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THE $(\triangle, \square)$-EDGE GRAPH $G_{\triangle,\square}$ OF A GRAPH G

GH. NASIRIBOROUJENI, M. MIRZAVAZIRI* AND A. ERFANIAN

ABSTRACT. To a simple graph $G = (V, E)$, we correspond a simple graph $G_{\triangle,\square}$ whose vertex set is $V_{\triangle,\square} = \{\{x, y\} : x, y \in V\}$ and two vertices $\{x, y\}$ and $\{z, w\}$ are adjacent if and only if $\{x, z\}, \{x, w\}, \{y, z\}, \{y, w\} \in E$ or correspond to a vertex of V. The graph $G_{\triangle,\square}$ is called the $(\triangle, \square)$-edge graph of the graph G. In this paper, our ultimate goal is to provide a link between the connectedness of G and $G_{\triangle,\square}$.

1. INTRODUCTION

In the recent years, the commuting graph of group have become a topic of research for many mathematicians (see, for example [2, 4, 9]). This graph is precisely the complement of the non-commuting graph of a group, denoted by $\Delta(G)$, considered in [1]. Some authors gave some generalization of the commuting graph and non-commuting graph (see, for example [3, 6, 7, 10]). In 2017, the authors have generalized the notion of a commuting graph to the commuting graph of subsets of a finite group G, $\Gamma(n_1, n_2, \ldots, n_k, G)$ whose vertices are the n_i-subsets of G such that two vertices X and Y are adjacent if and only if $xy = yx$ for all $x \in X$ and $y \in Y$.

Taking idea from this, we correspond a graph $G_{\triangle,\square}$ to a simple graph G. The main idea is to correspond some edges to triangles and squares of G in a new graph. More precisely, if $G = (V, E)$ is a simple graph, then we define a new graph $G_{\triangle,\square}$ whose vertices are

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*Corresponding author.

2-subsets of V. Then if (x, y, z) is a triangle in G, we draw the triangle $(\{x, y\}, \{x, z\}, \{y, z\})$ in $G_{\triangle,\square}$ and if (x, y, z, w) is a square in G, with xy, yz, zw, wx as its edges, we draw an edge $\{x, z\}\{y, w\}$ in $G_{\triangle,\square}$. Roughly speaking, we can say that edges of $G_{\triangle,\square}$ are derived from $\triangle$'s and $\square$'s of G.

The ultimate goal of the present paper is to find a link between connectivity of G and $G_{\triangle,\square}$. Some simple facts can be easily proved in on direction when we want to think about the connectedness of G using the same property for $G_{\triangle,\square}$. We do this in Section 2 together with giving some general facts about $G_{\triangle,\square}$.

On the other hand, from connectivity of G to the same property of $G_{\triangle,\square}$, the problem seems to be hard. We give some partial answer for the problem in Section 3.

We recall certain graph theoretic terminologies (see, for example, [5, 8, 11]). Note that all graphs considered here are simple graphs, i.e., undirected graphs without loop or multiple edges. Moreover, by (x, y, z) we mean the triangle with vertices x, y, z and by (x, y, z, w) we mean the square with vertices x, y, z, w and edges xy, yz, zw, wx. The complete graph K_n is the graph with n vertices and all possible edges. The complete bipartite graph $K_{n,n'}$ is the graph with two partition sets A and A' with $|A| = n$ and $|A'| = n'$ such that there is no edges inside a partition and contains all possible edges between two partitions.

2. Triangles and squares as edges

Let us start by the definition of $G_{\triangle,\square}$.

Definition 2.1. Let $G = (V, E)$ be a graph. The $(\triangle, \square)$-edge graph of G is the graph $G_{\triangle,\square} = (V_{\triangle,\square}, E_{\triangle,\square})$ whose vertex set is $V_{\triangle,\square} = \{\{x, y\} : x, y \in V\}$ and two vertices $\{x, y\}, \{z, w\} \in V_{\triangle,\square}$ are adjacent if and only if $\{x, z\}, \{x, w\}, \{y, z\}, \{y, w\} \in E$ or correspond to a vertex of V.

For the sake of simplicity we write $\overline{xy}$ instead of $\{x, y\}$.

