

$$\langle x, y, z \mid x^{p^n} = y^{p^m} = z^p = 1, [x,y] = z, [x,z] = [y,z] = 1 \rangle,$$

where $n \geq m \geq 1$ and if $p = 2$, then $m + n > 2$.
Let $L_1(G) = L(G)$, and for $n \geq 2$, define $L_n(G)$ inductively as

$$L_n(G) = \{g \in G \mid [g, \alpha_1, \alpha_2, ..., \alpha_n] = 1, \forall \alpha_1, \alpha_2, ..., \alpha_n \in \mathrm{Aut}(G)\}.$$

A group G is called the autonilpotent of class n if n is the smallest natural number such that $L_n(G) = G$. Finally, let G and H be any two groups. We denote by $\mathrm{Hom}(G, H)$ the set of all homomorphisms from G into H. Clearly, if H is an abelian group, then $\mathrm{Hom}(G, H)$ forms an abelian group under the following operation $(fg)(x) = f(x)g(x)$, for all $f, g \in \mathrm{Hom}(G, H)$ and $x \in G$.

The following lemma is well-known and will be used in the proof of our results.

Lemma 2.1. *Let U, V and W be finite abelian groups. Then*

(i) $\mathrm{Hom}(U \times V, W) \cong \mathrm{Hom}(U, W) \times \mathrm{Hom}(V, W)$;
(ii) $\mathrm{Hom}(U, V \times W) \cong \mathrm{Hom}(U, V) \times \mathrm{Hom}(U, W)$;
(iii) $\mathrm{Hom}(C_m, C_n) \cong C_f$, *where f is the greatest common divisor of m and n.*

The following preliminary lemma is well-known result [10, Lemma 2.2].

Lemma 2.2. *Let G be a group and M, N be normal subgroups of G with $N \leq M$ and $C_N(M) \leq Z(G)$. Then*

$$C_{\mathrm{Aut}^N(G)}(M) \cong \mathrm{Hom}(G/M, C_N(M)).$$

The following useful result can be found in [11].

Theorem 2.3. [11, Theorem 2.5]. *Let G be a finite p-group different from C_2. Then $\mathrm{Aut}^L(G) \cong \mathrm{Hom}(G, L(G))$.*

3. Proofs

Lemma 3.1. *Let G be a finite non-abelian p-group such that $G/L(G)$ is abelian and* $\mathrm{Aut}_c(G) = \mathrm{Aut}^L(G)$. *Then*

(i) $Z(G) \leq \Phi(G)$ *and*

$$\mathrm{Aut}_c(G) \cong \mathrm{Hom}(G/Z(G), G') \cong \mathrm{Hom}(G/Z(G), L(G)) \cong \mathrm{Hom}(G/G', L(G));$$

(ii) $|\mathrm{Aut}_c(G)| = \prod_{1\leq i\leq n} |\Omega_{e_i}(G')|$.

Proof. (i) Suppose, on the contrary, that there exists a maximal subgroup M of G such that $Z(G) \nleq M$. Then $G = M\langle z\rangle$, for some z in $Z(G)\backslash M$. We choose an element u in $\Omega_1(L(G))$. Then it is easy to see that the map $\alpha : hz^i \mapsto h(zu)^i$, where $h \in M$ and $0 \leq i < p$, is an absolute central automorphism of G which is not class preserving. Hence $Z(G) \leq \Phi(G)$. Next, since $\mathrm{Aut}_c(G) \leq C_{\mathrm{Aut}^{G'}(G)}(Z(G)) \leq C_{\mathrm{Aut}^L(G)}(Z(G)) \leq \mathrm{Aut}^L(G) = \mathrm{Aut}_c(G)$, by Lemma 2.2 and Theorem 2.3, we have

$$\mathrm{Aut}_c(G) = C_{\mathrm{Aut}^{G'}(G)}(Z(G)) \cong \mathrm{Hom}(G/Z(G), G') \cong \mathrm{Hom}(G/Z(G), L(G)) \cong \mathrm{Hom}(G/G', L(G)).$$

