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# UPPER BOUNDS FOR FINITENESS OF GENERALIZED LOCAL COHOMOLOGY MODULES

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Abstract. Let $R$ be a commutative Noetherian ring with non-zero identity and $\mathfrak{a}$ an ideal of $R$. Let $M$ be a finite $R$–module of finite projective dimension and $N$ an arbitrary finite $R$–module. We characterize the membership of the generalized local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ in certain Serre subcategories of the category of modules from upper bounds. We define and study the properties of a generalization of cohomological dimension of generalized local cohomology modules. Let $\mathcal{S}$ be a Serre subcategory of the category of $R$–modules and $n > \operatorname{pd} M$ be an integer such that $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ belongs to $\mathcal{S}$ for all $i > n$. Then, for any ideal $\mathfrak{b} \supseteq \mathfrak{a}$, it is also shown that the module $\mathrm{H}^n_{\mathfrak{a}}(M,N)/\mathfrak{b}\,\mathrm{H}^n_{\mathfrak{a}}(M,N)$ belongs to $\mathcal{S}$.

## 1. Introduction

Throughout this paper $R$ is a commutative Noetherian ring. Let $\mathfrak{a}$ be an ideal of $R$, $M$ be a finite $R$–module of finite projective dimension and $N$ an arbitrary finite $R$–module. The notion of generalized local cohomology was introduced by J. Herzog [18]. The $i$–th generalized local cohomology modules of $M$ and $N$ with respectt to $\mathfrak{a}$ is defined by

$$\mathrm{H}^i_{\mathfrak{a}}(M,N) \cong \varinjlim_{n} \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N).$$

It is clear that $\mathrm{H}^i_{\mathfrak{a}}(R,N)$ is just the the ordinary local cohomology module $\mathrm{H}^i_{\mathfrak{a}}(N)$. This concept was studied in the articles [22], [6] and [24].

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For ordinary local cohomology module, there is the important concept of *cohomological dimension* of an $R$–module $N$ with respect to an ideal $\mathfrak{a}$ of $R$. It is defined by

$$\operatorname{cd}_{\mathfrak{a}}(N) := \sup\{i \geqslant 0 | \operatorname{H}^i_{\mathfrak{a}}(N) \neq 0\}.$$

This notion has been studied by several authors; see, for example [16], [17], [21], [19] and [15].

Hartshorn [17] has defined the notion $\operatorname{q}_{\mathfrak{a}}(R)$ as the greatest integer $i$ such that $\operatorname{H}^i_{\mathfrak{a}}(R)$ is not Artinian. Dibaei and Yassemi [13] extended this notion to arbitrary finite $R$–modules as

$$\operatorname{q}_{\mathfrak{a}}(N) := \sup\{i \geqslant 0 | \operatorname{H}^i_{\mathfrak{a}}(N) \text{ is not Artinian}\}$$

Recall that a subclass of the class of all modules is called Serre class, if it is closed under taking submodules, quotients and extensions. Examples are given by the class of finite modules, Artinian modules and etc. In [1, Theorems 3.1 and 3.3] the author and Melkersson characterized the membership of ordinary local cohomology modules in certain Serre class of the class of modules from upper bounds. They also introduced *Serre cohomological dimension of a module with respect to an ideal* [1, Definition 3.5] as

$$\operatorname{cd}_{(\mathfrak{a},\mathcal{S})}(N) := \sup\{i \geq 0 | \operatorname{H}^i_{\mathfrak{a}}(N) \text{ is not in } \mathcal{S}\}.$$

see also [4, Definition 3.4]. Note that when $\mathcal{S} = \{0\}$ then $\operatorname{cd}_{(\mathfrak{a},\mathcal{S})}(N) = \operatorname{cd}_{\mathfrak{a}}(N)$ and when $\mathcal{S}$ is the class of Artinian modules, then $\operatorname{cd}_{(\mathfrak{a},\mathcal{S})}(N) = \operatorname{q}_{\mathfrak{a}}(N)$.

