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GENERALIZED UNI-SOFT INTERIOR IDEALS IN ORDERED SEMIGROUPS

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Abstract. For all $\mathcal{M}, \mathcal{N} \in P(U)$ such that $\mathcal{M} \subset \mathcal{N}$, we first introduced the definitions of $(\mathcal{M}, \mathcal{N})$-uni-soft ideals and $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideals of an ordered semigroup and studied them. When $\mathcal{M} = \emptyset$ and $\mathcal{N} = U$, we meet the ordinary soft ones. Then we proved that in regular and in intra-regular ordered semigroups the concept of $(\mathcal{M}, \mathcal{N})$-uni-soft ideals and the $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideals coincide. Finally, we introduced $(\mathcal{M}, \mathcal{N})$-uni-soft simple ordered semigroup and characterized the simple ordered semigroups in terms of $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideals.

1. Introduction

An ideal of a semigroup is a special subsemigroup satisfying certain conditions. The best way to know an algebraic structure is to begin with a special substructure of it. There are plenty of papers on ideals. After Zadeh's introduction of fuzzy set in 1965 [20], the fuzzy sets have been used in the reconsideration of classical mathematics. For example, Meng and Guo [15] researched fuzzy ideals of BCK/BCI-algebras, Koguep [13] researched fuzzy ideals of hyperlattices, and Kehayopulu and Tsingelis [9] researched fuzzy interior ideals of ordered semigroups.

This inadequacy is removed by Molodtsov [16], by the invention of soft set theory in 1999. He introduced parameterization tools to tackle various uncertainties. Due to the beauty of parameterization tools,

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several researchers attracted towards this direction. Many papers have been published in this regard. In [14], Maji et al. studied various operations on soft sets. Some new operations on soft sets have been introduced by Ali et al. in [2]. Aktas and Cagman [1], compared soft sets to the related concepts of fuzzy sets and rough sets. Also, Feng and Li [4], considered soft product operations. Jun et al., [6], applied the concept of soft set theory to ordered semigroups. Khan et al. [10, 11, 12], characterized different classes of ordered semigroups by using soft-union quasi-ideals and soft-union ideals.

In this paper, we introduced the concept of $(\mathcal{M},\mathcal{N})$-uni-soft ideals and $(\mathcal{M},\mathcal{N})$-uni-soft interior ideals of an ordered semigroup and studied them. We also proved that in regular and in intra-regular ordered semigroups the concept of $(\mathcal{M},\mathcal{N})$-uni-soft ideals and the $(\mathcal{M},\mathcal{N})$-uni-soft interior ideals coincide. Lastly, we introduced $(\mathcal{M},\mathcal{N})$-uni-soft simple ordered semigroup and characterized the simple ordered semigroups in terms of $(\mathcal{M},\mathcal{N})$-uni-soft interior ideals.

2. Basic definitions and preliminaries

An ordered semigroup $(S,\cdot,\leq)$ is a *Poset* $(S,\leq)$ equipped with a binary operation "$\cdot$" such that

(1) $(S,\cdot)$ is a semigroup,

(2) If $x,a,b\in S$, then $a\leq b \Longrightarrow \begin{cases} xa\leq xb \\ ax\leq bx. \end{cases}$

Let $(S,\cdot,\leq)$ be an ordered semigroup. For subsets A and B of an ordered semigroup S, we denote

$$AB := \{ab \mid a\in A, b\in B\}.$$

If A is a subset of S, we denote by $(A]$ the subset of S defined as follows

$$(A] := \{t\in S \mid t\leq h \text{ for some } h\in A\}.$$

For $a\in S$, we write $(a]$ instead of $(\{a\}]$. For subsets A and B of an ordered semigroup S, we have $A\subseteq (A]$. If $A\subseteq B$, then $(A]\subseteq (B]$, $(A](B]\subseteq (AB]$, $((A]] = (A]$ and $((A](B]]\subseteq (AB]$.

Let $(S,\cdot,\leq)$ be an ordered semigroup. A non-empty subset A of S is called a *subsemigroup* of S if $A^2\subseteq A$.

A non-empty subset A of S is called a *right* (resp., *left*) *ideal* of S if:

(1) $AS\subseteq A$ (resp., $SA\subseteq A$) and

(2) if $a\in A$ and $S\ni b\leq a$, then $b\in A$.

If A is both a right and a left ideal of S, then it is called an ideal of S.

A non-empty subset A of S is called a *interior ideal* of S if:

(1) $SAS\subseteq A$

(2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$.

An ordered semigroup S is said to be regular if for every $x \in S$ there exist $a \in S$ such that $a \leq axa$.

An ordered semigroup S is said to be intra-regular if for all $a \in S$ there exists $x, y \in S$ such that $a \leq xa^2y$.

An ordered semigroup S is said to be left (resp., right) simple if it contains no proper left (resp., right) ideal.

An ordered semigroup S is said to be simple if it contains no proper two-sided ideal.

