NEW METHODS FOR CONSTRUCTING GENERALIZED GROUPS, TOPOLOGICAL GENERALIZED GROUPS, AND TOP SPACES

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ABSTRACT. The purpose of this paper is to introduce new methods for constructing generalized groups, generalized topological groups and top spaces. We study some properties of these structures and present some relative concrete examples. Moreover, we obtain generalized groups by using of Hilbert spaces and tangent spaces of Lie groups, separately.

1. INTRODUCTION

Generalized groups as an extension of groups were recently introduced by Molaei in [8]. A generalized group is a non-empty set G admitting an operation called multiplication, which satisfies the following conditions:

i) $x(yz) = (xy)z$, for all $x, y \in G$;

ii) For each $x \in G$, there exists a unique $e_x \in G$ such that $xe_x = e_x x = x$ (existence and uniqueness of identity element);

iii) For each $x \in G$, there exists $x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = e_x$ (existence of inverse element) [8].

In [2], Araujo and Konieczny proved that the generalized groups are equivalent to the notion of completely simple semigroups. In fact a semigroup S is called a completely simple semigroup if for all $a \in S$, $SaS = S$, and if e and f are idempotents in S such that $ef = fe$ then

$e = f$. Here we call them as generalized groups. Various applications of these algebraic structures are studied in the recent papers [1, 3, 4, 5, 6, 7, 11].

In this paper, we shall introduce some methods for constructing generalized groups, generalized rings, topological generalized groups and top spaces, and investigate some properties of them. Our motivation is to introduce methods that identify the relationship between algebraic, geometric and topological structures. In particular, we present some concrete examples of these new methods. Meanwhile, we establish generalized groups by using of Hilbert spaces and tangent spaces of Lie groups in separate manners.

In the following, we give a brief review on the definitions which are useful for establishing our main results.

Definition 1.1. [9] A generalized group G is called a topological generalized group provided that G is a Hausdorff topological space and the mappings

$$\mathcal{I} : G \rightarrow G,$$
$$g \mapsto g^{-1},$$

and

$$\mathcal{P} : G \times G \rightarrow G,$$
$$(g, h) \mapsto gh,$$

are continuous.

Definition 1.2. [12] A non-empty set R with two different operations

$$(x, y) \mapsto x + y \ and \ (x, y) \mapsto xy$$

is called a generalized ring if the following conditions are satisfied:

i) $(R, +)$ is a generalized group;
ii) $x(yz) = (xy)z$, for all $x, y, z \in R$;
iii) For all $x, y, z \in R$,

$$x(y + z) = xy + xz \ and \ (x + y)z = xz + yz.$$

Definition 1.3. [10] A normal topological generalized group G is called top space if

i) The topological space T is a smooth manifold;
ii) The mappings

$$\mathcal{I} : T \rightarrow T,$$
$$x \mapsto x^{-1},$$

and

$$\mathcal{P} : T \times T \rightarrow T,$$

$$(x, y) \mapsto xy,$$

are C^∞ maps.

It is noted that throughout the paper, we assume that $M_n(K)$ is the set of $n \times n$ matrices and $GL(n, K)$ is the set of all nonsingular matrices over K, where K is the field of real or complex numbers.

2. Main results

In this section, we introduce a new method for constructing generalized groups, topological generalized groups and top spaces, and stablish some results about them.

Theorem 2.1. *Let G be a non-empty subset of $M_n(K)$ and let $f : G \longrightarrow K \setminus \{0\}$ be a map such that $f(\lambda B) = \lambda f(B)$, for all $\lambda \in K \setminus \{0\}$. Define $A * B := f(A)B$, for all $A, B \in G$. Then $(G, *)$ is a generalized group.*

Proof. Let $A, B, C \in G$. So, we have

$$(A * B) * C = f(f(A)B)C = f(A)f(B)C,$$

and

$$
\begin{aligned}
A * (B * C) &= A * (f(B)C) \\
&= f(A)(f(B)C) \\
&= f(A)f(B)C \\
&= (A * B) * C.
\end{aligned}
$$

Therefore $(G, *)$ is associative.