Amjadi and Naghipour in [3] (resp. Asgharzadeh, Divaani-Aazar and Tousi in [5]) extended $\operatorname{cd}_{\mathfrak{a}}(N)$ (resp. $\operatorname{q}_{\mathfrak{a}}(N)$) to generalized local cohomology modules as

$$\operatorname{cd}_{\mathfrak{a}}(M,N) := \sup\{i \geqslant 0 | \operatorname{H}^i_{\mathfrak{a}}(M,N) \neq 0\}$$

$$(\text{resp. } \operatorname{q}_{\mathfrak{a}}(M,N) := \sup\{i \geqslant 0 | \operatorname{H}^i_{\mathfrak{a}}(M,N) \text{ is not Artinian}\}).$$

They also proved basic results about related notions. Also, there are some other attempts to study generalized local cohomology modules from upper bounds; see [10, Corollary 2.7] and [11, Theorem 5.1, Lemma 5.2 and Corollary 5.3 ].

Our objective in this paper is to characterize the membership of generalized local cohomology modules in certain Serre class of the category of $R$–modules from upper bounds. We will do it in Section 2. Our main results in this section are Theorems 2.1, 2.4 and 2.7. In Section 3, we will define and study the Serre cohomological dimension of two modules with respect to an ideal. Our definition and results in this paper improve and generalize all of the above mentioned one. For unexplained terminology, we refer the reader to [8] and [9].

## 2. Main results

The following theorem characterize the membership of generalized local cohomology modules in a certain Serre class from upper bounds.

**Theorem 2.1.** *Let $\mathcal{S}$ be a Serre subcategory of the category of $R$–modules. Let $\mathfrak{a}$ an ideal of $R$, $M$ a finite $R$–module of finite projective dimension and $N$ an arbitrary finite $R$–module. Let $n \geqslant \operatorname{pd} M$ be a non-negative integer. Then the following statements are equivalent:*

(i) $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ *is in* $\mathcal{S}$ *for all* $i > n$.

(ii) $\mathrm{H}^i_{\mathfrak{a}}(M,L)$ *is in* $\mathcal{S}$ *for all* $i > n$ *and for every finite* $R$*–module* $L$ *such that* $\operatorname{Supp}_R(L) \subseteq \operatorname{Supp}_R(N)$.

(iii) $\mathrm{H}^i_{\mathfrak{a}}(M,R/\mathfrak{p})$ *is in* $\mathcal{S}$ *for all* $\mathfrak{p} \in \operatorname{Supp}_R(N)$ *and all* $i > n$.

(iv) $\mathrm{H}^i_{\mathfrak{a}}(M,R/\mathfrak{p})$ *is in* $\mathcal{S}$ *for all* $\mathfrak{p} \in \operatorname{Min}\operatorname{Ass}_R(N)$ *and all* $i > n$.

*Proof.* We use descending induction on $n$. So, we may assume that all conditions are equivalent when $n$ is replaced by $n+1$ using [24, Theorem 2.5].

(i)⇒(iii). We want to show that $\mathrm{H}^{n+1}_{\mathfrak{a}}(M,R/\mathfrak{p})$ is in $\mathcal{S}$ for each $\mathfrak{p} \in \operatorname{Supp}_R(N)$. Suppose the contrary and let $\mathfrak{p} \in \operatorname{Supp}_R(N)$ be maximal of those $\mathfrak{p} \in \operatorname{Supp}_R(N)$ such that $\mathrm{H}^{n+1}_{\mathfrak{a}}(M,R/\mathfrak{p})$ is not in $\mathcal{S}$. Since $\mathfrak{p} \in \operatorname{Supp}_R(N)$, there is by [7, Chap.(ii), § 4, $n^o$ 4, Proposition 20] a nonzero map $f : N \longrightarrow R/\mathfrak{p}$. Let $\mathfrak{b} \supsetneqq \mathfrak{p}$ be the ideal of $R$ such that $\operatorname{Im} f = \mathfrak{b}/\mathfrak{p}$. The exact sequence $0 \to \operatorname{Ker} f \to N \to \operatorname{Im} f \to 0$, yields the exact sequence

$$\mathrm{H}^{n+1}_{\mathfrak{a}}(M,N) \longrightarrow \mathrm{H}^{n+1}_{\mathfrak{a}}(M,\operatorname{Im} f) \longrightarrow \mathrm{H}^{n+2}_{\mathfrak{a}}(M,\operatorname{Ker} f).$$