In the following, we assume that U is an initial universe set, E is a set of parameters, $P(U)$ denotes the power set of U and $A, B, C, \ldots \subseteq E$. And we will assume that $\emptyset \subseteq \mathcal{M} \subset \mathcal{N} \subseteq U$.

A soft set theory is introduced by Molodstov [16], and Çağman [3] provided new definitions and various results on soft set theory.

Definition 2.1. [16, 3] A *soft set* f_A over U is defined to be the set of ordered pairs

$$f_A = \{(x, f_A(x)) \mid x \in E, f_A(x) \in P(U)\},$$

where $f_A : E \longrightarrow P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$.

The function f_A is also called an *approximation function*.

It is clear from Definition 2.1, that a soft set is a *parameterized family* of subsets of U. Note that the set of all soft sets over U will be denoted $S(U)$.

Define an ordered relation "$\widetilde{\subseteq}_{(\mathcal{M},\mathcal{N})}$" on $P(U)$ as follows:

For any $f_A, f_B \in S(U)$, $\emptyset \subseteq \mathcal{M} \subset \mathcal{N} \subseteq U$, we defined

$$f_A \widetilde{\subseteq}_{(\mathcal{M},\mathcal{N})} f_B \Longleftrightarrow f_A(x) \cap \mathcal{N} \subseteq f_B(x) \cup \mathcal{M},$$

and we define a relation "$=_{(\mathcal{M},\mathcal{N})}$" as follows:

$$f_A =_{(\mathcal{M},\mathcal{N})} f_B \Longleftrightarrow f_A \widetilde{\subseteq}_{(\mathcal{M},\mathcal{N})} f_B \text{ and } f_B \widetilde{\subseteq}_{(\mathcal{M},\mathcal{N})} f_A.$$

The *soft union* of f_A and f_B, denoted by $f_A \widetilde{\cup} f_B = f_{A\cup B}$, is defined by

$$\left(f_A \widetilde{\cup} f_B\right)(x) = f_A(x) \cup f_B(x) \text{ for all } x \in E.$$

The *soft intersection* of f_A and f_B, denoted by $f_A \widetilde{\cap} f_B = f_{A\cap B}$, is defined by

$$\left(f_A \widetilde{\cap} f_B\right)(x) = f_A(x) \cap f_B(x) \text{ for all } x \in E.$$

For a soft f_A over U and $\delta \subseteq U$. The $\delta-$exclusive set of (f_A, S), denoted by $e_A(f_A; \delta)$, is defined as

$$e_A(f_A; \delta) = \{x \in L \,|\, f_A(x) \subseteq \delta\}$$

For a non-empty subset A of S, the *characteristic soft set* (χ_A, S) over U is a soft set defined as follows:

$$\chi_A : S \longrightarrow P(U), x \longmapsto \begin{cases} U, & \text{if } x \in A, \\ \emptyset, & \text{if } x \in S\backslash A. \end{cases}$$

For the characteristic soft set (χ_A, S) over U, the soft set (χ_A^c, S) over U is given as follows:

$$\chi_A^c : S \longrightarrow P(U), x \longmapsto \begin{cases} \emptyset, & \text{if } x \in A, \\ U, & \text{if } x \in S\backslash A. \end{cases}$$

3. $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideals of ordered semigroups

In this section, we define $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideals of ordered semigroups and study their properties as regards soft set operations and soft uni-product. Also, it is shown that every $(\mathcal{M}, \mathcal{N})$-uni-soft ideal is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal.

Definition 3.1. Let $(S, \cdot, \leq)$ be an ordered semigroup. A soft set (f_S, S) over U is called $(\mathcal{M}, \mathcal{N})$-uni-soft left ideal over U if:

(1) $x \leq y \Longrightarrow f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$ for all $x, y \in S$ and
(2) $f_S(xy) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$ For all $x, y \in S$.

A soft set (f_S, S) over U is called $(\mathcal{M}, \mathcal{N})$-uni-soft right ideal over U if:

(1) $x \leq y \Longrightarrow f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$ for all $x, y \in S$ and
(2) $f_S(xy) \cap \mathcal{N} \subseteq f_S(x) \cup \mathcal{M}$ For all $x, y \in S$.

A soft set f_S of S over U is called a $(\mathcal{M}, \mathcal{N})$-uni-soft ideal of S over U if it is both a $(\mathcal{M}, \mathcal{N})$-uni-soft left and a $(\mathcal{M}, \mathcal{N})$-uni-soft right ideal of S over U.

