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ON GRADED INJECTIVE DIMENSION

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Abstract. There are remarkable relations between the graded homological dimensions and the ordinary homological dimensions. In this paper, we study the injective dimension of a complex of graded modules and derive its some properties. In particular, we define the *dualizing complex for a graded ring and investigate its consequences.

1. Introduction

Let R be a Noetherian $\mathbb{Z}$-graded ring. In [5] and [6], Fossum and Fossum-Foxby have studied the graded homological dimension of graded modules and compare them with classical homological dimensions. They showed that for a graded R-module M, one has

$$^*\operatorname{id}_R M \leq \operatorname{id}_R M \leq {}^*\operatorname{id}_R M + 1,$$

where $\operatorname{id}_R M$ (resp., $^*\operatorname{id}_R M$) denotes the injective dimension of M in the category of R-modules (resp., category of graded R-modules). It is natural to ask how these inequalities hold for the injective dimension of a complex of graded modules and homogeneous homomorphisms. Section 2 of this paper is devoted to review some hyper-homological algebra for the derived category of the graded ring R. In Section 3, we define the *injective dimension of complexes of graded modules and homogeneous homomorphisms, and derive its some properties. Among other results, we prove the generalization of the dual version of Auslander-Buchbaum equality, which implies the known inequality

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$^*\operatorname{id}_R X \leq \operatorname{id}_R X \leq {}^*\operatorname{id}_R X + 1$ for a complex of graded modules and homogeneous homomorphisms X. Throughout this paper, R is commutative and all complexes are chain complexes, that is their indexes increase to left. For more details on graded rings and modules, see [2] and [6].

2. Derived category of complexes of graded modules

Let X be a complex of R-modules and R-homomorphisms. The *supremum* and the *infimum* of a complex X, denoted by $\sup(X)$ and $\inf(X)$, are defined by the supremum and the infimum of the set $\{i \in \mathbb{Z} | \mathrm{H}_i(X) \neq 0\}$. For an integer m, $\Sigma^m X$ denotes the complex X *shifted* m degrees to the left; it is given by

$$(\Sigma^m X)_\ell = X_{\ell-m} \text{ and } \partial_\ell^{\Sigma^m X} = (-1)^m \partial_{\ell-m}^X,$$

for $\ell \in \mathbb{Z}$.

The symbol $\mathcal{D}(R)$ denotes the *derived category* of R-complexes. The full subcategories $\mathcal{D}_\sqsubset(R)$, $\mathcal{D}_\sqsupset(R)$, $\mathcal{D}_\square(R)$ and $\mathcal{D}_0(R)$ of $\mathcal{D}(R)$ consist of R-complexes X while $\mathrm{H}_\ell(X) = 0$, for respectively $\ell \gg 0$, $\ell \ll 0$, $|\ell| \gg 0$ and $\ell \neq 0$. Homology isomorphisms are marked by the sign $\simeq$. The right derived functor of the homomorphism functor of R-complexes and the left derived functor of the tensor product of R-complexes are denoted by $\mathbf{R}\operatorname{Hom}_R(-,-)$ and $- \otimes_R^{\mathbf{L}} -$, respectively.

Let $M = \oplus_{n\in\mathbb{Z}} M_n$ and $N = \oplus_{n\in\mathbb{Z}} N_n$ be two graded R-modules. The $^*\operatorname{Hom}$ functor is defined by $^*\operatorname{Hom}_R(M,N) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_i(M,N)$, such that $\operatorname{Hom}_i(M,N)$ is a $\mathbb{Z}$-submodule of $\operatorname{Hom}_R(M,N)$ consisting of all $\varphi : M \to N$ such that $\varphi(M_n) \subseteq N_{n+i}$ for all $n \in \mathbb{Z}$. In general, $^*\operatorname{Hom}_R(M,N) \neq \operatorname{Hom}_R(M,N)$, but equality holds if M is finitely generated, see [6, Lemma 4.2]. Also, the tensor product $M \otimes_R N$ of M and N is a graded module with $(M \otimes_R N)_n$ is generated (as a $\mathbb{Z}$-module) by elements $m \otimes n$ with $m \in M_i$ and $n \in N_j$ where $i + j = n$.