Since $\operatorname{Supp}_R(\operatorname{Ker} f) \subseteq \operatorname{Supp}_R(N)$, by induction $\mathrm{H}^{n+2}_{\mathfrak{a}}(M,\operatorname{Ker} f)$ belongs to $\mathcal{S}$. It follows that $\mathrm{H}^{n+1}_{\mathfrak{a}}(M,\operatorname{Im} f)$ belongs to $\mathcal{S}$. There is a filtration

$$0 = N_t \subset N_{t-1} \subset N_{t-2} \subset \cdots \subset N_0 = R/\mathfrak{b}$$

of submodules of $R/\mathfrak{b}$, such that for each $0 \leqslant i \leqslant t$, $N_{i-1}/N_i \cong R/\mathfrak{q}_i$ where $\mathfrak{q}_i \in \mathrm{V}(\mathfrak{b})$. Then by the maximality of $\mathfrak{p}$, $\mathrm{H}^{n+1}_{\mathfrak{a}}(M,R/\mathfrak{q}_i)$ is in $\mathcal{S}$. Use the exact sequences $0 \to N_i \to N_{i-1} \to R/\mathfrak{q}_i \to 0$, to conclude that $\mathrm{H}^{n+1}_{\mathfrak{a}}(M,R/\mathfrak{b})$ is in $\mathcal{S}$. Next the exact sequence $0 \to \operatorname{Im} f \to R/\mathfrak{p} \to R/\mathfrak{b} \to 0$, yields the exact sequence

$$\mathrm{H}^{n+1}_{\mathfrak{a}}(M,\operatorname{Im} f) \longrightarrow \mathrm{H}^{n+1}_{\mathfrak{a}}(M,R/\mathfrak{p}) \longrightarrow \mathrm{H}^{n+1}_{\mathfrak{a}}(M,R/\mathfrak{b}).$$

It follows that $\mathrm{H}^{n+1}_{\mathfrak{a}}(M,R/\mathfrak{p})$ is in $\mathcal{S}$ which is a contradiction.

(iii)⇒(ii). Use a filtration for $L$ as above.

(iv)$\Rightarrow$(iii). Let $\mathfrak{p} \in \operatorname{Supp}_R(N)$. Then $\mathfrak{p} \supseteq \mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Min}\operatorname{Ass}_R(N)$. Hence $\mathfrak{p} \in \operatorname{Supp}_R(R/\mathfrak{q})$. Applying (i)$\Rightarrow$ (iii), it follows that $\mathrm{H}^{n+1}_{\mathfrak{a}}(M, R/\mathfrak{p})$ is in $\mathcal{S}$.

Since the implications (ii)$\Rightarrow$(i) and (iii)$\Rightarrow$(iv) are clear, the proof is complete. □

**Corollary 2.2.** *Let $\mathcal{S}$ be a Serre subcategory of the category of $R$–modules. Let $\mathfrak{a}$ an ideal of $R$ and $M$ a finite $R$–module of finite projective dimension. Let $n \geqslant \operatorname{pd} M$ be a non-negetive integer. If $L$ and $N$ are finite $R$–modules such that $\operatorname{Supp}_R(L) = \operatorname{Supp}_R(N)$, then $\mathrm{H}^i_{\mathfrak{a}}(M, L)$ is in $\mathcal{S}$ for all $i > n$ if and only if $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ is in $\mathcal{S}$ for all $i > n$.*

**Definition 2.3.** (*See* [1, Definition 2.1] *and* [2, Definition 3.1].) *Let $\mathcal{M}$ be a Serre subcategory of the category of $R$–modules. We say that $\mathcal{M}$ is a Melkersson subcategory with respect to the ideal $\mathfrak{a}$ if for any $\mathfrak{a}$–torsion $R$–module $X$, $0 :_X \mathfrak{a}$ is in $\mathcal{M}$ implies that $X$ is in $\mathcal{M}$. $\mathcal{M}$ is called Melkersson subcategory when it is a Melkersson subcategory with respect to all ideals of $R$.*

When $\mathcal{M}$ is Melkersson subcategory of the category of $R$–modules, we are able to weaken the condition $(iii)$ in Theorem 2.1 to require that $\mathrm{H}^i_{\mathfrak{a}}(M, R/\mathfrak{p})$ is in $\mathcal{M}$ for all $\mathfrak{p} \in \operatorname{Supp}_R(N)$, just for $i = n + 1$.