Example 2.2. Let $G = (x, y, z, w) \cup \{xz\}$, namely the square (x, y, z, w) with an extra edge xz. Then

$$\begin{aligned} V_{\triangle,\square} &= \{\overline{xy}, \overline{xz}, \overline{xw}, \overline{yz}, \overline{yw}, \overline{zw}\}, \\ E_{\triangle,\square} &= (\overline{xy}, \overline{xz}, \overline{yz}) \cup (\overline{xz}, \overline{xw}, \overline{zw}) \cup \{\overline{xz}\ \overline{yw}\}. \end{aligned}$$

A simple verification shows that triangles of G produce triangles in $G_{\triangle,\square}$ and squares of G give edges in $G_{\triangle,\square}$. We therefore can immediately deduce the following.

Proposition 2.3. *Let G be a triangle-free and square-free graph. Then $G_{\triangle,\square}$ has no edges. In particular, for a tree T, the $(\triangle,\square)$-edge graph $T_{\triangle,\square}$ has no edge.*

We can evidently see that if G and G' are two isomorphic graphs then $G_{\triangle,\square}$ is isomorphic to $G'_{\triangle,\square}$. A natural question is about the converse: if $G_{\triangle,\square}$ is isomorphic to $G'_{\triangle,\square}$, can we deduce that G and G' are isomorphic? Proposition 2.3 shows that the converse is not true in general. To see this, let T and T' be two non-isomorphic trees with the same order. Then $T_{\triangle,\square}$ and $T'_{\triangle,\square}$ are empty graphs with the same order which are clearly isomorphic to each other.

Proposition 2.4. *For positive integers $n \geq 2$ and $n' \geq 2$,*

$$(K_n)_{\triangle,\square} = K_{\binom{n}{2}}, \quad (K_{n,n'})_{\triangle,\square} = K_{\binom{n}{2},\binom{n'}{2}} \cup \overline{K_{nn'}}.$$

Proof. Note that $(K_n)_{\triangle,\square}$ has $\binom{n}{2}$ vertices with all possible edges. Furthermore, note that $(K_{n,n'})_{\triangle,\square}$ has $\binom{n+n'}{2}$ vertices and its edges are derived from squares of $K_{n,n'}$ since the bipartite graph $K_{n,n'}$ is triangle-free. Thus if $\overline{xy}$ and $\overline{zw}$ are two vertices with x, y in one partition set and z, w belong to the other partition set, then $\overline{xy}\ \overline{zw}$ is an edge in $(K_{n,n'})_{\triangle,\square}$ and if $\overline{xy}$ is a vertex such that x and y are in two different partition sets, then $\overline{xy}$ is an isolated vertex in $(K_{n,n'})_{\triangle,\square}$. □

Lemma 2.5. *Let G be a graph and $\overline{xy}$ be a non-isolated vertex of $G_{\triangle,\square}$. Then*

a. *if xy is an edge of G then x and y have at least a common neighbour in G, i.e., xy is on a triangle in G;*
b. *if xy is not an edge of G then x and y have at least two common neighbours in G, i.e., x and y are on opposite corners of a square in G.*

Proof. Since $\overline{xy}$ is not isolated, there is a neighbour $\overline{zw}$ for it. There are two cases:

a. $\{x, y\}$ and $\{z, w\}$ has a common vertex, say $x = w$. By the definition of an edge in $G_{\triangle,\square}$ we can therefore deduce that $\{x, z\}, \{x\}, \{y, z\}, \{y, x\} \in V \cup E$. This means that (x, y, z) is a triangle in G;
b. $\{x, y\}$ and $\{z, w\}$ has no common vertex. By the definition of an edge in $G_{\triangle,\square}$ we can therefore deduce that $\{x, z\}, \{x, w\}, \{y, z\}, \{y, w\} \in E$. This means that (x, y, z, w) is a square in G.

□

Corollary 2.6. *If edge xy is on m triangles and x and y are on opposite corners of k squares in G, then in $G_{\triangle,\square}$ we have $deg(\overline{xy}) = 2m + k$.*

Corollary 2.7. *Let G be a graph of order at least 3 and let $G_{\triangle,\square}$ be connected. Then*

a. *G has a triangle or a square;*
b. *G has no pendant or isolated vertex.*

Proof. Since $G_{\triangle,\square}$ is connected and is not a singleton, it can not have isolated vertex. Now we can apply Lemma 2.5. $\square$

One can easily see that the converse of Corollary 2.7 is not true in general.