Let $\{M_\alpha\}_{\alpha\in I}$ be a family of graded R-modules. Then, $\bigoplus_\alpha M_\alpha$ becomes a graded R-module with $(\bigoplus_\alpha M_\alpha)_n = \bigoplus_\alpha (M_\alpha)_n$, for all $n \in \mathbb{Z}$, see [6, Page 289]. Recall that the direct products exist in the category of graded modules. Then the direct product is denoted by $^*\prod_\alpha M_\alpha$ and $(^*\prod_\alpha M_\alpha)_n = \prod_\alpha (M_\alpha)_n$ for all $n \in \mathbb{Z}$, see [6, Page 289]. In this case, there are the following bijections [6, Page 289]

$$^*\operatorname{Hom}_R(\bigoplus_\alpha M_\alpha, -) \xrightarrow{\cong} {}^*\prod_\alpha {}^*\operatorname{Hom}_R(M_\alpha, -),$$

$$^*\operatorname{Hom}_R(-, {}^*\prod_\alpha M_\alpha) \xrightarrow{\cong} {}^*\prod_\alpha {}^*\operatorname{Hom}_R(-, M_\alpha).$$

Likewise, direct limits exist in the category of graded modules with

$$({}^*\varinjlim M_\alpha)_n = \varinjlim (M_\alpha)_n$$

for all $n \in \mathbb{Z}$, see [6, Page 289].

The symbol ${}^*\mathcal{C}(R)$ denotes the category of complexes of graded R-modules and homogeneous differentials. Remember that the category of graded modules is an abelian category. The derived category of ${}^*\mathcal{C}(R)$ will be denoted by ${}^*\mathcal{D}(R)$, (see [10]). Analogously we have ${}^*\mathcal{C}_\sqsubset(R)$, ${}^*\mathcal{C}_\sqsupset(R)$, ${}^*\mathcal{C}_\square(R)$ and ${}^*\mathcal{C}_0(R)$ (resp. ${}^*\mathcal{D}_\sqsubset(R)$, ${}^*\mathcal{D}_\sqsupset(R)$, ${}^*\mathcal{D}_\square(R)$ and ${}^*\mathcal{D}_0(R)$) which are the full subcategories of ${}^*\mathcal{C}(R)$ (resp. ${}^*\mathcal{D}(R)$).

For R-complexes X and Y of graded modules, with homogeneous differentials ∂^X and ∂^Y the *homomorphism complex* ${}^*\operatorname{Hom}_R(X,Y)$ is defined as:

$${}^*\operatorname{Hom}_R(X,Y)_\ell = {}^*\prod_{p\in\mathbb{Z}} {}^*\operatorname{Hom}_R(X_p, Y_{p+\ell})$$

and when $\psi = (\psi_p)_{p\in\mathbb{Z}}$ belongs to ${}^*\operatorname{Hom}_R(X,Y)_\ell$, then the family $\partial_\ell^{{}^*\operatorname{Hom}_R(X,Y)}(\psi)$ in ${}^*\operatorname{Hom}_R(X,Y)_{\ell-1}$ has p-th component

$$\partial_\ell^{{}^*\operatorname{Hom}_R(X,Y)}(\psi)_p = \partial^Y_{p+\ell}\psi_p - (-1)^\ell \psi_{p-1}\partial^X_p.$$

When $X \in {}^*\mathcal{C}^f_\sqsupset(R)$ and $Y \in {}^*\mathcal{C}_\sqsubset(R)$ all the products

$${}^*\prod_{p\in\mathbb{Z}} {}^*\operatorname{Hom}_R(X_p, Y_{p+\ell})$$

are finite. Note that for each $p \in \mathbb{Z}$, X_p is finitely generated R-module, thus ${}^*\operatorname{Hom}_R(X_p, Y_{p+\ell}) = \operatorname{Hom}_R(X_p, Y_{p+\ell})$, see [6, Lemma 4.2]. Therefore

$${}^*\prod_{p\in\mathbb{Z}} {}^*\operatorname{Hom}_R(X_p, Y_{p+\ell}) = \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R(X_p, Y_{p+\ell}),$$

for every $\ell \in \mathbb{Z}$. Therefore ${}^*\operatorname{Hom}_R(X,Y) = \operatorname{Hom}_R(X,Y)$.

Also the *tensor product complex* $X \otimes_R Y$ is defined as:

$$(X \otimes_R Y)_\ell = \bigoplus_{p\in\mathbb{Z}} (X_p \otimes_R Y_{\ell-p})$$

and the ℓ-th differential $\partial_\ell^{X\otimes_R Y}$ is given on a generator $x_p \otimes y_{\ell-p}$ in $(X \otimes_R Y)_\ell$, where x_p and $y_{\ell-p}$ are homogeneous elements, by

$$\partial_\ell^{X\otimes_R Y}(x_p \otimes y_{\ell-p}) = \partial^X_p(x_p) \otimes y_{\ell-p} + (-1)^p x_p \otimes \partial^Y_{\ell-p}(y_{\ell-p}).$$

If X and Y are R-complexes of graded modules, then ${}^*\operatorname{Hom}_R(X,-)$, ${}^*\operatorname{Hom}_R(-,Y)$, and $X \otimes_R -$ are graded functors on ${}^*\mathcal{C}(R)$.