**Theorem 2.4.** *Let $\mathcal{M}$ be a Melkersson subcategory of the category of $R$–modules. Let $\mathfrak{a}$ an ideal of $R$ and $M$ a finite $R$–module of finite projective dimension. Let $n \geqslant \operatorname{pd} M$ be a non-negative integer. Then for each finite $R$–module $N$ the conditions in Theorem 2.1 are equivalent to:*

(v) *$\mathrm{H}^{n+1}_{\mathfrak{a}}(M, R/\mathfrak{p})$ is in $\mathcal{M}$ for all $\mathfrak{p} \in \operatorname{Supp}_R(N)$.*

*Proof.* (v)$\Rightarrow$(iv). We prove by induction on $i \geq n+2$ that $\mathrm{H}^i_{\mathfrak{a}}(M, R/\mathfrak{p})$ is in $\mathcal{M}$ for all $\mathfrak{p} \in \operatorname{Supp}_R(N)$. It is enough to treat the case $i = n + 2$. Suppose that $\mathrm{H}^{n+2}_{\mathfrak{a}}(M, R/\mathfrak{p})$ is not in $\mathcal{M}$ for some $\mathfrak{p} \in \operatorname{Supp}_R(N)$. It follows that $\mathfrak{a} \not\subseteq \mathfrak{p}$, since otherwise $\mathrm{H}^{n+2}_{\mathfrak{a}}(M, R/\mathfrak{p}) = 0$, because $n + 2 > \operatorname{pd} M$. Take $x \in \mathfrak{a} \setminus \mathfrak{p}$ and put $L = R/(\mathfrak{p} + xR)$. Then $\operatorname{Supp}_R(L) \subset \operatorname{Supp}_R(N)$. We have a finite filtration

$$0 = L_t \subset L_{t-1} \subset L_{t-2} \subset \cdots \subset L_0 = L$$

such that $L_{i-1}/L_i \cong R/\mathfrak{p}_i$ for each $1 \leq i \leq t$ where $\mathfrak{p}_i \in \operatorname{Supp}_R(N)$. Using the exact sequence

$$\mathrm{H}^{n+1}_{\mathfrak{a}}(M, L_i) \longrightarrow \mathrm{H}^{n+1}_{\mathfrak{a}}(M, L_{i-1}) \longrightarrow \mathrm{H}^{n+1}_{\mathfrak{a}}(M, R/\mathfrak{p}_i)$$

for each $1 \leq i \leq t$, shows that $\mathrm{H}^{n+1}_{\mathfrak{a}}(M, L)$ is in $\mathcal{M}$. Consider the exact sequence $0 \to R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \to L \to 0$, which induces the following exact sequence

$$\mathrm{H}^{n+1}_{\mathfrak{a}}(M, L) \longrightarrow \mathrm{H}^{n+2}_{\mathfrak{a}}(M, R/\mathfrak{p}) \xrightarrow{x} \mathrm{H}^{n+2}_{\mathfrak{a}}(M, R/\mathfrak{p}).$$

This shows that $0 :_{\mathrm{H}^{n+2}_{\mathfrak{a}}(M,R/\mathfrak{p})} x$ is in $\mathcal{M}$. Since $\mathrm{H}^{n+2}_{\mathfrak{a}}(M, R/\mathfrak{p})$ is $(x)$–torsion, $\mathrm{H}^{n+2}_{\mathfrak{a}}(M, R/\mathfrak{p})$ is in $\mathcal{M}$, which is a contradiction. □

**Remark 2.5.** *In Theorems 2.1 and 2.4 we may specialize $\mathcal{S}$ to any of the Melkersson subcategories, given in* [1, Example 2.4] *to obtain characterizations of Artinianness, vanishing, finiteness of the support etc.*

Chu and Tang gave some parts of them in [10, Theorems 2.5, 2.6 and Corrolary 2.7] for the case of Artinianness in local rings. In the case of vanishing, i.e., when $\mathcal{S}$ merely consists of zero modules and finiteness of support (in local case) some parts of them were studied in [3, Theorem B] and [11, Theorem 5.1 part (a)]. See also [12, Theorems 3.1 and 3.2]. These authors used Gruson's Theorem, [23, Theorem 4.1], while we just used the maximal condition in a Noetherian ring.