Example 3.2. Let $S = \{e, a\}$ be an ordered semigroup defined by the order relation $e \leq a$ with the following multiplication table:

$\cdot$	e	a
e	e	a
a	a	e

Define a soft set (f_S, S) over U as follows:

$$f_S : S \longrightarrow P(U),\ x \longmapsto f_S(x) = \begin{cases} \gamma & \text{if } x = e, \\ \gamma & \text{if } x = a. \end{cases}$$

Where $\mathcal{M} \subseteq \gamma \subset \mathcal{N}$. Then $f_S(xy) \cap \mathcal{N} = f_S(e) \cap \mathcal{N} = \gamma \cap \mathcal{N} = \gamma = \gamma \cup \mathcal{M} = f_S(e) \cup \mathcal{M} = f_S(y) \cup \mathcal{M}$, for every $x, y \in S$. Therefore, (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft left (resp., right) ideal over U.

Definition 3.3. Let $(S, \cdot, \leq)$ be an ordered semigroup. A soft set (f_S, S) over U is called $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over U if:

(1) $x \leq y \Longrightarrow f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$ for all $x, y \in S$ and
(2) $f_S(xay) \cap \mathcal{N} \subseteq f_S(a) \cup \mathcal{M}$ For all $a, x, y \in S$.

In Example 3.2, one can easily show that (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over U.

Example 3.4. [11] Let $S = \{a, b, c, d\}$ be an ordered semigroup with the following multiplication table and the ordered relation:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, d)\}.$$

Let (f_S, S) be a soft set over $U = \mathbb{Z}$ defined by

$$f_S : S \longrightarrow P(U),\ x \longmapsto f_S(x) = \begin{cases} 6\mathbb{N} & \text{if } x = a, \\ 3\mathbb{Z} & \text{if } x \in \{b, d\}, \\ 3\mathbb{N} & \text{if } x = c. \end{cases}$$

Where $\mathcal{M} \subseteq 6\mathbb{N} \subset 3\mathbb{N} \subset 3\mathbb{Z} \subseteq \mathcal{N}$. Then (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft inteior ideal over U.

Theorem 3.5. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal of S over U if and only if the δ-exclusive set of (f_S, S) is an interior ideal of S for all $\delta \in P(U)$, where $\mathcal{M} \subset \delta \subset \mathcal{N}$.*

Proof. Suppose that (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal of S over U and $\delta \in P(U)$. First, we need to show that $xay \in e_S(f_S; \delta)$, for all $a \in e_S(f_S; \delta)$, $x, y \in S$. By hypothesis, we have $f_S(xay) \cap \mathcal{N} \subseteq f_S(a) \cup \mathcal{M} \subseteq \delta \cup \mathcal{M} = \delta$ and $\delta \subset \mathcal{N}$, which means that $f_S(xay) \subseteq \delta$, that is $xay \in e_S(f_S; \delta)$. Now, for all $x \in S$ and $y \in e_S(f_S; \delta)$ such that $x \leq y$, since (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal of S over U, so from $x \leq y$ we have $f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M} \subseteq \delta \cup \mathcal{M} = \delta$, we conclude that $f_S(y) \subseteq \delta$, that is $y \in e_S(f_S; \delta)$. Hence the δ-exclusive set of (f_S, S) is an interior ideal of S for all $\delta \in P(U)$ where $\mathcal{M} \subset \delta \subset \mathcal{N}$.

Conversely, assume that the δ-exclusive set of (f_S, S) is an interior ideal of S for all $\delta \in P(U)$ where $\mathcal{M} \subset \delta \subset \mathcal{N}$ and $a, x, y \in S$. Let $f_S(xay) \cap \mathcal{N} \supset \delta = f_S(a) \cup \mathcal{M}$ for $\delta \in P(U)$. It follows that $f_S(a) \subseteq \delta$ and $f_S(xay) \supset \delta$, that is $a \in e_S(f_S; \delta)$ and $xay \notin e_S(f_S; \delta)$. Which is a contradiction to the fact that $e_S(f_S; \delta)$ is an interior ideal of S. Hence$f_S(xay) \cap \mathcal{N} \subseteq f_S(a) \cup \mathcal{M}$ for all $a, x, y \in S$. If there are $x, y \in S$

such that $x \leq y$. Let $f_S(x) \cap \mathcal{N} \supset \delta = f_S(y) \cup \mathcal{M}$ then $\delta \subseteq P(U)$, which means that $f_S(y) \subseteq \delta$ and $f_S(x) \supset \delta$, that is $y \in e_S(f_S; \delta)$ and $x \notin e_S(f_S; \delta)$. Which is again a contradiction to the fact that $e_S(f_S; \delta)$ is an interior ideal of S. Hence if $x \leq y$, then $f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$, for all $x, y \in S$. Therefore (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal of S over U. □

Lemma 3.6. *Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset I of S is an ideal of S if and only if the characteristic function (χ_I^c, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft ideal over U.*

Proof. It follows from Theorem 3.5. □

Theorem 3.7. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the soft union of two $(\mathcal{M}, \mathcal{N})$-uni-soft interior (resp., left, right) ideals over U is also a $(\mathcal{M}, \mathcal{N})$-uni-soft interior (resp., left, right) ideal over U.*