Note that any object of ${}^*\mathcal{C}_\sqsubset(R)$ has an *injective resolution by [10, Page 47], and any object of ${}^*\mathcal{C}_\sqsupset(R)$ has an *projective resolution by [10, Page 48]. The right derived functor of the ${}^*\operatorname{Hom}$ functor in the category of graded complexes is denoted by $\mathbf{R}\,{}^*\operatorname{Hom}_R(-,-)$ and set ${}^*\operatorname{Ext}^i_R(-,-) = \operatorname{H}_{-i}(\mathbf{R}\,{}^*\operatorname{Hom}_R(-,-))$. It is easily seen that if R is a Noetherian $\mathbb{Z}$-graded ring and $X \in {}^*\mathcal{C}^f_{\sqsupset}(R)$ and $Y \in {}^*\mathcal{C}_{\sqsubset}(R)$ then $\mathbf{R}\,{}^*\operatorname{Hom}_R(X,Y) = \mathbf{R}\operatorname{Hom}_R(X,Y)$. Also the left derived functor of $-\otimes_R -$ in the category of graded complexes is denoted by $-\otimes^{\mathbf{L}^*}_R -$. Since *projective graded R-modules coincide with projective R-modules by [6, Proposition 3.1] we easily see that $-\otimes^{\mathbf{L}^*}_R -$ coincides with the ordinary left derived functor of $-\otimes_R -$ in the category of complexes. So we use $-\otimes^{\mathbf{L}}_R -$ instead of $-\otimes^{\mathbf{L}^*}_R -$.

We recall the definition of the *depth* and *width* of complexes. Let $\mathfrak{a}$ be an ideal in a ring R and X a complex of graded R-modules. The $\mathfrak{a}$-depth and $\mathfrak{a}$-width of X over R are defined respectively by

$$\operatorname{depth}(\mathfrak{a},X) := -\sup \mathbf{R}\operatorname{Hom}_R(R/\mathfrak{a},X),$$
$$\operatorname{width}(\mathfrak{a},X) := \inf(R/\mathfrak{a}\otimes^{\mathbf{L}}_R X).$$

For a local ring $(R,\mathfrak{m})$ set $\operatorname{depth}_R X := \operatorname{depth}(\mathfrak{m},X)$; $\operatorname{width}_R X := \operatorname{width}(\mathfrak{m},X)$. Let $(R,\mathfrak{m})$ be a *local graded ring and X be a complex of graded R-modules. By [2, Proposition 1.5.15(c)], $-\otimes_R R_{\mathfrak{m}}$ is a faithfully exact functor on the category of graded R-modules. Then we have

$$\begin{aligned}\operatorname{width}(\mathfrak{m},X) &= \inf\{i|\operatorname{H}_i(R/\mathfrak{m}\otimes^{\mathbf{L}}_R X)\neq 0\}\\ &= \inf\{i|\operatorname{H}_i(R/\mathfrak{m}\otimes^{\mathbf{L}}_R X)\otimes_R R_{\mathfrak{m}}\neq 0\}\\ &= \inf\{i|\operatorname{H}_i(R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}\otimes^{\mathbf{L}}_{R_{\mathfrak{m}}} X_{\mathfrak{m}})\neq 0\}\\ &= \operatorname{width}(\mathfrak{m}R_{\mathfrak{m}},X_{\mathfrak{m}}) = \operatorname{width}_{R_{\mathfrak{m}}} X_{\mathfrak{m}}.\end{aligned}$$

Likewise we have $\operatorname{depth}(\mathfrak{m},X) = \operatorname{depth}_{R_{\mathfrak{m}}} X_{\mathfrak{m}}$.

3. *INJECTIVE DIMENSION

The injective dimension of a complex X, denoted by $\operatorname{id}_R X$, is defined and studied in [1]. A graded module J is called *injective if it is an injective object in the category of graded modules. In other words, the functor ${}^*\operatorname{Hom}_R(-,J)$ is exact in this category. A long exact sequence of *injective modules is called *injective resolution. The injective dimension of a graded module M in the category of graded modules, is denoted by ${}^*\operatorname{id}_R M$ (cf. [6, 2]). In this section we study the *injective dimension of homologically left bounded complexes of graded modules.

Let $n \in \mathbb{Z}$. A homologically left bounded complex of graded modules X, is said to have *injective dimension at most n, denoted by ${}^*\operatorname{id}_R X \leq n$, if there exists an *injective resolution $X \to I$, such that $I_i = 0$ for

$i < -n$. If ${}^*\operatorname{id}_R X \leq n$ holds, but ${}^*\operatorname{id}_R X \leq n-1$ does not, we write ${}^*\operatorname{id}_R X = n$. If ${}^*\operatorname{id}_R X \leq n$ for all $n \in \mathbb{Z}$ we write ${}^*\operatorname{id}_R X = -\infty$. If ${}^*\operatorname{id}_R X \leq n$ for no $n \in \mathbb{Z}$ we write ${}^*\operatorname{id}_R X = \infty$. The following theorem inspired by [1, Theorem 2.4.I and Corollary 2.5.I].