**Lemma 2.6.** *Let $M$ and $N$ be two finite $R$–modules such that*

$$\mathrm{Supp}_R(M) \cap \mathrm{Supp}_R(N) \subseteq \mathrm{V}(\mathfrak{a}).$$

*Then $\mathrm{H}^i_{\mathfrak{a}}(M, N) \cong \mathrm{Ext}^i_R(M, N)$ for all $i \geq 0$.*

*Proof.* See [14, Corollary 2.8 (i)]. □

**Theorem 2.7.** *Let $\mathcal{S}$ be a Serre subcategory of the category of $R$–modules and $M$ a finite $R$–module of finite projective dimension. Let $N$ be a finite $R$–module, $\mathfrak{a}$ an ideal of $R$ and $n > \mathrm{pd}\, M$ be an integer such that $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ belongs to $\mathcal{S}$ for all $i > n$. Then, for any ideal $\mathfrak{b} \supseteq \mathfrak{a}$, $\mathrm{H}^n_{\mathfrak{a}}(M, N)/\mathfrak{b}\, \mathrm{H}^n_{\mathfrak{a}}(M, N)$ belongs to $\mathcal{S}$.*

*Proof.* Suppose $\mathrm{H}^n_{\mathfrak{a}}(M, N)/\mathfrak{b}\, \mathrm{H}^n_{\mathfrak{a}}(M, N)$ is not in $\mathcal{S}$. Let $L$ be a maximal submodule of $N$ such that $\mathrm{H}^n_{\mathfrak{a}}(M, N/L) \otimes_R R/\mathfrak{b}$ is not in $\mathcal{S}$. Let $T \supseteq L$ be such that $\Gamma_{\mathfrak{b}}(N/L) = T/L$. Since $\mathrm{Supp}_R(T/L) \subseteq \mathrm{V}(\mathfrak{b}) \cap \mathrm{Supp}_R(N) = \mathrm{Supp}_R(N/\mathfrak{b}N) \subseteq \mathrm{Supp}_R(N)$, $\mathrm{H}^i_{\mathfrak{a}}(M, T/L)$ belongs to $\mathcal{S}$ for all $i > n$ by Theorem 2.1. Also, by Lemma 2.6, we have

$$\mathrm{H}^n_{\mathfrak{a}}(M, T/L) \cong \mathrm{Ext}^n_R(M, T/L) = 0.$$

From the exact sequence $0 \to T/L \to N/L \to N/T \to 0$, we get the exact sequence

$$0 = \mathrm{H}^n_{\mathfrak{a}}(M, T/L) \longrightarrow \mathrm{H}^n_{\mathfrak{a}}(M, N/L) \xrightarrow{f} \mathrm{H}^n_{\mathfrak{a}}(M, N/T) \longrightarrow \mathrm{H}^{n+1}_{\mathfrak{a}}(M, T/L).$$

$\operatorname{Tor}_i^R(R/\mathfrak{b}, \operatorname{Ker} f)$ and $\operatorname{Tor}_i^R(R/\mathfrak{b}, \operatorname{Coker} f)$ are in $\mathcal{S}$ for all $i$, because $\operatorname{Ker} f(=0)$ and $\operatorname{Coker} f$ are in $\mathcal{S}$. It follows from [20, Lemma 3.1], that $\operatorname{Ker}(f \otimes R/\mathfrak{b})$ and $\operatorname{Coker}(f \otimes R/\mathfrak{b})$ are in $\mathcal{S}$. Since $\mathrm{H}^n_{\mathfrak{a}}(M, N/L) \otimes_R R/\mathfrak{b}$ is not in $\mathcal{S}$, the module $\mathrm{H}^n_{\mathfrak{a}}(M, N/T) \otimes_R R/\mathfrak{b}$ can not be in $\mathcal{S}$. By the maximality of $L$, we get $T = L$. We have shown that $\Gamma_{\mathfrak{b}}(N/L) = 0$ and therefore we can take $x \in \mathfrak{b}$ such that the sequence $0 \to N/L \xrightarrow{x} N/L \to N/(L+xN) \to 0$ is exact. Thus we get the exact sequence

$$\mathrm{H}^n_{\mathfrak{a}}(M, N/L) \xrightarrow{x} \mathrm{H}^n_{\mathfrak{a}}(M, N/L) \to \mathrm{H}^n_{\mathfrak{a}}(M, N/L + xN) \to \mathrm{H}^{n+1}_{\mathfrak{a}}(M, N/L).$$

This yields the exact sequence

$$0 \to \mathrm{H}^n_{\mathfrak{a}}(M, N/L)/x\, \mathrm{H}^n_{\mathfrak{a}}(M, N/L) \to \mathrm{H}^n_{\mathfrak{a}}(M, N/L + xN) \to C \to 0,$$

where $C \subseteq \mathrm{H}^{n+1}_{\mathfrak{a}}(M, N/L)$ and thus $C$ is in $\mathcal{S}$.