Theorem 2.8. *Let G be a graph and let $G_{\triangle,\square}$ be connected. Then G is also connected.*

Proof. If G is not connected, then there are two vertices x and y which are belonged to two different components of G. Thus xy is not an edge and they have no common neighbour. This contradicts to Lemma 2.5. $\square$

Again, notice that the converse of the above facts are not true in general. Figure 1 gives an appropriate example to show this.

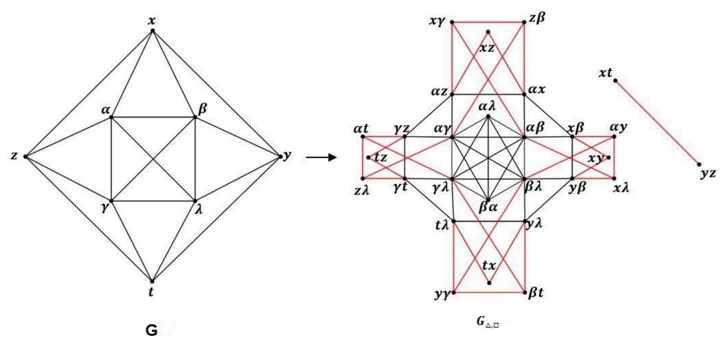

FIGURE 1.

Theorem 2.9. *Let G be a graph and let $G_{\triangle,\square}$ be connected. Then $|E| \geqslant 2|V| - 3$.*

Proof. The result is obviously true, even with $>$ instead of $\geqslant$, if $\deg(x) \geqslant 4$ for each $x \in G$. Thus we can assume that there is a vertex $x_1 \in G$ with $\deg(x_1) \leqslant 3$. Corollary 2.7 now implies that $\deg(x_1) = 2$ or 3.

Let $\deg(x_1) = 2$ for some x_1 and let x_2 and x_3 be the only neighbours of x_1. Thus the remaining vertices $x_4, \ldots, x_{|V|}$ are not neighbours of x_1 and Lemma 2.5 guarantees that each of these vertices should have at least two common neighbours with x_1. Since x_2 and x_3 are the only neighbours of x_1 we can therefore deduce that both x_2 and x_3 are neighbours of $x_4,\dots,x_{|V|}$. We now have

$$\deg(x_1) \geqslant 2, \deg(x_2), \deg(x_3) \geqslant |V|-2, \deg(x_4),\dots,\deg(x_{|V|}) \geqslant 2.$$

Summing these and regarding to the face that the degree sum of vertices is $2|E|$ gives $|E| \geqslant 2|V|-4$. But if these are the only edges of G then G is triangle-free which contradicts to the part (a) of Lemma 2.5. This implies the result.

Otherwise, if there is no x with $\deg(x) = 2$, suppose $\deg(x_1) = 3$ for some x_1 and let x_2, x_3 and x_4 be the only neighbours of x_1. Thus the remaining vertices $x_5,\dots,x_{|V|}$ are not neighbours of x_1 and Lemma 2.5 guarantees that each of these vertices should have at least two common neighbours with x_1. Since x_2, x_3 and x_4 are the only neighbours of x_1 we can therefore deduce that both two of the vertices x_2, x_3 and x_4 are neighbours of $x_5,\dots,x_{|V|}$. This guarantees the existence of at least $(|V|-4)\times 2$ edges between the sets $\{x_5,\dots,x_{|V|}\}$ and $\{x_2,x_3,x_4\}$. We also have 3 extra edges for x_1. Moreover, since each of the vertex $\{x_5,\dots,x_{|V|}\}$ has degree at least 3 we should have at least $\lfloor\frac{|V|-4}{2}\rfloor$ extra edges. Furthermore, there should be at least 2 edges for the set $\{x_2,x_3,x_4\}$, since the 3 edges incidence on x_1 should be belonged to some triangles.

We now have at least

$$(|V|-4)\times 2+3+\lfloor\frac{|V|-4}{2}\rfloor+2 = 2|V|-5+\lfloor\frac{|V|-4}{2}\rfloor+2 \geqslant 2|V|-3$$

edges in G. Note that the equality can only occurs just for the case $|V| = 4$, but if $|V| = 4$ we have $|E| = 6$. Thus we can say that $|E| > 2|V|-3$ is this case. $\square$

Theorem 2.10. *Let G be a graph with $|E| = 2|V|-3$ and let $G_{\triangle,\square}$ be connected. Then G is a union of $|V|-2$ triangles with a unique common edge.*