Proof. Let (f_S, S) and (g_S, S) be $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideals over U. For any $x, a, y \in S$, we have

$$\begin{aligned} \left(f_S \widetilde{\cup} g_S\right)(xay) \cap \mathcal{N} &= \left(f_S(xay) \cup g_S(xay)\right) \cap \mathcal{N} \\ &= \left(f_S(xay) \cap \mathcal{N}\right) \cup \left(g_S(xay) \cap \mathcal{N}\right) \\ &\subseteq \left(f_S(a) \cup \mathcal{M}\right) \cup \left(g_S(a) \cup \mathcal{M}\right) \\ &= \left(f_S(a) \cup g_S(a)\right) \cup \mathcal{M} \\ &= \left(f_S \widetilde{\cup} g_S\right)(a) \cup \mathcal{M}. \end{aligned}$$

Furtheremore, let $x, y \in S$ such that $x \leq y$. Since (f_S, S) and (g_S, S) are $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideals over U, we have

$$\begin{aligned} \left(f_S \widetilde{\cup} g_S\right)(x) \cap \mathcal{N} &= \left(f_S(x) \cup g_S(x)\right) \cap \mathcal{N} \\ &= \left(f_S(y) \cap \mathcal{N}\right) \cup \left(g_S(y) \cap \mathcal{N}\right) \\ &\subseteq \left(f_S(y) \cup \mathcal{M}\right) \cup \left(g_S(y) \cup \mathcal{N}\right) \\ &= \left(f_S(y) \cup g_S(y)\right) \cup \mathcal{M} \\ &= \left(f_S \widetilde{\cup} g_S\right)(y) \cup \mathcal{M}. \end{aligned}$$

Therefore $\left(f_S \widetilde{\cup} g_S, S\right)$ is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over U. In a similar way, $\left(f_S \widetilde{\cup} g_S, S\right)$ is a $(\mathcal{M}, \mathcal{N})$-uni-soft left (resp., right) ideal over U. □

Let $(S, \cdot, \leq)$ and $(T, \cdot, \leq)$ be two ordered semigroups. Under the coordinatewise multiplication, i.e.,

$$(x, a)(y, b) = (xy, ab)$$

where $(x, a), (y, b) \in S \times T$, the Cartesian product

$$S \times T = \{(x, a) \,|x \in S, a \in T\}$$

is a semigroup. Define a partial order $\leq$ on $S \times T$ by

$$(x,a) \leq (y,b) \text{ if and only if } x \leq y \text{ and } a \leq b,$$

where $(x,a),(y,b) \in S \times T$. Then, $(S \times T, \cdot, \leq)$ is an ordered semigroup.

For uni-soft sets (f_S, S) and (f_T, T) over U, we consider a uni-soft set $(f_{S\vee T}, S \times T)$ over U in which $f_{S\vee T}$ is defined as follows:

$$f_{S\vee T} : S \times T \longrightarrow P(U), \ \ (x,a) \longmapsto f_S(x) \cup f_T(a).$$

Theorem 3.8. *Let $(S, \cdot, \leq)$ be an ordered semigroup. If (f_S, S) and (f_T, T) are $(\mathcal{M}, \mathcal{N})$-uni-soft interior (resp., left, right) ideals over U, then $(f_{S\vee T}, S \times T)$ is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior (resp., left, right) ideal over U.*

Proof. Let $(x,a),(y,b),(z,c) \in S \times T$. Then

$$\begin{aligned} f_{S\vee T}((x,a),(y,b),(z,c)) \cap \mathcal{N} &= f_{S\vee T}(xyz, abc) \cap \mathcal{N} \\ &= (f_S(xyz) \cup f_T(abc)) \cap \mathcal{N} \\ &= (f_S(xyz) \cap \mathcal{N}) \cup (f_T(abc) \cap \mathcal{N}) \quad (\#). \end{aligned}$$

Since (f_S, S) and (f_T, T) are $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over U, we have $f_S(xyz) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$ and $f_T(abc) \cap \mathcal{N} \subseteq f_T(b) \cup \mathcal{M}$. Hence from eqution (#) we have

$$\begin{aligned} (f_S(xyz) \cap \mathcal{N}) \cup (f_T(abc) \cap \mathcal{N}) &\subseteq (f_S(y) \cup \mathcal{M}) \cup (f_T(b) \cup \mathcal{M}) \\ &= (f_S(y) \cup f_T(b)) \cup \mathcal{M} \\ &= f_{S\vee T}(y,b) \cup \mathcal{M}. \end{aligned}$$

Furthermore, let $(x,a),(y,b) \in S \times T$ be such that $(x,a) \leq (y,b)$. Then

$$\begin{aligned} f_{S\vee T}(x,a) \cap \mathcal{N} &= (f_S(x) \cup f_T(a)) \cap \mathcal{N} \\ &= (f_S(x) \cap \mathcal{N}) \cup (f_T(a) \cap \mathcal{N}) \\ &\subseteq (f_S(y) \cup \mathcal{M}) \cup (f_T(b) \cup \mathcal{M}) \\ &= (f_S(y) \cup f_T(b)) \cup \mathcal{M} \\ &= f_{S\vee T}(y,b) \cup \mathcal{M}. \end{aligned}$$