Note that $x \in \mathfrak{b}$. Hence we get the exact sequence

$$\operatorname{Tor}_1^R(R/\mathfrak{b}, C) \longrightarrow \mathrm{H}^n_{\mathfrak{a}}(M, N/L) \otimes_R R/\mathfrak{b} \longrightarrow \mathrm{H}^n_{\mathfrak{a}}(M, N/(L + xN)) \otimes_R R/\mathfrak{b}$$

However, $L \subsetneqq (L + xN)$ and therefore $\mathrm{H}^n_{\mathfrak{a}}(M, N/(L + xN)) \otimes_R R/\mathfrak{b}$ belongs to $\mathcal{S}$ by the maximality of $L$. Consequently,

$$\mathrm{H}^n_{\mathfrak{a}}(M, N/L) \otimes_R R/\mathfrak{b}$$

is in $\mathcal{S}$ which is a contradiction. □

Asgharzadeh, Divaani-Aazar and Tousi, in [5, Theorem 3.1] proved the following corollary when $\mathcal{S}$ is the category of Artinian $R$–modules with an strong assumption that $N$ has finite Krull dimension. This condition is very near to local case, while it is a simple conclusion of Theorem 2.7 without that strong assumption.

**Corollary 2.8.** *Let $\mathcal{S}$ be a Serre subcategory of the category of $R$–modules and $M$ be a finite $R$–module of finite projective dimension. Let $N$ be a finite $R$–module, $\mathfrak{a}$ be an ideal of $R$ and $n > \operatorname{pd} M$ be an integer such that $\mathrm{H}^i_{\mathfrak{a}}(M, N)$ belongs to $\mathcal{S}$ for all $i > n$, then $\mathrm{H}^n_{\mathfrak{a}}(M, N)/\mathfrak{a}\, \mathrm{H}^n_{\mathfrak{a}}(M, N)$ belongs to $\mathcal{S}$.*

## 3. Cohomological dimension with respect to Serre class

In the following, we introduce the last integer such that the generalized local cohomology modules belong to a Serre class.

**Definition 3.1.** *Let $\mathcal{S}$ be a Serre subcategory of the category of $R$–modules. Let $\mathfrak{a}$ be an ideal of $R$ and $M, N$ two $R$–modules. We define*

$$\operatorname{cd}_{(\mathfrak{a},\mathcal{S})}(M, N) := \sup\{i \geq 0 \mid \mathrm{H}^i_{\mathfrak{a}}(M, N) \text{ is not in } \mathcal{S}\}.$$

*with the usual convention that the suprimum of the empty set of is interpreted as* $-\infty$.

*For example when* $\mathcal{S}=\{0\}$, *then* $\mathrm{cd}_{(\mathfrak{a},\mathcal{S})}(M,N)=\mathrm{cd}_{\mathfrak{a}}(M,N)$ *and when* $\mathcal{S}$ *is the class of Artinian modules, then* $\mathrm{cd}_{(\mathfrak{a},\mathcal{S})}(M,N)=\mathrm{q}_{\mathfrak{a}}(M,N)$ *as in* [3] *and* [5].

In the following we study the main properties of this invariant.

**Proposition 3.2.** (Compare [12, Theorems 3.1 and 3.2] ) Let $\mathcal{S}$ be a Serre subcategory of the category of $R$–modules. Let $\mathfrak{a}$ be an ideal of $R$ and $M$ a finite $R$–module of finite projective dimension. The following statements hold.

(a) Let $\mathcal{S}_1,\mathcal{S}_2$ be two Serre subcategories of the category of $R$–modules such that $\mathcal{S}_1\subset\mathcal{S}_2$. Then $\mathrm{cd}_{(\mathfrak{a},\mathcal{S}_2)}(M,N)\leq\mathrm{cd}_{(\mathfrak{a},\mathcal{S}_1)}(M,N)$ for every finite $R$–module $N$. In particular $\mathrm{cd}_{(\mathfrak{a},\mathcal{S})}(M,N)\leq\mathrm{cd}_{\mathfrak{a}}(M,N)$ for each Serre subcategory $\mathcal{S}$ of the category of $R$–modules.