Moreover, $A * (\frac{A}{f(A)}) = f(A)(\frac{A}{f(A)}) = A$. Since $f(\lambda A) = \lambda f(A)$, put $\lambda = \frac{1}{f(A)}$, so we have

$$\left(\frac{A}{f(A)}\right) * A = f(\frac{A}{f(A)})A = \left(\frac{1}{f(A)}f(A)\right)A = A.$$

Therefore, $e_A = \frac{A}{f(A)}$ is the identity element of A which is unique. If not, let E_A be an another identity element of A, then

$$
\begin{aligned}
A * E_A = A * e_A &= A \\
&\Rightarrow f(A)E_A = f(A)e_A \\
&\Rightarrow E_A = e_A.
\end{aligned}
$$

And finally, it is easily seen that $A^{-1} = \frac{A}{f(A)^2}$. □

Remark 2.2. Similary, if $A * B = A f(B)$, for all $A, B \in G$, then $(G, *)$ is a generalized group.

Example 2.3. Suppose that $T = \{A \in M_n(\mathbb{R}) | tr(A) \neq 0\}$, where $tr(A)$ is the trace of matrix A. Define $A \star B := tr(A)B$, for all $A, B \in T$, then $(T, \star)$ is a generalized group. Indeed, for all $A \in T$ and $\lambda \in \mathbb{R} \backslash \{0\}$ we have $tr(\lambda A) = \lambda tr(A)$. Therefore

$$(A \star B) \star C = tr(tr(A)B).C = tr(A)tr(B)C,$$

and

$$\begin{aligned} A \star (B \star C) &= A \star (tr(B)C) \\ &= tr(A)(tr(B)C) \\ &= tr(A)tr(B)C = (A \star B) \star C, \end{aligned}$$

so $(\mathbb{R}, \star)$ is associative. It is easy to check that $e_A = \dfrac{A}{tr(A)}$ is the unique identity element and $A^{-1} = \frac{A}{tr^2(A)}$ is the inverse element of A.

Example 2.4. Let $A, B \in GL(n, \mathbb{C})$. Define $A * B := \sqrt[n]{det(A)}\ B$. Then $(GL(n, \mathbb{C}), *)$ is a generalized group. Indeed, for all $A \in GL(n, \mathbb{C})$ and $\lambda \in \mathbb{C} \backslash \{0\}$ we have $\sqrt[n]{det(\lambda A)} = \lambda \sqrt[n]{det(A)}$. So, we have

$$(A * B) * C = \sqrt[n]{det(\sqrt[n]{det(A)}B)}\ C = \sqrt[n]{det(A)}\sqrt[n]{det(B)}\ C$$

and

$$\begin{aligned} A * (B * C) = \sqrt[n]{detA}(\sqrt[n]{detB}\ C) &= \sqrt[n]{det(A)}\sqrt[n]{det(B)}\ C \\ &= (A * B) * C, \end{aligned}$$

so $(GL(n, \mathbb{C}), *)$ is associative. Moreover

$$A * \frac{A}{\sqrt[n]{det(A)}} = \sqrt[n]{det(A)}\frac{A}{\sqrt[n]{det(A)}} = A,$$

and

$$\frac{A}{\sqrt[n]{det(A)}} * A = \sqrt[n]{det(\frac{A}{\sqrt[n]{det(A)}})}\ A = A.$$

So $e_A = \dfrac{A}{\sqrt[n]{det(A)}}$ is the identity element of A. Simply we see that $\frac{A}{\sqrt[n]{det(A)^2}}$ is the inverse of A.