Therefore $(f_{S\vee T}, S \times T)$ is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over U. Similarly, we show that If (f_S, S) and (f_T, T) are $(\mathcal{M}, \mathcal{N})$-uni-soft left (resp., right) ideals over U, then $(f_{S\vee T}, S \times T)$ is a $(\mathcal{M}, \mathcal{N})$-uni-soft left (resp., right) ideal over U. □

Theorem 3.9. *Let $\varphi : S \longrightarrow T$ be a homomorphism of an ordered semigroup. If (f_S, S) is a uni-soft interior (resp., left, right) ideal over*

U, the pre image $(\varphi^{-1}(f_S), S)$ of (f_S, T) under φ is auni-soft interior (resp., left, right) ideal over U, where $\varphi^{-1}(f_S)$ is given as follows:

$$\varphi^{-1}(f_S) : S \longrightarrow P(U),\ x \longmapsto f_S(\varphi(x)).$$

Proof. Let $(S, \cdot, \leq)$ be an ordered semigroup and $\varphi : S \longrightarrow T$ be a homomorphism. Let $x, y \in S$ and $x \leq y$. Since φ is a homomorphism of ordered semigroups from S to T, we have $\varphi(x) \leq \varphi(y)$. Since (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over U, we have $f_S(\varphi(x)) \cap \mathcal{N} \subseteq f_S(\varphi(y)) \cup \mathcal{M}$. Hence

$$\varphi^{-1}(f_S)(x) \cap \mathcal{N} = f_S(\varphi(x)) \cap \mathcal{N} \subseteq f_S(\varphi(y)) \cup \mathcal{M} = \varphi^{-1}(f_S)(y) \cup \mathcal{M}.$$

Furtheremore, for any $x, y, z \in S$, we have

$$\begin{aligned} \varphi^{-1}(f_S)(xyz) \cap \mathcal{N} &= f_S(\varphi(xyz)) \cap \mathcal{N} \\ &= f_S(\varphi(x)\varphi(y)\varphi(z)) \cap \mathcal{N} \\ &\subseteq f_S(\varphi(y)) \cup \mathcal{M} \\ &= \varphi^{-1}(f_S)(y) \cup \mathcal{M}. \end{aligned}$$

Therefore $(\varphi^{-1}(f_S), S)$ is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over U. Similarly, we can show that $(\varphi^{-1}(f_S), S)$ is a $(\mathcal{M}, \mathcal{N})$-uni-soft left (resp., right) ideal over U. □

Lemma 3.10. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then every $(\mathcal{M}, \mathcal{N})$-uni-soft two-sided ideal over U is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over U.*

Proof. Let $x, y, a \in S$. Since (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft right ideal over U, we have,

$$f_S(xay) \cap \mathcal{N} = f_S((xa)y) \cap \mathcal{N} \subseteq f_S(xa) \cup \mathcal{M}, \tag{1}$$

and since (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft right ideal over U, we have

$$f_S(xay) \cap \mathcal{N} = f_S(x(ay)) \cap \mathcal{N} \subseteq f_S(ay) \cup \mathcal{M}. \tag{2}$$

From (1) and (2), we get $f_S(xay) \cap \mathcal{N} = f_S(x(ay) \cap \mathcal{N}) \cap \mathcal{N} \subseteq (f_S(ay) \cup \mathcal{M}) \cap \mathcal{N} = (f_S(ay) \cap \mathcal{N}) \cup (\mathcal{M} \cap \mathcal{N}) \subseteq (f_S(a) \cup \mathcal{M}) \cup \mathcal{M} = f_S(a) \cup \mathcal{M}$.

Let $x, y \in S$ such that $x \leq y$. Then $f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$, because (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft two-sided ideal over U. Thus (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over U. □

The following example shows that the converse of the Lemma 3.10, is not true in general.

Example 3.11. In Example 3.4, soft set (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft inteior ideal over U. But it is not a uni-soft left ideal over U, since $f_S(dc) \cap \mathcal{N} = f_S(b) \cap \mathcal{N} = 3\mathbb{Z} \cap \mathcal{N} = 3\mathbb{Z} \nsubseteq 3\mathbb{N} = 3\mathbb{N} \cup \mathcal{M} = f_S(c) \cup \mathcal{M}$, for every $c, d \in S$, and hence it is not a uni-soft two-sided ideal over U.

4. $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideals of regular/intra-regular ordered semigroups

In this section, we prove that in regular and in intra-regular ordered semigroups, the concepts of $(\mathcal{M}, \mathcal{N})$-uni-soft ideals and $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideals coincide.