Theorem 3.1. *For $X \in {}^*\mathcal{D}_{\sqsubset}(R)$ and $n \in \mathbb{Z}$ the following are equivalent:*

(1) ${}^*\operatorname{id}_R X \leq n$.
(2) $n \geq -\sup U - \inf(\mathbf{R}^*\operatorname{Hom}_R(U, X))$ *for all* $U \in {}^*\mathcal{D}_{\square}(R)$ *and* $H(U) \neq 0$.
(3) $n \geq -\inf X$ *and* ${}^*\operatorname{Ext}^{n+1}_R(R/J, X) = 0$ *for every homogeneous ideal J of R.*
(4) $n \geq -\inf X$ *and for any (resp. some) *injective resolution I of X, the graded R-module* $\operatorname{Ker}(\partial_{-n} : I_{-n} \to I_{-n-1})$ *is *injective.*

Moreover the following holds:

$$\begin{aligned} {}^*\operatorname{id}_R X &= \sup\{j \in \mathbb{Z} \mid {}^*\operatorname{Ext}^j_R(R/J, X) \neq 0 \textit{ for some homogeneous ideal } J\} \\ &= \sup\{-\sup(U) - \inf(\mathbf{R}^*\operatorname{Hom}_R(U, X)) \mid U \not\cong 0 \textit{ in } {}^*\mathcal{D}_{\square}(R)\}. \end{aligned}$$

Proof. (1) $\Rightarrow$ (2) Let $t := \sup U$ and I be an *injective resolution of X, such that, for all $i < -n$, $I_i = 0$. Then we have

$$ {}^*\operatorname{Ext}^i_R(U, X) \cong \mathrm{H}_{-i}({}^*\mathrm{Hom}_R(U, I)). $$

Since ${}^*\operatorname{Hom}_R(U, I)_{-i} = 0$ for $-i < -n - t$, the assertion follows.

(2) $\Rightarrow$ (3) It is trivial that ${}^*\operatorname{Ext}^{n+1}_R(R/J, X) = 0$ for every homogeneous ideal J of R. For the second assertion let $U = R$ in (2). So that $\operatorname{Ext}^i_R(R, X) = {}^*\operatorname{Ext}^i_R(R, X) = 0$ for $i > n$. Now by [1, Lemma 1.9(b)], we have $\mathrm{H}_{-i}(X) = 0$ for $-i < -n$. This means that $n \geq -\inf X$.

(3) $\Rightarrow$ (4) By hypothesis of (4) $\mathrm{H}_i(I) = 0$ for $i < -n$. Thus the complex

$$ \cdots \to 0 \to 0 \to I_{-n} \to I_{-n-1} \to \cdots \to I_i \to I_{i-1} \to \cdots $$

gives an *injective resolution of $\operatorname{Ker}\partial_{-n}$. In particular

$$ {}^*\operatorname{Ext}^1_R(R/J, \operatorname{Ker}\partial_{-n}) = \mathrm{H}_{-n-1}{}^*\operatorname{Hom}_R(R/J, I) = {}^*\operatorname{Ext}^{n+1}_R(R/J, X) = 0 $$

for every homogeneous ideal J of R. Thus $\operatorname{Ker}\partial_{-n}$ is *injective by [6, Corollary 4.3].

(4) $\Rightarrow$ (1) Let I be any *injective resolution of X. By assumption, the module $\operatorname{Ker}\partial_{-n}$ is *injective. Thus ${}^*\operatorname{id}_R X < -n$ by definition.

The last equalities are easy consequences of (1),. . .,(4). □

For a local ring $(R, \mathfrak{m}, k)$ and for an R-complex X and $i \in \mathbb{Z}$ the ith *Bass number* and *Betti number* of X are defined respectively by

$\mu^i_R(X) := \dim_k \mathrm{H}_{-i}(\mathbf{R}\operatorname{Hom}_R(k,X))$ and $\beta^R_i(X) := \dim_k \mathrm{H}_i(k \otimes^{\mathbf{L}}_R X)$. It is well-known that for $X \in \mathcal{D}_{\sqsubset}(R)$ one has (cf. [1, Proposition 5.3.I])

$$\operatorname{id}_R X = \sup\{m \in \mathbb{Z} | \exists \mathfrak{p} \in \operatorname{Spec}(R); \mu^m_{R_\mathfrak{p}}(X_\mathfrak{p}) \neq 0\}.$$

As a graded analogue we have:

Proposition 3.2. *For $X \in {}^*\mathcal{D}_{\sqsubset}(R)$ we have the following equality*

$${}^*\operatorname{id}_R X = \sup\{m \in \mathbb{Z} | \exists \mathfrak{p} \in^* \operatorname{Spec}(R); \mu^m_{R_\mathfrak{p}}(X_\mathfrak{p}) \neq 0\}.$$

Proof. The argument is the same as proof of [1, Proposition 5.3.I] with some changes. Denote the supremum by i. By Theorem 3.1, we have ${}^*\operatorname{id}_R X \geq i$. Hence the equality holds if $i = \infty$. Thus assume that i is finite. By Theorem 3.1 we have to show that if ${}^*\operatorname{Ext}^j_R(M,X) \neq 0$ for some finitely generated graded R-module M, then $j \leq i$; this implies that ${}^*\operatorname{id}_R X \leq i$. The elements of $Ass(M)$ are homogeneous prime ideals. Thus there exists a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ of graded submodules of M such that for each i we have $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ with $\mathfrak{p}_i \in \operatorname{Supp} M$ and $\mathfrak{p}_i$ is homogeneous. From the long exact sequence of ${}^*\operatorname{Ext}^j_R(-,X) \neq 0$ the set

$$\{\mathfrak{q} \in \operatorname{Spec}(R) | \text{ there is an } h \geq j \text{ such that } {}^*\operatorname{Ext}^h_R(R/\mathfrak{q},X) \neq 0\},$$

turns to be non empty. Let $\mathfrak{p}$ be maximal in this set and for a homogeneous $x \in R \backslash \mathfrak{p}$ consider the exact sequence

$$0 \to R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \to R/(\mathfrak{p}+Rx) \to 0.$$

It induces an exact sequence

$$\begin{aligned}&{}^*\operatorname{Ext}^h_R(R/(\mathfrak{p}+Rx),X) \to {}^*\operatorname{Ext}^h_R(R/\mathfrak{p},X) \xrightarrow{x} {}^*\operatorname{Ext}^h_R(R/\mathfrak{p},X)\\ &\to {}^*\operatorname{Ext}^{h+1}_R(R/(\mathfrak{p}+Rx),X)\end{aligned}$$

in which the left-hand term is trivial because of the maximality of $\mathfrak{p}$. Thus ${}^*\operatorname{Ext}^h_R(R/\mathfrak{p},X) \xrightarrow{x} {}^*\operatorname{Ext}^h_R(R/\mathfrak{p},X)$ is injective for all homogeneous elements $x \in R\backslash\mathfrak{p}$, hence so is the homogeneous localization homomorphism ${}^*\operatorname{Ext}^h_R(R/\mathfrak{p},X) \to {}^*\operatorname{Ext}^h_R(R/\mathfrak{p},X)_{(\mathfrak{p})}$. Thus the free $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})}$-module ${}^*\operatorname{Ext}^h_R(R/\mathfrak{p},X)_{(\mathfrak{p})}$ is nonzero. Consequently

$$({}^*\operatorname{Ext}^h_R(R/\mathfrak{p},X)_{(\mathfrak{p})})_{\mathfrak{p}R_{(\mathfrak{p})}} \cong \operatorname{Ext}^h_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p},X_\mathfrak{p})$$

is nonzero. This implies that $j \leq h \leq i$. ∎

Remark 3.3. (1) A graded module is called *projective if it is a projective object in the category of graded modules. By [6, Proposition 3.1] the *projective graded R-modules coincide with projective R-modules. The projective dimension of a graded module M in the category of graded modules, is denoted by ${}^*\operatorname{pd}_R M$ (cf. [6]). Let $n \in \mathbb{Z}$. A homologically right bounded complex of graded modules X, is said to have *projective dimension at most n, denoted by ${}^*\operatorname{pd}_R X \leq n$, if there exists a *projective resolution $P \to X$, such that $P_i = 0$ for $i > n$. If ${}^*\operatorname{pd}_R X \leq n$ holds, but ${}^*\operatorname{pd}_R X \leq n-1$ does not, we write ${}^*\operatorname{pd}_R X = n$. If ${}^*\operatorname{pd}_R X \leq n$ for all $n \in \mathbb{Z}$ we write ${}^*\operatorname{pd}_R X = -\infty$. If ${}^*\operatorname{pd}_R X \leq n$ for no $n \in \mathbb{Z}$ we write ${}^*\operatorname{pd}_R X = \infty$.

(2) For $X \in {}^*\mathcal{D}_{\sqsupset}(R)$ by the same method as in [1, Theorem 2.4.P and Corollary 2.5.P] we have

$$\begin{aligned} {}^*\operatorname{pd}_R X &= \sup\{j \in \mathbb{Z} | {}^*\operatorname{Ext}^j_R(X, N) \neq 0 \text{ for some graded } R\text{-module } N\} \\ &= \sup\{\inf(U) - \inf(\mathbf{R}\,{}^*\operatorname{Hom}_R(X, U)) | U \not\simeq 0 \text{ in } {}^*\mathcal{D}_{\square}(R)\}. \end{aligned}$$

(3) It is easy to see that for $X \in {}^*\mathcal{D}_{\sqsupset}(R)$, we have ${}^*\operatorname{pd}_R X \leqslant \operatorname{pd}_R X$.