(b) If $L$ and $N$ are finite $R$–modules such that $\mathrm{Supp}_R(L)\subset\mathrm{Supp}_R(N)$, then $\mathrm{cd}_{(\mathfrak{a},\mathcal{S})}(M,L)\leq\mathrm{cd}_{(\mathfrak{a},\mathcal{S})}(M,N)$ and equality holds if

$$\mathrm{Supp}_R(L)=\mathrm{Supp}_R(N).$$

(c) Let $0\to N'\to N\to N''\to 0$ be an exact sequence of finite $R$–modules. Then

$$\mathrm{cd}_{(\mathfrak{a},\mathcal{S})}(M,N)=\max\{\mathrm{cd}_{(\mathfrak{a},\mathcal{S})}(M,N'),\mathrm{cd}_{(\mathfrak{a},\mathcal{S})}(M,N'')\}.$$

(d) $\mathrm{cd}_{(\mathfrak{a},\mathcal{S})}(M,R)=\sup\{\mathrm{cd}_{(\mathfrak{a},\mathcal{S})}(M,N)|N \text{ is a finite } R\text{–module }\}$.

(e) $\mathrm{cd}_{(\mathfrak{a},\mathcal{S})}(M,N)=\sup\{\mathrm{cd}_{(\mathfrak{a},\mathcal{S})}(M,R/\mathfrak{p})|\mathfrak{p}\in\mathrm{Supp}_R(N)\}$.

(f) $\mathrm{cd}_{(\mathfrak{a},\mathcal{S})}(M,N)=\sup\{\mathrm{cd}_{(\mathfrak{a},\mathcal{S})}(M,R/\mathfrak{p})|\mathfrak{p}\in\mathrm{Min}\,\mathrm{Ass}_R(N)\}$.

If $\mathcal{M}$ is Melkersson subcategory, then the following statements hold:

(g) $\mathrm{cd}_{(\mathfrak{a},\mathcal{M})}(M,N)=\min\{r\geq 0|\,\mathrm{H}^{r+1}_{\mathfrak{a}}(M,R/\mathfrak{p})\in\mathcal{M} \text{ for all } \mathfrak{p}\in\mathrm{Supp}_R(N)\}$.

(h) For each integer $i$ with $1+\mathrm{pd}\,M\leq i\leq\mathrm{cd}_{(\mathfrak{a},\mathcal{M})}(M,N)+\mathrm{pd}\,M$, there exists $\mathfrak{p}\in\mathrm{Supp}_R(N)$ with $\mathrm{H}^i_{\mathfrak{a}}(M,R/\mathfrak{p})$ not in $\mathcal{M}$.

(i) $\mathrm{cd}_{(\mathfrak{a},\mathcal{M})}(M,R)=\min\{r\geq 0|\,\mathrm{H}^{r+1}_{\mathfrak{a}}(M,R/\mathfrak{p})\in\mathcal{M} \text{ for all } \mathfrak{p}\in\mathrm{Spec}(R)\}$.

(j) $\mathrm{cd}_{(\mathfrak{a},\mathcal{M})}(M,R)=\min\{r\geq 0|\,\mathrm{H}^{r+1}_{\mathfrak{a}}(M,N)\in\mathcal{M} \text{ for all finite } R\text{–modules } N\}$.

*Proof.* (a) By definition.

(b) Follows from Theorem 2.1.

(c) The inequality "$\geq$", holds by (b) and we get the opposite inequality from the following exact sequence

$$\ldots\longrightarrow\mathrm{H}^i_{\mathfrak{a}}(M,N')\longrightarrow\mathrm{H}^i_{\mathfrak{a}}(M,N)\longrightarrow\mathrm{H}^i_{\mathfrak{a}}(M,N'')\longrightarrow\ldots$$

The assertions (d), (e) and (f) follow from Theorem 2.1 $(i) \Leftrightarrow (ii)$, $(i) \Leftrightarrow (ii)$ and $(i) \Leftrightarrow (iv)$, respectively.

(g), (h), (i) and (j) follow from Theorem 2.4. □

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