Proposition 2.5. *Let* $x \oplus_n y := x + \dfrac{[10^n y]}{10^n}$, *for every* $x, y \in \mathbb{R}$. *Then* $(\mathbb{R}, \oplus_n)$ *is a generalized group.*

Proof. Let $x, y, z \in \mathbb{R}$. Then

$$(x \underset{n}{\oplus} y) \underset{n}{\oplus} z = x + \frac{[10^n y]}{10^n} + \frac{[10^n z]}{10^n},$$

and

$$\begin{aligned}
x \underset{n}{\oplus} (y \underset{n}{\oplus} z) = x \underset{n}{\oplus} (y + \frac{[10^n z]}{10^n}) &= x + \frac{10^n(y + \frac{[10^n z]}{10^n})}{10^n} \\
&= x + \frac{[10^n y + [10^n z]]}{10^n} \\
&= x + \frac{[10^n y] + [10^n z]}{10^n}.
\end{aligned}$$

Therefore $(\mathbb{R}, \underset{n}{\oplus})$ is associative. Now, we show that $e_x = x - \frac{[10^n x]}{10^n}$ is the unique identitiy element of x. Indeed, we have

$$\begin{aligned}
x \underset{n}{\oplus} (x - \frac{[10^n x]}{10^n}) &= x + \frac{[10^n(x - \frac{[10^n x]}{10^n})]}{10^n} \\
&= x + \frac{[10^n x] - [10^n x]}{10^n} = x.
\end{aligned}$$

Also

$$(x - \frac{[10^n x]}{10^n}) \underset{n}{\oplus} x = (x - \frac{[10^n x]}{10^n}) + \frac{[10^n x]}{10^n} = x.$$

One can easily check that e_x is unique and $x^{-1} = x - \frac{2[10^n x]}{10^n}$ is the inverse of x. $\qquad\square$

Lemma 2.6. *If α is an arbitrary element of $\mathbb{R}$, then $\lim\limits_{n \to \infty} (\frac{[10^n \alpha]}{10^n}) = \alpha$.*

Proof. Let $\epsilon > 0$. We choose $m \in \mathbb{N}$, such that $1 < 10^m \epsilon$. So, for all $n \geq m$, we have

$$\begin{aligned}
|\frac{[10^n \alpha]}{10^n} - \alpha| &= |\frac{[10^n \alpha] - 10^n \alpha}{10^n}| \\
&\leq \frac{1}{10^n} \\
&\leq \frac{1}{10^m} < \epsilon.
\end{aligned}$$

$\qquad\square$

Corollary 2.7. *Let $x, y \in \mathbb{R}$. Then $\lim\limits_{n \to \infty} (x \underset{n}{\oplus} y) = x + y$.*

Proposition 2.8. *Let $x \underset{n}{\oplus} y := x + \dfrac{[10^n y]}{10^n}$ and $x \star y := x$, for all $x, y \in \mathbb{R}$. Then $(\mathbb{R}, \star, \underset{n}{\oplus})$ is a generalized ring.*

Proof. Obviously, $(\mathbb{R}, \star)$ is a generalized group. By using of Proposition 2.5, $(\mathbb{R}, \underset{n}{\oplus})$ is a generalized group, so it is associative. The other properties can be checked as follows:

$$x \underset{n}{\oplus} (y \star z) = x \underset{n}{\oplus} y = x + \frac{[10^n y]}{10^n},$$

and

$$(x \underset{n}{\oplus} y) \star (x \underset{n}{\oplus} z) = x \underset{n}{\oplus} y = x + \frac{[10^n y]}{10^n},$$

So

$$x \underset{n}{\oplus} (y \star z) = (x \underset{n}{\oplus} y) \star (x \underset{n}{\oplus} z).$$

Also

$$(x \star y) \underset{n}{\oplus} z = x \underset{n}{\oplus} z = x + \frac{[10^n z]}{10^n},$$

and

$$(x \underset{n}{\oplus} z) \star (y \underset{n}{\oplus} z) = x \underset{n}{\oplus} z = x + \frac{[10^n z]}{10^n},$$

so $(x \star y) \underset{n}{\oplus} z = (x \underset{n}{\oplus} z) \star (y \underset{n}{\oplus} z)$. This completes the proof. $\square$