Theorem 4.1. *Let $(S, \cdot, \leq)$ be a regular ordered semigroup. Then every $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over U is a $(\mathcal{M}, \mathcal{N})$-uni-soft two-sided ideal over U.*

Proof. Let (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over U and let $x, y \in S$. Since S is a regular, then there exist $a, b \in S$ such that $x \leq xax$ and $y \leq yby$. We have

$$\begin{aligned} f_S(xy) \cap \mathcal{N} &\subseteq (f_S((xax)\,y) \cup \mathcal{M}) \cap \mathcal{N} \\ &= (f_S((xa)\,xy) \cap \mathcal{N}) \cup (\mathcal{M} \cap \mathcal{N}) \\ &\subseteq (f_S\,(x) \cup \mathcal{M}) \cup \mathcal{M} \\ &= f_S\,(x) \cup \mathcal{M}, \end{aligned}$$

and

$$\begin{aligned} f_S(xy) \cap \mathcal{N} &\subseteq (f_S(x\,(yby)) \cup \mathcal{M}) \cap \mathcal{N} \\ &= (f_S(xy\,(by)) \cap \mathcal{N}) \cup (\mathcal{M} \cap \mathcal{N}) \\ &\subseteq (f_S\,(y) \cup \mathcal{M}) \cup \mathcal{M} \\ &= f_S\,(y) \cup \mathcal{M}. \end{aligned}$$

Now, let $x, y \in S$ such that $x \leq y$. Then $f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$, because (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal of S over U. Therefore (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft two-sided ideal over U. □

By Lemma 3.10 and Theorem 4.1, we have the following:

Remark 4.2. In regular ordered semigroups the concepts of $(\mathcal{M}, \mathcal{N})$-uni-soft two-sided ideals and $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideals coincide.

Theorem 4.3. *Let $(S, \cdot, \leq)$ be an intra-regular ordered semigroup and (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over U. Then (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft two-sided ideal over U.*

Proof. Let (f_S, S) be a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over U. Let $x, y \in S$, since S is a intra-regular then there exists $a, b \in S$ such that $x \leq ax^2a$ and $y \leq by^2b$. Since (f_S, S) is an uni-soft interior ideal of S, we have

$$\begin{aligned} f_S(xy) \cap \mathcal{N} &\subseteq \left(f_S(\left(ax^2a\right) y) \cup \mathcal{M}\right) \cap \mathcal{N} \\ &= (f_S((ax)\, x\, (ay)) \cap \mathcal{N}) \cup (\mathcal{M} \cap \mathcal{N}) \\ &\subseteq (f_S(x) \cup \mathcal{M}) \cup \mathcal{M} \\ &= f_S(x) \cup \mathcal{M}, \end{aligned}$$

and

$$\begin{aligned} f_S(xy) \cap \mathcal{N} &\subseteq \left(f_S(x\left(by^2b\right)) \cup \mathcal{M}\right) \cap \mathcal{N} \\ &= (f_S((xb)\, y\, (yb)) \cap \mathcal{N}) \cup (\mathcal{M} \cap \mathcal{N}) \\ &\subseteq (f_S(y) \cup \mathcal{M}) \cup \mathcal{M} \\ &= f_S(y) \cup \mathcal{M}. \end{aligned}$$

Let $x, y \in S$ be such that $x \leq y$. Then $f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$, because (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal of S over U. Thus (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft two-sided ideal of S over U. □

By Lemma 3.10 and Theorem 4.3, we have the following:

Remark 4.4. In intra regular ordered semigroups, the concepts of $(\mathcal{M}, \mathcal{N})$-uni-soft two-sided ideals and $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideals coincide.

Theorem 4.5. *If S is a monoid with identity e. Then every $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over U is a $(\mathcal{M}, \mathcal{N})$-uni-soft two sided ideal over U.*

Proof. Let (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over U and $x, y \in S$. Then $f_S(xy) \cap \mathcal{N} = f_S(xye) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$ and $f_S(xy) \cap \mathcal{N} = f_S(exy) \cap \mathcal{N} \subseteq f_S(x) \cup \mathcal{M}$. Furthermore, let $x, y \in S$ be such that $x \leq y$. Then $f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$, because (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal of S over U. Therefore (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft two-sided ideal over U. □

5. $(\mathcal{M}, \mathcal{N})$-UNI-SOFT SIMPLE ORDERED SEMIGROUPS

In this section, we define $(\mathcal{M}, \mathcal{N})$-uni-soft simple ordered semigroups and characterize simple ordered semigroups in terms of $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideals.

Definition 5.1. An ordered semigroup S is called $(\mathcal{M}, \mathcal{N})$-uni-soft simple if for any $(\mathcal{M}, \mathcal{N})$-uni-soft ideal of S, we have $f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$, for all $x, y \in S$.