(ii) Let $G/Z(G) = \langle \bar{x_1}, \bar{x_2}, ..., \bar{x_n}\rangle$ such that $o(\bar{x_i}) = o(x_iZ(G)) = p^{e_i}$, for $1 \leq i \leq n$. Since $Z(G) \leq \Phi(G)$, by [13, Lemma 3.5], $\{x_1, x_2, ..., x_n\}$ is a minimal generating set for G. Now the correspondence $\sigma \to (x_1^\sigma, ..., x_n^\sigma)$ is one-to-one mapping from $\mathrm{Aut}_c(G)$ onto $x_1^G \times ... \times x_n^G$. Thus $|\mathrm{Aut}_c(G)| \leq \prod_{1\leq i\leq n} |x_i^G|$. By [13, Lemma 3.1], $\exp([x_i, G]) = o(\bar{x_i}) = o(x_iZ(G))$, for any $1 \leq i \leq n$. So $|\mathrm{Hom}(\langle\bar{x_i}\rangle, [x_i, G])| = |[x_i, G]|$. Hence,

$$|\mathrm{Aut}_c(G)| = |\mathrm{Hom}(G/Z(G), L(G))| \geq |\mathrm{Hom}(G/Z(G), G')|$$

$$= \prod_{1\leq i\leq n} |\mathrm{Hom}(\langle\bar{x_i}\rangle, G')| \geq \prod_{1\leq i\leq n} |\mathrm{Hom}(\langle\bar{x_i}\rangle, [x_i, G])| = \prod_{1\leq i\leq n} |[x_i, G]|$$

$$= \prod_{1\leq i\leq n} |x_i^G| \geq |\mathrm{Aut}_c(G)|.$$

Therefore, $|\mathrm{Aut}_c(G)| = \prod_{1\leq i\leq n} |x_i^G|$ and for each $1 \leq i \leq n$,

$$|\mathrm{Hom}(\langle\bar{x_i}\rangle, G')| = |\mathrm{Hom}(\langle\bar{x_i}\rangle, [x_i, G])|.$$

As $|\mathrm{Hom}(\langle\bar{x_i}\rangle, [x_i, G])| = |[x_i, G]|$, it follows from [13, Lemma 2.7], that

$$|[x_i, G]| = |\mathrm{Hom}(\langle\bar{x_i}\rangle, G')| = |\mathrm{Hom}(\langle\bar{x_i}\rangle, \Omega_{e_i}(G'))| = |\Omega_{e_i}(G')|.$$

On the other hand, $[x_i, G] \leq \Omega_{e_i}(G')$ and hence $[x_i, G] = \Omega_{e_i}(G')$, for $1 \leq i \leq n$. Since $|x_i^G| = |[x_i, G]|$ for $1 \leq i \leq n$, we have $|\mathrm{Aut}_c(G)| = \prod_{1\leq i\leq n} |\Omega_{e_i}(G')|$, which completes the proof. □

Proof of Theorem.

First assume that $\mathrm{Aut}_c(G) = \mathrm{Aut}^L(G)$. For any $x, y \in G$, by the inner automorphism $i_y \in \mathrm{Aut}_c(G)$, induced by $y \in G$, $x^{-1}x^{i_y} = [x, y] \in L(G)$ and thus $G' \leq L(G)$. So by Lemma 3.1(ii), $|\mathrm{Aut}_c(G)| = \prod_{1\leq i\leq n} |\Omega_{e_i}(G')|$, $Z(G) \leq \Phi(G)$ and $|\mathrm{Aut}_c(G)| = |\mathrm{Hom}(G/Z(G), G')|$. We claim that $G' = L(G)$. Suppose, on the contrary, that $G' < L(G)$. Then $G/L(G)$ is a proper quotient subgroup of G/G' and $|G/G'/G/L(G)| = |L(G)/G'| > 1$. It thus follows from [3, Lemma D] and $L(G) \leq Z(G)$ that $\mathrm{Hom}(G/Z(G), G')$ is isomorphic to a proper subgroup of $\mathrm{Hom}(G/G', L(G))$. Hence $|\mathrm{Aut}_c(G)| < |\mathrm{Aut}^L(G)|$, a contradiction. Therefore $G' = L(G)$.
Next, let $L(G) \neq Z(G)$. So $G' < Z(G)$. By Lemma 3.1(i), we have

$$|\mathrm{Hom}(G/Z(G), L(G))| = |\mathrm{Hom}(G/G', L(G))|$$

and so by Lemma 2.1,

$$\prod_{1\leq i\leq n, 1\leq j\leq l} p^{\min\{e_i, b_j\}} = \prod_{1\leq i\leq k, 1\leq j\leq l} p^{\min\{a_i, b_j\}}.$$