Theorem 2.9. *Let G be a group. For a non-empty set X, let ψ be a map from X to G. Define $(x, g) \star (x_1, g_1) := (x, g\psi(x_1)g_1)$, for all $(x, g), (x_1, g_1) \in X \times G$, then $(X \times G, \star)$ is a generalized group.*

Proof. It is easy to see that $(X \times G, \star)$ is associative. Now, we show that $(x, \psi(x)^{-1})$ is the identity element of (x, g). i.e.,

$$(x, g) \star (x, \psi(x)^{-1}) = (x, g\psi(x)\psi(x)^{-1}) = (x, g),$$

and also

$$(x, \psi(x)^{-1}) \star (x, g) = (x, \psi(x)^{-1}\psi(x)g) = (x, g).$$

One can easily check that the identity element is unique and the inverse element of (x, g) is $(x, \psi(x)^{-1}g^{-1}\psi(x)^{-1})$. This completes the proof. $\square$

Example 2.10. Let $X = \mathbb{Z}$, $G = (\mathbb{R}, +)$ and $\psi : \mathbb{Z} \longrightarrow \mathbb{R}$ be the inclusion map. Define $(n, g) \star (n_1, g_1) := (n, g + n_1 + g_1)$, for all $(n, g), (n_1, g_1) \in \mathbb{Z} \times \mathbb{R}$. Then $(\mathbb{Z} \times \mathbb{R}, \star)$ is a generalized group.

Example 2.11. Let $X = \mathbb{Z}$ and let G be a group. Suppose that $0 \neq g_0 \in G$ and $\psi : \mathbb{Z} \longrightarrow G$; $\psi(n) = g_0^n$, for all $n \in \mathbb{Z}$. Define $(n, g) \star (n_1, g_1) := (n, g\psi(n)g_1)$, for all $(n, g), (n_1, g_1) \in \mathbb{Z} \times G$. Then $(\mathbb{Z} \times G, \star)$ is a generalized group.

In what follows, I_n denotes the $n \times n$ identity matrix.

Example 2.12. Let $X = \{kI_n | \ k \in \mathbb{Z}\}$, $G = M_n(\mathbb{R})$ and let $\psi : X \longrightarrow G$ be such that $\psi(kI_n) = kI_n$. We define $(kI_n, A) \star (k'I_n, A') := (kI_n, A+k'I_n+A')$, for all $(kI_n, A), (k'I_n, A') \in X \times G$. Then $(X \times G, \star)$ is a generalized group.

Example 2.13. Let $X = \{rI_n | \ r \in \mathbb{R} - \{0\}\}$, $G = GL(n, \mathbb{R})$ and $\psi : X \longrightarrow G$ be such that $\psi(rI_n) = rI_n$. Define $(rI_n, A) \star (sI_n, B) := (rI_n, AsI_nB) = (rI_n, sAB)$, for all $(rI_n, A), (sI_n, B) \in X \times G$. Then $(X \times G, \star)$ is a generalized group.

Similar to Theorem 2.9, in the following theorems we introduce some new types of generalized topological groups and top spaces.

Theorem 2.14. *Let G be a Hausdorff topological group, X be a Hausdorff topological space and $\phi : X \to G$ be a continuous map. Define $(x, g) * (x_1, g_1) := (x, g\phi(x_1)g_1)$, for all $(x, g), (x_1, g_1) \in X \times G$. Then $(X \times G, *)$ is a topological generalized group.*