(4) The notions of *flat module and *flat dimension are obtained by replacing 'projective' by 'flat' in (1). By [6, Proposition 3.2] the *flat graded R-modules coincide with flat R-modules. Therefore for a homologically right bounded complex of graded modules X, we have ${}^*\operatorname{fd}_R X \leqslant {}^*\operatorname{pd}_R X$.

The proof of the following proposition is easy so we omit it (see [2, Theorem 1.5.9]). Let J be an ideal of the graded ring R. Then the graded ideal J^* denotes the ideal generated by all homogeneous elements of J. It is well-known that if $\mathfrak{p}$ is a prime ideal of R, then $\mathfrak{p}^*$ is a homogeneous prime ideal of R by [2, Lemma 1.5.6].

Proposition 3.4. *Let $X \in {}^*\mathcal{D}_{\square}(R)$ and $\mathfrak{p}$ is a non-homogeneous prime ideal in R. Then $\mu^{i+1}_{R_\mathfrak{p}}(X_\mathfrak{p}) = \mu^i_{R_{\mathfrak{p}^*}}(X_{\mathfrak{p}^*})$ and $\beta^{R_\mathfrak{p}}_i(X_\mathfrak{p}) = \beta^{R_{\mathfrak{p}^*}}_i(X_{\mathfrak{p}^*})$ for any integer $i \geq 0$.*

Corollary 3.5. *Let $X \in {}^*\mathcal{D}_{\square}(R)$ and $\mathfrak{p}$ be a non-homogeneous prime ideal in R. Then*

$$\operatorname{depth} X_\mathfrak{p} = \operatorname{depth} X_{\mathfrak{p}^*} + 1.$$

Proof. Using Proposition 3.4, we can assume that both $\operatorname{depth} X_\mathfrak{p}$ and $\operatorname{depth} X_{\mathfrak{p}^*}$ are finite. So the equality follows from the fact that over a local ring $(R, \mathfrak{m}, k)$ we have $\operatorname{depth}_R X = \inf\{i \in \mathbb{Z} | \mu^i_R(X) \neq 0\}$. □

Foxby defined the *small support* of a homologically right bounded complex X over a Noetherian ring R, denoted by $\operatorname{supp}_R X$, as the set of prime ideal of R such that $R_{(\mathfrak{p})}/\mathfrak{p}R_{(\mathfrak{p})} \otimes^L_R X$ is non-trivial complex (See [7]). It is well known that;

$$\operatorname{supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R | \exists m \in \mathbb{Z} : \beta^{R_\mathfrak{p}}_m(X_\mathfrak{p}) \neq 0\}.$$

Let ${}^*\operatorname{supp}_R X$ be a subset of $\operatorname{supp}_R X$ consisting of homogeneous prime ideals of $\operatorname{supp}_R X$. Then from Proposition 3.4 we see that $\mathfrak{p} \in \operatorname{supp}_R X$

if and only if $\mathfrak{p}^* \in {}^*\operatorname{supp}_R X$. Also using [12, Lemma 2.3] for $\mathfrak{p} \in \operatorname{supp}_R X$ we have $\operatorname{depth} X_\mathfrak{p} < \infty$ if and only if $\operatorname{width}_{R_\mathfrak{p}} X_\mathfrak{p} < \infty$. Therefore by corollary 3.5 we get;

$$\operatorname{width}_{R_\mathfrak{p}} X_\mathfrak{p} < \infty \Leftrightarrow \operatorname{width}_{R_{\mathfrak{p}^*}} X_{\mathfrak{p}^*} < \infty.$$

Proposition 3.6. *Let $X \in {}^*\mathcal{D}_\square(R)$ and $\mathfrak{p}$ is a non-homogeneous prime ideal in R. Then*

$$\operatorname{width}_{R_\mathfrak{p}} X_\mathfrak{p} = \operatorname{width}_{R_{\mathfrak{p}^*}} X_{\mathfrak{p}^*}.$$

Proof. We can assume that both $\operatorname{width} X_\mathfrak{p}$ and $\operatorname{width} X_{\mathfrak{p}^*}$ are finite numbers, and the argument would be dual to the proof of [2, Theorem 1.5.9]. □

The ungraded version of the following theorem was proved for modules by Chouinard [3, Corollary 3.1], and extended to complexes by Yassemi [12, Theorem 2.10].