Proof. Using the same notation as in Theorem 2.9, we can say that equality occurred just for the case $\deg(x_1) = 2$ and just provided that the last extra edge is between x_2 and x_3, since this is the only edge which guarantees that $|E| = 2|V|-3$. Note that if we have no edge between x_2 and x_3 then for $G_{\triangle,\square}$ to be connected we need at least two edges and so we have $|E| > 2|V|-3$ that conrary to assumption. $\square$

Theorem 2.11. *If G has m triangles and k squares, then $|E(G_{\triangle,\square})| = 3m+k$.*

Proof. For each triangle (x,y,z) in G, there is a triangle $(\overline{xy},\overline{xz},\overline{yz})$ in $G_{\triangle,\square}$ which has three edge and for each square (x,y,z,w) in G there is a edge in $G_{\triangle,\square}$ between vertices $\overline{xz}, \overline{yw}$. All the edges obtained from these two methods are separate and so $|E(G_{\triangle,\square})| = 3m + k$. $\square$

Corollary 2.12. *For each natural number m we have* $3\binom{m+1}{4} = \binom{\binom{m}{2}}{2}$

Proof. Let $G = K_m$, we know that this graph has $\binom{m}{3}$ triangles and $\frac{p(m,4)}{8} = 3\binom{m}{4}$ squares. Therefore, according to Theorem 2.11, the number of edges $G_{\triangle,\square}$ is equal to $3(\binom{m}{3} + \binom{m}{4}) = 3\binom{m+1}{4}$. On the other hand, $G_{\triangle,\square} = K_{\binom{m}{2}}$ and so has $\binom{\binom{m}{2}}{2}$ edges. Consequently, $3\binom{m+1}{4} = \binom{\binom{m}{2}}{2}$. $\square$

3. Sufficient conditions for the connectedness of $G_{\triangle,\square}$

In this section, we study some sufficient conditions for the connectedness of $G_{\triangle,\square}$.

Theorem 3.1. *Let G be a graph with $|V| \geq 3$. If $|E| \geq \binom{n-1}{2} + 2$, then $G_{\triangle,\square}$ is connected.*

Proof. The result is obviously true for $|V| = 3$. Now, suppose that the theorem is true for graph G with $|V| = n-1$ and we prove it for graph G with $|V| = n$. We consider the following two cases:
Case 1. G has a vertex a of degree $n-2$. In this case we remove the vertex a and all the edges attached to it, then we obtain the graph H with $n-1$ vertices. In the graph H we have $|E(H)| \geqslant \binom{n-1}{2} + 2 - (n-2) = \binom{n-2}{2} + 2$.
Consequently, graph $H_{\triangle,\square}$ is connected. If vertices of graph H are $x_1, x_2, \dots, x_{n-1}$. Then graph $G_{\triangle,\square}$ contains graph $H_{\triangle,\square}$ and $n-1$ vertices $\overline{ax_1}, \overline{ax_2}, \dots, \overline{ax_{n-1}}$, to show that the graph $G_{\triangle,\square}$ is connected, it is sufficient to show that each of these $n-1$ vertices is at least one of the $H_{\triangle,\square}$ vertices is adjacent. Because $deg(a) = n-2$, assume that a is adjacent to $x_1, x_2, \dots, x_{n-2}$. We consider vertex $\overline{ax_i}(i = 1, 2, \dots, n-2)$. Because $deg(x_i)$ in graph H at least 2, there is $1 \leqslant j \leqslant n-2, j \neq i$ such that vertex x_j adjacent to x_i and so $\overline{ax_i}, \overline{ax_j}, \overline{x_ix_j}$ form a triangle in $G_{\triangle,\square}$, consequently $\overline{ax_i}$ is adjacent to $\overline{x_ix_j}$. Now we consider vertex $\overline{ax_{n-1}}$, because $deg(x_{n-1}) \geqslant 2$, there is two vertices x_k, x_l that with x_{n-1} are adjacent and so vertex $\overline{ax_{n-1}}$ with vertex $\overline{x_kx_l}$ adjacent.
Case 2. G has not a vertex of degree $n-2$. So, G has at least two vertices of degree $n-1$, because otherwise it has a maximum of one vertex of degree $n-1$ and we have: $|E(G)| \leqslant \dfrac{n-1+(n-1)(n-3)}{2} = \dfrac{(n-1)(n-2)}{2} = \binom{n-1}{2} < \binom{n-1}{2} + 2$, which contradicts assumption.