Proof. In Theorem 2.9, we showed that $(X \times G, *)$ is a generalized group. Since X and G are Hausdorff topological spaces, so $X \times G$ is a Hausdorff topological space. It is enough to show that $\mathcal{P} : (X \times G) \times (X \times G) \to (X \times G)$ such that

$$\mathcal{P}((x, g), (x_1)g_1)) = (x, g) * (x_1, g_1) = (x, g\phi(x_1)g_1),$$

and $\mathcal{I} : X \times G \to X \times G$ such that

$$\mathcal{I}(x, g) = (x, \phi(x)^{-1}g^{-1}\phi(x)^{-1}),$$

are continuous maps. Suppose that $h : (X \times G) \times (X \times G) \to X \times G \times X \times G$ such that $h((x, g), (x_1, g_1)) = (x, g, x_1, g_1)$, $I : X \to X$ is the identity map, $f : G \times X \times G \to G$ such that $f(g, x_1, g_1) = m_1(m_1(g, \phi(x_1)), g_1)$, where m_1 is the multiplication map of G and $h' : X \times G \times X \times G \to X \times G$ such that $h'(x, g, x_1, g_1) = (I(x), f(g, x_1, g_1))$. Since the following diagram is commutative, and h and h' are continuous, therefore $\mathcal{P}$ is continuous.

$$
\begin{array}{ccc}
(X \times G) \times (X \times G) & \xrightarrow{\ \ h\ \ } & X \times G \times X \times G \\
& \searrow{\scriptstyle \mathcal{P}} & \ \ \downarrow{\scriptstyle h'} \\
& & X \times G
\end{array}
$$

Let $L_g : G \to G$ and $R_g : G \to G$ be left and right translation maps on G, respectively. Then

$$\mathcal{I}(x, g) = (I(x), L_{\phi^{-1}(x)} \circ R_{\phi^{-1}(x)} \circ m_2(g)),$$

where m_2 is the inverse map of G. So the inverse map $\mathcal{I}$ is continuous. $\square$

Example 2.15. Let $X = \mathbb{Z}$ be equipped with discrete topology and $G = \mathbb{R}$ be equipped with standard topology. So $(X \times G, \star)$ in Example 2.10, is a topological generalized group according to Theorem 2.14.

Theorem 2.16. *Let G be a Lie group, X be a smooth manifold and $\phi : X \to G$ be a smooth map. Define $(x, g) * (x_1, g_1) := (x, g\phi(x_1)g_1)$, for all $(x, g), (x_1, g_1) \in X \times G$. Then $(X \times G, *)$ is a top space.*

Proof. In Theorem 2.9, we showed that $(X \times G, *)$ is a generalized group where the identity element of (x, g) is $(x, \psi(x)^{-1})$ and the inverse element of (x, g) is $(x, \psi(x)^{-1}g^{-1}\psi(x)^{-1})$. Now we show that $X \times G$ is normal. Since for all $(x, g), (x_1, g_1) \in X \times G$, we have $e((x, g) * (x_1, g_1)) = e(x, g\phi(x_1)g_1) = (x, \phi(x)^{-1})$, and also $e((x, g)) * e((x_1, g_1)) = (x, \phi(x)^{-1}) * (x_1, \phi(x_1)^{-1}) = (x, \phi(x)^{-1}\phi(x_1)\phi(x_1)^{-1}) = (x, \phi(x)^{-1})$. We observe that $e((x, g) * (x_1, g_1)) = e((x, g)) * e((x_1, g_1))$. As obviously, $X \times G$ is a smooth manifold, it is enough to show that $\mathcal{P} : (X \times G) \times (X \times G) \to (X \times G)$ such that

$$\mathcal{P}((x, g), (x_1)g_1)) = (x, g) * (x_1, g_1) = (x, g\phi(x_1)g_1),$$

and $\mathcal{I} : X \times G \to X \times G$ such that

$$\mathcal{I}(x, g) = (x, \phi(x)^{-1}g^{-1}\phi(x)^{-1}),$$

are smooth maps. Suppose that $h : (X \times G) \times (X \times G) \to X \times G \times X \times G$ such that $h((x, g), (x_1, g_1)) = (x, g, x_1, g_1)$, $I : X \to X$ is the identity map, $f : G \times X \times G \to G$ such that $f(g, x_1, g_1) = m_1(m_1(g, \phi(x_1)), g_1))$ where m_1 is the multiplication map of G and $h' : X \times G \times X \times G \to X \times G$ such that $h'(x, g, x_1, g_1) = (I(x), f(g, x_1, g_1))$. Since the following diagram is commutative, and h and h' are smooth, therefore $\mathcal{P}$ is smooth.

$$(X \times G) \times (X \times G) \xrightarrow{\ h\ } X \times G \times X \times G$$

with $\mathcal{P}$ and h' mapping to $X \times G$.