Theorem 5.2. *Let $(S,\cdot \leq)$ be an ordered semigroup. Then S is $(\mathcal{M},\mathcal{N})$-uni-soft simple if and only if for any $(\mathcal{M},\mathcal{N})$-uni-soft ideal (f_S,S) of S, if $e_S(f_S,S)\neq\emptyset$, then $e_S(f_S,\delta)=S$, for all $\delta\in P(U)$ where $\mathcal{M}\subset\delta\subset\mathcal{N}$.*

Proof. Let (f_S,S) be a $(\mathcal{M},\mathcal{N})$-uni-soft ideal of S over U, and $e_S(f_S,\delta)\neq\emptyset$. We need to show that $x\in e_S(f_S,\delta)$ for all $x\in S$, where $\mathcal{M}\subset\delta\subset\mathcal{N}$. Since $e_S(f_S,\delta)\neq\emptyset$, then there exists $y\in e_S(f_S,\delta)$, that is $f_S(y)\subseteq\delta$. By hypothesis we have,

$$f_S(x)\cap\mathcal{N}\subseteq f_S(y)\cup\mathcal{M}\subseteq\delta\cup\mathcal{M}=\delta.$$

Notice that $\delta\subset\mathcal{N}$, which means that $f_S(x)\subseteq\delta$, that is $x\in e_S(f_S,\delta)$.

Conversely, suppose that for any $(\mathcal{M},\mathcal{N})$-uni-soft ideal (f_S,S) over U, we have $e_S(f_S,\delta)=S$ for all $\mathcal{M}\subset\delta\subset\mathcal{N}$. We need to show that $f_S(x)\cap\mathcal{N}\subseteq f_S(y)\cup\mathcal{M}$ for all $x,y\in S$. Let if

$$f_S(x)\cap\mathcal{N}\supset\delta=f_S(y)\cup\mathcal{M}.$$

Then $f_S(y)\subseteq\delta$ and $f_S(x)\supset\delta$, which means that $x\notin e_S(f_S,\delta)=S$, a contradiction. So $f_S(x)\cap\mathcal{N}\subseteq f_S(y)\cup\mathcal{M}$ for all $x,y\in S$. □

Theorem 5.3. *Ley $(S,\cdot,\leq)$ be an ordered semigroup and soft set (f_S,S) a $(\mathcal{M},\mathcal{N})$-uni-soft right (resp., left) ideal over U. Then $I_x=\{y\in S\,|f_S(y)\cap\mathcal{N}\subseteq f_S(x)\cup\mathcal{M}\}$ is a right (resp., left) ideal of S, for all $x\in S$.*

Proof. Let $x\in S$. Then $I_x\neq\emptyset$ since $x\in I_x$. Suppose that $y\in I_x$ and $s\in S$, Then $ys\in I_x$. Since (f_S,S) is a $(\mathcal{M},\mathcal{N})$-uni-soft ideal over U and $y,s\in S$, we have

$$\begin{aligned} f_S(ys)\cap\mathcal{N} &= (f_S(ys)\cap\mathcal{N})\cap\mathcal{N} \\ &\subseteq (f_S(y)\cup\mathcal{M})\cap\mathcal{N} \\ &= (f_S(y)\cap\mathcal{N})\cup(\mathcal{M}\cap\mathcal{N}) \\ &\subseteq (f_S(x)\cup\mathcal{M})\cup\mathcal{M} && \text{since } y\in I_x \\ &= f_S(x)\cup\mathcal{M}. \end{aligned}$$

Hence $ys\in I_x$.

Furthermore, let $s, y \in S$ be such that $s \leq y$. If $y \in I_x$, Then $s \in I_x$. Since (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft ideal over U and $s \leq y$, we have

$$\begin{aligned} f_S(s) \cap \mathcal{N} &= (f_S(s) \cap \mathcal{N}) \cap \mathcal{N} \\ &\subseteq (f_S(y) \cup \mathcal{M}) \cap \mathcal{N} \\ &= (f_S(y) \cap \mathcal{N}) \cup (\mathcal{M} \cap \mathcal{N}) \\ &\subseteq (f_S(x) \cup \mathcal{M}) \cup \mathcal{M} && \text{since } y \in I_x \\ &= f_S(x) \cup \mathcal{M}. \end{aligned}$$

So $s \in I_x$. Therefore, I_x is a right ideal of S, for all $x \in S$. □

By left right dual of Theorem 5.3, we have the following result.

Theorem 5.4. *Ley $(S, \cdot, \leq)$ be an ordered semigroup and soft set (f_S, S) a $(\mathcal{M}, \mathcal{N})$-uni-soft ideal over U. Then for all $x \in S$, the set*

$$I_x = \{y \in S \,|\, f_S(y) \cap \mathcal{N} \subseteq f_S(x) \cup \mathcal{M}\}$$

is a right ideal of S.

Theorem 5.5. *An ordered semigroup S is simple if and only if it is $(\mathcal{M}, \mathcal{N})$-uni-soft simple.*

Proof. Assume that S is simple. Let (f_S, S) be a $(\mathcal{M}, \mathcal{N})$-uni-soft ideal over U and $x, y \in S$. By Theorem 5.4, the set I_x is an ideal of S. Since S is simple, we have $I_x = S$. Then $b \in I_x$, from which we have that $f_S(y) \cap \mathcal{N} \subseteq f_S(x) \cup \mathcal{M}$. Thus S is $(\mathcal{M}, \mathcal{N})$-uni-soft simple.