Now, if vertices of degree $n-1$ in G are a, b. In $G_{\triangle,\square}$, each other vertex is adjacent to $\overline{ab}$ and so $G_{\triangle,\square}$ is connected. $\square$

Definition 3.2. Let G be a graph and $x, y \in G$. The common neighbourhood of x and y, denoted by $N(x, y)$, is defined by

$$N(x, y) = \{z|zx \text{ and } zy \in E\}.$$

The closed common neighbourhood of x and y, denoted by $N'(x, y)$, is then defined by $N'(x, y) = N(x, y) \cup \{x, y\}$ if $xy \in E$ and $N'(x, y) = N(x, y)$ if $xy \notin E$.

Lemma 3.3. *Let G be a graph. Then $G_{\triangle,\square}$ has an isolated vertex as a connected component if and only if $N(x, y) = \emptyset$ for an edge xy of G or $|N(x, y)| \leq 1$*

Proof. A vertex $\overline{xy} \in G_{\triangle,\square}$ is an isolated vertex if and only if it has no neighbour. An argument similar to the proof of Lemma 2.5 shows that this happens when xy is an edge which is not in a triangle or xy is not an edge and x and y have at most one common neighbour. $\square$

Lemma 3.4. *Let G be a graph. Then $G_{\triangle,\square}$ has an edge as a connected component if and only if $N'(x, y) = \{z, w\}$ and $N'(z, w) = \{x, y\}$ for some different vertices $x, y, z, w \in G$.*

Proof. Let $\overline{xy}\,\overline{zw}$ be a connected component of $G_{\triangle,\square}$. If $\{x, y\} \cap \{z, w\}$ is non-empty, say $x = w$, then (x, y, z) should be a triangle in G and thus $(xy, xz, yz) = (xy, wz, yz)$ is a triangle in $G_{\triangle,\square}$, which contradicts the fact that $\overline{xy}\,\overline{zw}$ is a connected component of $G_{\triangle,\square}$. We can therefore deduce that x, y, z, w are different vertices of G and (x, z, y, w) is a square in G.

If $xy \in E$ then $\overline{xz}$ is another neighbour of $\overline{xy}$ which again contradicts the fact that $\overline{xy}\,\overline{zw}$ is a connected component of $G_{\triangle,\square}$. The same reason shows that $zw \notin E$.

We have now shown that $\{z, w\} \subseteq N(x, y)$ and $\{x, y\} \subseteq N(z, w)$ and $xy, zw \notin E$. Now note that if $u \in N(x, y)$ and $u \neq z, w$ then $\overline{zu}$ is another neighbour of $\overline{xy}$ which contradicts the fact that $\overline{xy}\,\overline{zw}$ is a connected component of $G_{\triangle,\square}$. This completes one part of the result. The other part is clear. $\square$

Lemma 3.5. *Let G be a graph. Then $G_{\triangle,\square}$ has a triangle as a connected component if and only if $N'(x, y) = N'(x, z) = N'(y, z) = \{x, y, z\}$ for some different vertices $x, y, z \in G$.*

Proof. Let $(\overline{xy}, \overline{zw}, \overline{uv})$ be a connected component of $G_{\triangle,\square}$. If $xy \notin E$ then (x, z, y, w) and (x, u, y, v) should be squares in G. Thus (x, u, y, z) is also a square. This implies that $\overline{uz}$ is another neighbour of $\overline{xy}$ which contradicts the fact that $(\overline{xy}, \overline{zw}, \overline{uv})$ is a connected component of $G_{\triangle,\square}$. Thus we have $xy \in E$ and the same argument shows that $zw, uv \in E$.

Note that if x, y, z, w are different vertices of G then $\overline{xy}$ has at least three neighbours $\overline{xz}, \overline{yz}$ and $\overline{zw}$ which contradicts the fact that $(\overline{xy}, \overline{zw}, \overline{uv})$ is a connected component of $G_{\triangle,\square}$. Thus we can assume that $x = w$. The same argument shows that $\overline{uv} = \overline{yz}$. Thus the triangle is $(\overline{xy}, \overline{xz}, \overline{yz})$.