We claim that $n = k$. Suppose, by way of contradiction, that $n < k$. Since $e_j \leq a_j$ for $1 \leq j \leq n$, we have

$$\begin{aligned}|\mathrm{Hom}(G/Z(G), L(G))| &= |\mathrm{Hom}(C_{p^{e_1}} \times C_{p^{e_2}} \times ... \times C_{p^{e_n}}, L(G))| \\ &\leq |\mathrm{Hom}(C_{p^{a_1}} \times C_{p^{a_2}} \times ... \times C_{p^{a_n}}, L(G))| \\ &< |\mathrm{Hom}(C_{p^{a_1}} \times C_{p^{a_2}} \times ... \times C_{p^{a_n}}, L(G))| \\ &\times |\mathrm{Hom}(C_{p^{a_{n+1}}} \times ... \times C_{p^{a_k}}, L(G))| \\ &= |\mathrm{Hom}(C_{p^{a_1}} \times C_{p^{a_2}} \times ... \times C_{p^{a_k}}, L(G))| \\ &= |\mathrm{Hom}(G/G', L(G))|,\end{aligned}$$

which is impossible. Therefore $n = k$, as required. Since $a_i \geq e_i$ for $1 \leq i \leq n$, we have $\min\{a_i, b_j\} \geq \min\{e_i, b_j\}$, where $1 \leq i \leq n, 1 \leq j \leq l$. Thus $\min\{a_i, b_j\} = \min\{e_i, b_j\}$, for $1 \leq i \leq n, 1 \leq j \leq l$. Next since $G' < Z(G)$, there exists some $1 \leq j \leq n$ such that $e_j < a_j$. Let t be the largest integer between 1 and n such that $e_t < a_t$. We claim that $b_1 \leq e_t$. Suppose, on the contrary, that $b_1 > e_t$. Then by the above equality, $\min\{a_t, b_1\} = \min\{e_t, b_1\} = e_t$, which is impossible. Thus $b_1 \leq e_t$. Now by [8, Lemma 0.4], $e_1 = \exp(G/Z(G)) = \exp(G') = \exp(L(G)) = b_1$. Hence $b_1 \leq e_t \leq ... \leq e_2 \leq e_1 = b_1$ and so $b_1 = e_t$.

Conversely, let $G' = L(G) = Z(G)$ and $|\mathrm{Aut}_c(G)| = \prod_{1\leq i\leq n} |\Omega_{e_i}(G')|$. Then by [13, Theorem 3.12], $\mathrm{Aut}_c(G) = \mathrm{Aut}^Z(G) = \mathrm{Aut}^L(G)$. Now let $G' = L(G) < Z(G) \leq \Phi(G)$, $|\mathrm{Aut}_c(G)| = \prod_{1\leq i\leq n} |\Omega_{e_i}(G')|$, $n = k$ and $b_1 = e_t$, where t is the largest integer between 1 and n such that $a_t > e_t$. Let $G/Z(G) = \langle \bar{x_1}, \bar{x_2}, ..., \bar{x_n}\rangle$, where $\bar{x_i} = x_i Z(G)$ and $o(\bar{x_i}) = p^{e_i}$ for $0 < i \leq n$. Since $Z(G) \leq \Phi(G)$, by [13, Lemma 3.5], $\{x_1, ..., x_n\}$ is a minimal generating set for G. We observe that

$$|\mathrm{Hom}(G/Z(G), L(G))| = \prod_{1\leq i\leq n, 1\leq j\leq l} p^{\min\{e_i, b_j\}}$$

and

$$|\mathrm{Hom}(G/G', L(G))| = \prod_{1\leq i\leq k, 1\leq j\leq l} p^{\min\{a_i, b_j\}}.$$

Now since $b_1 = e_t$, then $e_1 \geq e_2 \geq ... \geq e_{t-1} \geq e_t = b_1 \geq b_2 \geq ... \geq b_l > 0$. Therefore, $b_j \leq e_i \leq a_i$ for all $1 \leq j \leq l$ and $1 \leq i \leq t$. So $\min\{a_i, b_j\} = b_j = \min\{e_i, b_j\}$ for $1 \leq i \leq t$ and $1 \leq j \leq l$. Next, since $a_i = e_i$, where $i > t$, we have $\min\{a_i, b_j\} = \min\{e_i, b_j\}$ for all $t + 1 \leq i \leq k$ and $1 \leq j \leq l$. Thus $\min\{a_i, b_j\} = \min\{e_i, b_j\}$ for all $1 \leq i \leq k$ and $1 \leq j \leq l$, which shows that $|\mathrm{Hom}(G/Z(G), L(G))| = |\mathrm{Hom}(G/G', L(G))|$. Hence

$$\begin{aligned}|\mathrm{Aut}_c(G)| &= \prod_{1\leq i\leq n} |\Omega_{e_i}(G')| = \prod_{1\leq i\leq n} |\mathrm{Hom}(\langle \bar{x_i}\rangle, \Omega_{e_i}(G'))| \\ &= \textstyle\prod_{1\leq i\leq n} |\mathrm{Hom}(\langle \bar{x_i}\rangle, G')| = |\mathrm{Hom}(G/Z(G), L(G))| \\ &= |\mathrm{Hom}(G/G', L(G))| = |\mathrm{Aut}^L(G)| = |\mathrm{Aut}^{G'}(G)|.\end{aligned}$$