Theorem 3.7. *Let $X \in {}^*\mathcal{D}_\square(R)$. If ${}^*\operatorname{id}_R X < \infty$, then*

$${}^*\operatorname{id}_R X = \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{width} X_\mathfrak{p} | \mathfrak{p} \in {}^*\operatorname{Spec}(R)\}.$$

Proof. We have the following computations

$$\begin{aligned} {}^*\operatorname{id}_R X &= \sup\{m \in \mathbb{Z} | \exists \mathfrak{p} \in {}^*\operatorname{Spec}(R) : \mu^m_{R_\mathfrak{p}}(X_\mathfrak{p}) \neq 0\} \\ &= \sup\{m \in \mathbb{Z} | \exists \mathfrak{p} \in {}^*\operatorname{Spec}(R) : \operatorname{H}_m(\mathbf{R}\operatorname{Hom}_{R_\mathfrak{p}}(\kappa(p), X_\mathfrak{p})) \neq 0\} \\ &= \sup\{-\inf \mathbf{R}\operatorname{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), X_\mathfrak{p}) | \mathfrak{p} \in {}^*\operatorname{Spec}(R)\} \\ &= \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} X_\mathfrak{p} | \mathfrak{p} \in {}^*\operatorname{Spec}(R)\}. \end{aligned}$$

The first equality holds by Proposition 3.2, and the last one holds by [12, Lemma 2.6(a)], since $\operatorname{id}_R X < \infty$ by Propositions 3.2 and 3.4. □

The following corollary was already known for graded modules in [6, Corollary 4.12].

Corollary 3.8. *For every $X \in {}^*\mathcal{D}_\square(R)$, we have*

$${}^*\operatorname{id}_R X \leq \operatorname{id}_R X \leq {}^*\operatorname{id}_R X + 1.$$

Proof. First of all note that by Proposition 3.4, $\operatorname{id}_R X < \infty$ if and only if ${}^*\operatorname{id}_R X < \infty$. The first inequality is clear by Theorem 3.7 and [12, Theorem 2.10]. For the second one let $\mathfrak{p} \in \operatorname{Spec} R$ be such that $\operatorname{id}_R X = \operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p}$ by [12, Theorem 2.10]. By Corollary 3.5 and Proposition 3.6 we have

$$\operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} \leq \operatorname{depth} R_{\mathfrak{p}^*} - \operatorname{width}_{R_{\mathfrak{p}^*}} M_{\mathfrak{p}^*} + 1 \leq {}^*\operatorname{id}_R X + 1,$$

where the second inequality holds by Theorem 3.7. □

Here we define the *dualizing complex for a graded ring and prove some related results.

Definition 3.9. A *dualizing complex D for a graded ring R is a homologically finite and bounded complex of graded R-modules, such that $^*\operatorname{id}_R D < \infty$ and the homothety morphism $\psi : R \to \mathbf{R}\,^*\operatorname{Hom}_R(D, D)$ is invertible in $^*\mathcal{D}(R)$.

Corollary 3.10. *Any *dualizing complex for R is a dualizing complex for R.*

The proof of the following lemma is the same as [10, Chapter V, Proposition 3.4].

Lemma 3.11. *Let $(R, \mathfrak{m}, k)$ be a *local ring and D is a *dualizing complex of R. Then there exists an integer t such that $H^i(\boldsymbol{R}\,^*\operatorname{Hom}_R(k, D))= 0$ for $i \neq t$ and $H^t(\boldsymbol{R}\,^*\operatorname{Hom}_R(k, D)) \cong k$.*

Assume that $(R, \mathfrak{m})$ is a *local ring. A *dualizing complex D is said to be *normalized *dualizing complex* if $t = 0$ in the lemma. It is easy to see that a suitable shift of any *dualizing complex is a normalized one. Also using [10, Chapter V, Proposition 3.4] we see that if D is a normalized *dualizing complex for $(R, \mathfrak{m})$, then $D_{\mathfrak{m}}$ is a normalized dualizing complex for $R_{\mathfrak{m}}$.

Lemma 3.12. *Let $(R, \mathfrak{m}, k)$ be a *local ring and that D is a normalized *dualizing complex for R. Then there exists a natural functorial isomorphism from the category of graded modules of finite length to itself*

$$\phi : H^0(\boldsymbol{R}\,^*\operatorname{Hom}_R(-, D)) \to {}^*\operatorname{Hom}_R(-, {}^*\operatorname{E}_R(k)),$$

where $^*E_R(k)$ is the *injective envelope of k over R.