Let $L_g : G \to G$ and $R_g : G \to G$ be left and right translation maps on G, respectively. Then

$$\mathcal{I}(x, g) = (I(x), L_{\phi^{-1}(x)} \circ R_{\phi^{-1}(x)} \circ m_2(g)),$$

where m_2 is the inverse map of G. So the inverse map $\mathcal{I}$ is smooth, and the proof is complete. $\qquad\square$

Proposition 2.17. *Let X, G, ψ be the same as Theorem 2.9 and $Y = \{(y,h)|e_{(y,h)} = e_{(x,g)}\}$, where (x,g) is a fixed element of $X \times G$. Then the following statements are valid:*

    i) $(Y,\star)$ *is a group;*
    ii) Y *and G are isomorphic.*

Proof. Since $e_{(y,h)} = e_{(x,g)}$, we have $x = y$ and hence $Y = \{(x,h)|h \in G\}$. We show that $(Y,\star)$ is a group. Associativity property of Y is trivial. We claim that $e_Y = (x, \psi(x)^{-1})$ is the identity element of Y, because

$$(x,h) \star e_Y = (x,h) \star (x, \psi(x)^{-1}) = (x, h\psi(x)\psi(x)^{-1}) = (x,h),$$

for all $(x,h) \in Y$. Also $(x, \psi(x)^{-1}h^{-1}\psi(x)^{-1})$ is the inverse of (x,h) for all $(x,h) \in Y$, because

$$
\begin{aligned}
(x,h) \star (x, \psi(x)^{-1}h^{-1}\psi(x)^{-1}) &= (x, \psi(x)^{-1}h^{-1}\psi(x)^{-1}) \star (x,h) \\
&= (x, \psi(x)^{-1}) \\
&= e_Y.
\end{aligned}
$$

Now, we show that Y is isomorphic with G. Let $\phi : Y \longrightarrow G$ be such that $\phi((x,g)) = \psi(x)g$. If $\phi(x,h) = \phi(x,h')$, then

$$
\begin{aligned}
\psi(x)h = \psi(x)h' &\implies \psi(x)^{-1}\psi(x)h = \psi(x)^{-1}\psi(x)h' \\
&\implies h = h' \\
&\implies (x,h) = (x,h').
\end{aligned}
$$

So, ϕ is one to one. Let h' be an arbitrary element of G. Now, by selecting $(x,h) = (x, \psi(x)^{-1}h') \in Y$, we have

$$\phi((x,h)) = \psi(x)h = \psi(x)\psi(x)^{-1}h' = h'.$$

Therefore ϕ is onto. Also ϕ is homomorphism, because

$$
\begin{aligned}
\phi((x,h) \star (x,h')) &= \phi(x, h\psi(x)h') \\
&= \psi xh\psi xh' \\
&= \phi((x,h))\phi((x,h')).
\end{aligned}
$$

So, the map ϕ is an isomorphism. $\qquad\square$

Similarly, we have the following two propositions.

Proposition 2.18. *Let X, G, ψ be the same as Theorem 2.14, and $Y = \{(y,h)|e_{(y,h)} = e_{(x,g)}\}$ such that (x,g) is a fixed element of $X \times G$. Then, the following statements are valid:*

i) $(Y, \star)$ *is a topological group;*

ii) Y *and* G *are isomorphic.*

Proposition 2.19. *Let* X, G, ψ *be the same as Theorem* 2.16. *Define* $Y := \{(y, h) | e_{(y,h)} = e_{(x,g)}\}$, *where* (x, g) *is a fixed element of* $X \times G$. *Then, the following statements are valid:*

i) $(Y, \star)$ *is a Lie group;*

ii) Y *and* G *are isomorphic.*

In the following theorem, we obtain a generalized group by using of a Hilbert space.