We have now proved that $\{z\} \subseteq N(x, y)$. If there is another vertex r with $r \in N(x, y)$ then $\overline{rx}$ is another neighbour of $\overline{xy}$ which again contradicts the fact that $(\overline{xy}, \overline{xz}, \overline{yz})$ is a connected component of $G_{\triangle,\square}$. This shows that $N(x, y) = \{z\}$. The other parts are proved similarly. $\square$

We now want to give a criterion for a graph G to determine that whether $G_{\triangle,\square}$ is connected or not. Prior to that we need some terminology.

Definition 3.6. Let G be a graph and $x, y \in G$. We say that (z, w) can be derived from (x, y) if there is a sequence $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ such that $(a_1, b_1) = (x, y)$ and $(a_k, b_k) = (z, w)$ and $a_{i+1}, b_{i+1} \in N'(a_i, b_i)$ for $i = 1, 2, \ldots, k-1$. The derived pairs of (x, y), denoted by $D(x, y)$, is the set of all (z, w) such that (z, w) can be derives from (x, y).

Theorem 3.7. *Let G be a graph. Then $G_{\triangle,\square}$ is disconnected if and only if $|D(x, y)| < \binom{|V|}{2}$ for some $x, y \in G$.*

Proof. Let $|D(x, y)| < \binom{|V|}{2}$ for some $x, y \in G$, then there is pairs of (z, w) such that $D(x, y)$ not included (z, w). Now let $G_{\triangle,\square}$ is connected, therefore there is a path from vertex $\overline{xy}$ to vertex $\overline{zw}$. If this path is $\overline{xy} = \overline{a_1b_1}, \overline{a_2b_2}, \ldots, \overline{a_kb_k} = \overline{zw}$, then according to the definition $a_{i+1}b_{i+1} \in N'(a_ib_i)$ for $i = 1, 2, \ldots, k-1$ consequently $(z, w) \in D(x, y)$ that is a contradiction. The same argument shows that the other part of theorem is also holds true. $\square$

Example 3.8. Let G be a graph and $N'(x, y) = \{z\}$ for some vertices $x, y, z \in G$. Then $D(x, y) = \{(x, y)\}$. Thus

$$|D(x, y)| = 1 < 3 \leqslant \binom{|V|}{2}.$$

This confirms the result of Lemma 3.3. Note that in this case $G_{\triangle,\square}$ has a connected component with $|D(x, y)|$ vertex, i.e., an isolated vertex.

Example 3.9. Let G be a graph and $N'(x, y) = \{z, w\}$ and $N'(z, w) = \{x, y\}$ for some different vertices $x, y, z, w \in G$. Then $D(x, y) =$

$\{(x,y),(z,w)\}$. Thus

$$|D(x,y)| = 2 < 6 \leqslant \binom{|V|}{2}.$$

This confirms the result of Lemma 3.4. Note that in this case $G_{\triangle,\square}$ has a connected component with $|D(x,y)|$ vertices, i.e., an edge.

Example 3.10. Let G be a graph with $|V| \geqslant 4$ and

$$N'(x,y) = N'(x,z) = N'(y,z) = \{x,y,z\},$$

for some different vertices $x,y,z \in G$. Then

$$D(x,y) = \{(x,y),(x,z),(y,z)\}.$$

Thus

$$|D(x,y)| = 3 < 6 \leqslant \binom{|V|}{2}.$$

This confirms the result of Lemma 3.5. Note that in this case $G_{\triangle,\square}$ has a connected component with $|D(x,y)|$ vertices, i.e., a triangle.

Example 3.11. Let G be a graph and

$$N'(x,y) = \{z,w,t\},\ N'(z,w) = N'(z,t) = N'(w,t) = \{x,y\},$$

for some different vertices $x,y,z,w,t \in G$. Then

$$D(x,y) = \{(x,y),(z,w),(z,t),(w,t)\}.$$

Thus

$$|D(x,y)| = 4 < 10 \leqslant \binom{|V|}{2}.$$

Note that $G_{\triangle,\square}$ has a $K_{1,3}$ component with the partition sets $\{\overline{xy}\}$ and $\{\overline{zw},\overline{zt},\overline{wt}\}$.

Let G be a graph with vertices $x_1,x_2,\ldots,x_{|V|}$. Recall that the adjacency matrix $A(G) = [a_{ij}]$ is a square $|V| \times |V|$ matrix such that a_{ij} is 1 when there is an edge from vertex x_i to vertex x_j, and 0 when there is no edge. Let $B(G) = [b_{ij}] = A(G)^2$. Then $b_{ii} = \deg(x_i)$ for $i = 1,2,\ldots,|V|$ and $b_{ij} = |N(x_i,x_j)|$ for $1 \leqslant i \neq j \leqslant |V|$. We can therefore say that $|N'(x_i,x_j)| = b_{ij} + 2a_{ij}$. We use these notations is the following theorem.