Now since $\mathrm{Aut}_c(G) \leq \mathrm{Aut}^{G'}(G)$, the proof is complete. □

Camina groups were introduced by Camina in [1]. Let G be a finite group and N be non-trivial proper normal subgroup of G. Then (G, N) is called a Camina pair if $xN \subseteq x^G$ for all $x \in G\backslash N$. It follows that (G, N) is a Camina pair if and only if $N \subseteq [x, G]$ for all $x \in G\backslash N$. A group G is called a Camina group if (G, G') is a Camina pair.

Corollary 3.2. *Let G be a non-abelian autonilpotent finite p-group of class 2 such that $\exp(L(G)) = p$. Then $\mathrm{Aut}_c(G) = \mathrm{Aut}^L(G)$ if and only if $G' = L(G)$ and G is a Camina p-group.*

Proof. First assume that $G' = L(G)$ and G is a Camina p-group. Then $\mathrm{Aut}_c(G) \leq \mathrm{Aut}^L(G)$. Now let $\sigma \in \mathrm{Aut}^L(G)$ and $x \in G$. If $x \in L(G)$, then $x^\sigma = x$ and if $x \in G\backslash L(G)$, then $x^{-1}x^\sigma \in L(G) \subseteq [x, G]$. This shows that $x^\sigma = y^{-1}xy$, for some $y \in G$. Therefore, $\sigma \in \mathrm{Aut}_c(G)$ and hence $\mathrm{Aut}_c(G) = \mathrm{Aut}^L(G)$.

To prove the converse, assume that $\mathrm{Aut}_c(G) = \mathrm{Aut}^L(G)$. By the main Theorem, [7, Corollary 3.7] and [9, Proposition 2.13], $G' = L(G) = \Phi(G)$ and $|\mathrm{Aut}_c(G)| = \prod_{1\le i\le n} |\Omega_{e_i}(G')|$. Next, since $\exp(G/Z(G)) = \exp(G') = p$, we have $|\mathrm{Aut}_c(G)| = p^{ns}$. Notice that any element $x \in G\backslash L(G)$ can be included in a minimal generating set $\{x = x_1, x_2, ..., x_n\}$ for G. If $[x, G] \subset L(G) = G'$, then $|x^G| < |G'|$, since $|x^G| = |[x, G]|$. Hence $|\mathrm{Aut}_c(G)| < p^{ns}$, which is a contradiction. Thus G is a Camina p-group. □

Corollary 3.3. *Let G be a finite non-abelian p-group such that G' is cyclic. Then $\mathrm{Aut}_c(G) = \mathrm{Aut}^L(G)$ if and only if $G' = L(G)$ and $Z(G) = L(G)G^{p^n}$ where $\exp(L(G)) = p^n$.*

Proof. If $G' = L(G)$ is cyclic, then by [12, Corollary 3.6], $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$. On the other hand, we observe that every element of $\mathrm{Aut}^L(G)$ fixes any element of $Z(G)$. Hence $\mathrm{Aut}^L(G) = C_{\mathrm{Aut}^L(G)}(Z(G)) = \mathrm{Inn}(G)$, by [10, Proposition 3.2]. Thus $\mathrm{Aut}_c(G) = \mathrm{Aut}^L(G)$. The converse follows from [10, Theorem 3.3]. □

By [7, Lemma 4.4], if G is an abelian p-group, then $|L(G)| = 1, 2$. Hence $\mathrm{Aut}^L(G) \cong C_2^d$, where $d = d(G)$ or $|\mathrm{Aut}^L(G)| = 1$. In the following corollary, we will characterize all finite non-abelian p-groups G of order $p^n (n \le 5)$, such that $\mathrm{Aut}^c(G) = \mathrm{Aut}^L(G)$.