Theorem 2.20. *Let* H *be a Hilbert space and* u, v *be two fixed vector in* H *such that* $\langle u, v \rangle = 1$. *Define* $x \oplus y := x + <y, u> v$, *for all* $x, y \in H$. *Then* $(H, \oplus)$ *is a generalized group.*

Proof. First we show that $(H, \oplus)$ is associative. we have

$$(x \oplus y) \oplus z = (x + <y, u> v) \oplus z = x + <y, u> v + <z, u> v,$$

and also

$$
\begin{aligned}
x \oplus (y \oplus z) &= x \oplus (y + <z, u> v) \\
&= x + <(y + <z, u> u), u> v \\
&= x + <y, u> v + <z, u><u, u> v \\
&= x + <y, u> v + <z, u> v.
\end{aligned}
$$

One can easily check that $e_x = x - <x, u> v$ and $x^{-1} = x - 2 <x, u> v$ are the identity and the inverse of $x \in H$, respectively. □

In the next theorem, we obtain a generalized group on a tangent space of a Lie group.

Theorem 2.21. *Let* G *be a Lie group equipped with Riemannian metric,* $T_p G$ *be the tangent space of* G *in* p *and* V *be a vector field on* G *such that* $V(g) \neq 0$, *for all* $g \in G$. *Define*

$$x_p \boxplus y_p := x_p + <y_p, \frac{V(p)}{\|(V(p)\|} > \frac{V(p)}{\|(V(p)\|},$$

for all $x_p, y_p \in T_p G$. *Then* $(T_p G, \boxplus)$ *is a generalized group for all* $p \in G$.

Proof. Set $E_p = \frac{V(p)}{\|(V(p)\|}$. Then

$$
\begin{aligned}
x_p \boxplus (y_p \boxplus z_p) &= x_p \boxplus (y_p + <z_p, E_p> E_p) \\
&= x_p + <y_p + <z_p, E_p> E_p, E_p> E_p \\
&= x_p + <y_p, E_p> E_p + <z_p, E_p> E_p.
\end{aligned}
$$

And also

$$
\begin{aligned}
(x_p \boxplus y_p) \boxplus z_p &= (x_p + <y_p, E_p> E_p) \boxplus z_p \\
&= x_p + <y_p, E_p> E_p + <z_p, E_p> E_p.
\end{aligned}
$$

So $(T_pG, \boxplus)$ is associative. Now, we can simply show that the identity of x_p is $x_p - <x_p, E_p> E_p$, and the inverse of x_p is $x_p - 2 <x_p, E_p> E_p$. $\qquad\square$

3. CONCLUSION

The aim of this paper was to find new methods for constructing generalized groups, topological generalized groups and top spaces which are the generalization of their corresponding algebraic concepts. We have studied some their properties and provided some corresponding examples.

Acknowledgments

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NEW METHODS FOR CONSTRUCTING GENERALIZED GROUPS, TOPOLOGICAL GENERALIZED GROUPS AND TOP SPACES

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روش‌های جدید برای ساختن گروه‌های تعمیم‌یافته، گروه‌های تعمیم‌یافته توپولوژیک و فضاهای تاپ

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هدف این مقاله معرفی روش‌های جدید جهت ساختن گروه‌های تعمیم‌یافته، گروه‌های تعمیم‌یافته توپولوژیک و فضاهای تاپ می‌باشد. ما در این مقاله خواص این ساختارها را بررسی و مثال‌هایی ملموس مرتبط با آن‌ها را ارائه می‌کنیم. به‌علاوه، با استفاده از فضاهای هیلبرت و فضاهای مماس برگروه‌های لی، گروه‌های تعمیم‌یافته‌ای بدست می‌آوریم.

کلمات کلیدی: گروه تعمیم‌یافته، گروه تعمیم‌یافته توپولوژیک، فضای تاپ، گروه لی.