Proof. Since D is a normalized *dualizing complex for R, the functor $T := \operatorname{H}^0(\mathbf{R}\,^*\operatorname{Hom}_R(-, D))$ is an additive contravariant exact functor from the category of graded modules of finite length to itself. Let M be a graded R-module and $m \in M$ is a homogeneous element of degree α. Then $\epsilon_m : R(-\alpha) \to M$ is a homogeneous morphism which sends 1 into m. Thus we have a homogeneous morphism $\psi(M) : T(M) \to {}^*\operatorname{Hom}_R(M, T(R))$ which sends a homogeneous element $x \in T(M)$ to a morphism $f_x \in {}^*\operatorname{Hom}_R(M, T(R))$ such that $f_x(m) = T(\epsilon_m)(x)$ for every homogeneous element $m \in M$. It is easy to see that it is functorial on M. Thus there exists a natural functorial morphism $\psi : T \to {}^*\operatorname{Hom}_R(-, T(R))$. Note that if M is a finite graded R-module, using a finite presentation of M, there is an isomorphism ${}^*\varinjlim{}^*\operatorname{Hom}_R(M, T(R/\mathfrak{m}^n)) \xrightarrow{\cong} {}^*\operatorname{Hom}_R(M, {}^*\varinjlim T(R/\mathfrak{m}^n))$. Therefore by the same method of [9, Lemma 4.4 and Propositions 4.5], there is a functorial isomorphism

$$\phi : \mathrm{H}^0(\mathbf{R}\, {}^*\operatorname{Hom}_R(-, D)) \to {}^*\operatorname{Hom}_R(-, {}^*\varinjlim T(R/\mathfrak{m}^n)),$$

from the category of graded modules of finite length to itself. Using the technique of proof of [9, Proposition 4.7] in conjunction with [6, Corollary 4.3], we see that ${}^*\varinjlim T(R/\mathfrak{m}^n)$ is an *injective R-module. Since D is a normalized *dualizing complex for R we have

$${}^*\operatorname{Hom}_R(k, {}^*\varinjlim T(R/\mathfrak{m}^n)) \cong \mathrm{H}^0(\mathbf{R}\, {}^*\operatorname{Hom}_R(k, D)) \cong k.$$

Particularly we can embed k to ${}^*\varinjlim T(R/\mathfrak{m}^n)$. In order to show that ${}^*\varinjlim T(R/\mathfrak{m}^n)$ is an *essential extension of k, let Q be a graded submodule of ${}^*\varinjlim T(R/\mathfrak{m}^n)$ such that $k \cap Q = 0$. Then ${}^*\operatorname{Hom}_R(k, Q)$ can be embed in

$${}^*\operatorname{Hom}_R(k, {}^*\varinjlim T(R/\mathfrak{m}^n)) \cong k.$$

Therefore ${}^*\operatorname{Hom}_R(k, Q) = 0$. On the other hand for each $n \in \mathbb{N}$ the set $V(\mathfrak{m})$ includes $\operatorname{Ass}(T(R/\mathfrak{m}^n))$. Now by [11, Proposition 2.1], the fact that each prime ideal of $\operatorname{Ass}({}^*\varinjlim T(R/\mathfrak{m}^n))$ is the annihilator of a homogeneous element [2, Lemma 1.5.6], and the definition of ${}^*\varinjlim$, we have

$$\operatorname{Ass}({}^*\varinjlim T(R/\mathfrak{m}^n)) \subseteq \bigcup_{n\in\mathbb{N}} \operatorname{Ass}(T(R/\mathfrak{m}^n)) \subseteq V(\mathfrak{m}).$$

Consequently Q has support in $V(\mathfrak{m})$, so that $Q = 0$. Therefore ${}^*\varinjlim T(R/\mathfrak{m}^n) \cong {}^*\mathrm{E}_R(k)$. □

Let $\mathfrak{a}$ be an ideal of R. The right derived *local cohomology functor* with support in $\mathfrak{a}$ is denoted by $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$. Its right adjoint, $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$, is the left derived *local homology functor* with support in $\mathfrak{a}$ (see [8] for detail).

Finally, we have the following proposition, its proof uses Lemma 3.12 and the argument is similar to [10, Chapter V, Proposition 6.1].

Proposition 3.13. *Let $(R, \mathfrak{m}, k)$ be a *local ring and that D be a normalized *dualizing complex for R. Then $\mathbf{R}\Gamma_{\mathfrak{m}}(D) \simeq {}^*\mathrm{E}_R(k)$.*

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ON GRADED INJECTIVE DIMENSION

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رابطه‌های قابل ذکری مابین بعدهای همولوژیکی و بعدهای همولوژیکی مدرج وجود دارد. در این مقاله، بعد انژکتیو همبافت از مدول‌های مدرج مورد مطالعه قرار گرفته و خواص آن بررسی می‌شود. به ویژه، همبافت دوگان‌ساز مدرج، برای یک حلقه مدرج را تعریف کرده و نتایج مربوطه را تعمیم می‌دهیم.

کلمات کلیدی: حلقه‌های مدرج، مدول‌های مدرج، بعد انژکتیو.