Corollary 3.4. *Let G be a finite non-abelian p-group of order $p^n, n \le 5$. Then $\mathrm{Aut}_c(G) = \mathrm{Aut}^L(G)$ if and only if $p = 2$, $G' = L(G)$ is cyclic and $Z(G) = \Phi(G)$.*

Proof. We can assume that $|G| = p^n, 3 \le n \le 5$. Let $\mathrm{Aut}_c(G) = \mathrm{Aut}^L(G)$. Then $G' = L(G)$, by the main Theorem. We distinguish two cases:

CASE I. p is an odd prime. If $|G| = p^3$, then G is an extraspecial p-group and $\mathrm{Inn}(G) \le \mathrm{Aut}_c(G) \le \mathrm{Aut}^{G'}(G) \cong \mathrm{Hom}(G/G', G') \cong \mathrm{Inn}(G)$, by Lemma 2.2. Hence $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$. If $|G| = p^4$ or $|G| = p^5$, then $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$, by [6] and [4, Theorem 2.3]. Thus $\mathrm{Inn}(G) = \mathrm{Aut}^L(G)$, which is a contradiction by [11, Section 4].

CASE II. Let $p = 2$. We discuss the following cases. If $|G| = 2^3$, then G is either D_8 or Q_8 and $G' = L(G) \cong C_2$. Hence by [12, Corollary 3.6], $\mathrm{Aut}_c(G) = \mathrm{Aut}^L(G) = \mathrm{Inn}(G)$ and so $Z(G) = \Phi(G)$, by Corollary 3.3. Next, assume that $|G| = 2^4$. We claim that $|Z(G)| = 4$. By way of contradiction, suppose that $|Z(G)| = 2$. Since G is of class $2, G' \le$

$Z(G) \cong C_2$. So G is an extraspecial 2-group, which is a contradiction since the order of G is not of the form 2^{2k+1}, for some $k \in \mathbb{N}$. Therefore $G/Z(G) \cong C_2^2$, and hence $|G'| = 2$. Thus $\mathrm{Aut}^L(G) = \mathrm{Aut}_c(G) = \mathrm{Inn}(G)$. Now by Corollary 3.3, $Z(G) = \Phi(G)$.
Finally, assume that $|G| = 2^5$. By [4, Theorem 2.3], $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$. Hence $\mathrm{Aut}^{G'}(G) = \mathrm{Aut}^L(G) = \mathrm{Inn}(G)$ and by [10, Theorem 3.3], G' is cyclic. Since $d(G/L(G)) > 1$, by [7, Theorem 5.1], we can assume that $|L(G)| = 2, 4$. First, suppose that $G' = L(G) \cong C_4$. Then $G/L(G)$ is one of the groups C_2^3 or $C_4 \times C_2$. In the first case, $\mathrm{Aut}^{G'}(G) = \mathrm{Inn}(G) \cong C_2^3$, by Lemma 2.2. Whence $G' = L(G) = Z(G) = \Phi(G) \cong C_4$ and $\exp(G') = \exp(G/Z(G)) = 4$, a contradiction. If $G/L(G) \cong C_4 \times C_2$, then by [7, Theorems 4.7 and 5.1], G is one of the groups $M_2(2, 3)$ or $M_2(3, 1, 1)$ and $L(G) \cong C_2^2$, which is impossible. Therefore, $|L(G)| = 2$ and by Corollary 3.3, $Z(G) = \Phi(G)$. The converse follows at once from Corollary 3.3. □

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ON EQUALITY OF ABSOLUTE CENTRAL AND CLASS PRESERVING AUTOMORPHISMS OF FINITE p–GROUPS

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بررسی تساوی خودریختی‌های مرکزی مطلق و حافظ رده در p–گروه‌های متناهی

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فرض کنیم G یک p–گروه متناهی غیرآبلی و $L(G)$ مرکز مطلق G باشد. هم‌چنین فرض کنیم $\mathrm{Aut}^L(G)$ و $\mathrm{Aut}_c(G)$ به‌ترتیب گروه خودریختی‌های مرکزی مطلق و حافظ رده‌ی G باشند. ما در این مقاله، شرط لازم و کافی را برای گروه G فراهم می‌کنیم به‌طوری‌که $\mathrm{Aut}^L(G) = \mathrm{Aut}_c(G)$. هم‌چنین تمام گروه‌های غیرآبلی از مرتبه‌ی p^n ($n \leq ۵$) را که هر خودریختی مرکزی مطلق آن‌ها حافظ رده باشد را دسته‌بندی می‌کنیم.

کلمات کلیدی: مرکز مطلق، خودریختی‌های مرکزی مطلق، خودریختی‌های حافظ رده، p–گروه‌های متناهی.