Conversely, suppose S contains proper ideals and let I be such ideal of S. By Lemma 3.6, we know that (χ_I^c, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft ideal of S. We have that $S \subseteq I$. Indeed, let $x \in S$. Since S is $(\mathcal{M}, \mathcal{N})$-uni-soft simple, $\chi_I^c(x) \cap \mathcal{N} \subseteq \chi_I^c(y) \cup \mathcal{M}$ for all $y \in S$. Now, let $z \in I$. Then we have

$$\chi_I^c(x) \cap \mathcal{N} \subseteq \chi_I^c(z) \cup \mathcal{M} = U \cup \mathcal{M} = U.$$

Notice that $\mathcal{M} \subset \mathcal{N}$, we conclude that $\chi_I^c(x) \subseteq U$, which implies that $\chi_I^c(x) = U$, that is $x \in I$. Thus we have that $S \subseteq I$, and so $S = I$. We get a contradiction to hypothesis that S contains proper ideals. □

Lemma 5.6. [8, 9] *An ordered semigroup S is simple if and only if for every $a \in S$, we have $S = (SaS]$.*

Theorem 5.7. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then S is simple if and only if for every $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal (f_S, S) of S, we have $f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$, for all $x, y \in S$.*

Proof. Supose S is simple. Let (f_S, S) be a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over U and $x, y \in S$. Since S is simple and $y \in S$, by Lemma 5.6, we have $S = (SyS]$. Since $x \in S$, we have $x \in (SyS]$, then $x \leq ayb$ for some $a, b \in S$. Since (f_S, S) is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over U, We have

$$\begin{aligned} f_S(x) \cap \mathcal{N} &= (f_S(x) \cap \mathcal{N}) \cap \mathcal{N} \\ &\subseteq (f_S(ayb) \cup \mathcal{M}) \cap \mathcal{N} \\ &= (f_S(ayb) \cap \mathcal{N}) \cup (\mathcal{M} \cap \mathcal{N}) \\ &\subseteq (f_S(y) \cup \mathcal{M}) \cup \mathcal{M} \\ &= f_S(y) \cup \mathcal{M}. \end{aligned}$$

Conversely, suppose that for every $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal (f_S, S) over U, we have $f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$, for all $x, y \in S$. Now let (f_S, S) be any $(\mathcal{M}, \mathcal{N})$-uni-soft ideal of S over U, then it is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal of S over U. So we have $f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$, for all $x, y \in S$. Thus S is $(\mathcal{M}, \mathcal{N})$-uni-soft simple, then by Theorem 5.5, we conclude that S is simple. □

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GENERALIZED UNI-SOFT INTERIOR IDEALS IN ORDERED SEMIGROUPS

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ایده‌آل‌های داخلی اجتماع-نرم تعمیم‌یافته در نیم‌گروه‌های مرتب

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برای هر $\mathcal{M},\mathcal{N} \in P(U)$ به گونه‌ای‌که $\mathcal{M} \subset \mathcal{N}$، ابتدا مفاهیم ایده‌آل‌های $(\mathcal{M},\mathcal{N})$-اجتماع-نرم و ایده‌آل‌های داخلی $(\mathcal{M},\mathcal{N})$-اجتماع-نرم برای نیم‌گروه‌های مرتب را معرفی کرده و آن‌ها را مورد مطالعه قرار می‌دهیم. در حالت خاص که $\mathcal{M} = \emptyset$ و $\mathcal{N} = U$، مفاهیم فوق با مفاهیم موجود قبلی ایده‌آل نرم و ایده‌آل داخلی نرم، سازگار می‌باشند. سپس ثابت می‌کنیم که در نیم‌گروه‌های مرتب منظم و نیم‌گروه‌های مرتب درون-منظم، مفاهیم تعریف شده فوق برهم منطبق می‌باشند. در انتها، ضمن معرفی مفهوم نیم‌گروه مرتب ساده $(\mathcal{M},\mathcal{N})$-اجتماع-نرم، به رده‌بندی نیم‌گروه‌های مرتب ساده بر حسب ایده‌آل‌های داخلی $(\mathcal{M},\mathcal{N})$-اجتماع-نرم می‌پردازیم.

کلمات کلیدی: مجموعه نرم، ایده‌آل‌های $(\mathcal{M},\mathcal{N})$-اجتماع-نرم، نیم‌گروه‌های مرتب درون-منظم، ایده‌آل‌های داخلی $(\mathcal{M},\mathcal{N})$-اجتماع-نرم، نیم‌گروه ساده $(\mathcal{M},\mathcal{N})$-اجتماع-نرم، منظم، نیم‌گروه مرتب ساده.