Theorem 3.12. *Let $G = \{x_1,x_2,\ldots,x_{|V|}\}$ be a graph, $A(G) = [a_{ij}]$ be its adjacency matrix and $B(G) = [b_{ij}] = A(G)^2$. Then*

$$\deg(\overline{x_ix_j}) = \binom{b_{ij}+2a_{ij}}{2} - a_{ij}$$

for each vertex $\overline{x_i x_j}$ in $G_{\triangle,\square}$. Furthermore,

$$\sum_{1\leqslant i<j\leqslant |V|} b_{ij} = \sum_{i=1}^{|V|} \binom{\deg(x_i)}{2}.$$

Theorem 3.13. *Let G be a graph. Then $G_{\triangle,\square}$ is a connected graph with $\operatorname{diam}(G_{\triangle,\square}) \leqslant 2$ if and only if for each four vertices $x, y, z, w \in G$ there are two vertices $u, v \in G$ such that $u, v \in \bigcap_{\alpha\beta\in\{x,y,z,w\}} N'(\alpha, \beta)$.*

Proof. Suppose $G_{\triangle,\square}$ is a connected graph with $\operatorname{diam}(G_{\triangle,\square}) \leqslant 2$ and let $x, y, z, w \in G$. If $\overline{xy}$ and $\overline{zw}$ are not adjacent in $G_{\triangle,\square}$ according to the assumption there is vertex $\overline{uv}$ in $G_{\triangle,\square}$ that $\overline{uv}$ is common neighbour for $\overline{xy}$ and $\overline{zw}$, therefore there are two vertices $u, v \in G$ such that $u, v \in \bigcap_{\alpha\beta\in\{x,y,z,w\}} N'(\alpha, \beta)$. But if $\overline{xy}$ and $\overline{zw}$ are adjacent, then if x and y are adjacent in G, we consider $x = u$ and $y = v$ and if x and y are not adjacent in G, then $\overline{xz}$ and $\overline{yw}$ are not adjacent in $G_{\triangle,\square}$ and with similar argument u, v are obtained. The other part of the theorem is also obviously true. □

References

1. A. Abdollahi, S. Akbari and H. R. Maimani, Non-commuting graph of a group, *J. Algebra*, **298** (2006), 468–492
2. S. Akbari, A. Mohammedian, H. Radjavi and P. Raja, On diameters of commuting graphs,*Linear Algebra Appl.*, **418** (2006), 161–176.
3. R. Barzgar, A. Erfanian and M. Farrokhi, Probability of mutually commuting two finite subsets of a finite group, *Ars. Combinatoria*, **124** (2016), 165–176.
4. C. Bates, D. Bundy, S. Hart and P. Rowley, A note on commuting graphs for symmetric groups, *Electron. J. Combin.*, **16** (2009), 1–13.
5. J. A. Bondy and J. S. R. Murty, *Graph theory with applications*, Elsevier, 1977.
6. A. Erfanian, B. Tolue and N. H. Sarmin, Some considerations on the n-th commutativity degree of finite groups, *Ars. Combinatoria*, **122** (2015), 257–267.
7. A. Erfanian, R. Rezaei, and P. Lescot, On the relative commutativity degree of a subgroup of a finite group, *Comm. Algebra*, **35** (2007), 4183–4198
8. C. Godsil and G. Royle, *Algebraic graph theory*, Springer, 2001.
9. A. Iranmanesh and A. Jafarzadeh, Characterization of finite groups by their commuting graph, *Acta Math. Acad. Paedagog. Nyhazi.(N.S)*, **23** (2007), 1–13.
10. A. R. Moghaddamfar, W. J. Shi, W. Zhou and A. R. Zokayi, On the non-commuting graph associated with a finite group, *Siberian Math. J.*, **46** (2005), 325–332.
11. D. B. West, *Introduction to group theory*, Second edition, Prentice Hall. Inc., 2001.

Ghorbanali Nasiriboroujeni
Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran.

Email: nasirighorbanali@gmail.com

Majid Mirzavaziri
Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran.
Email: mirzavaziri@gmail.com

Ahmad Erfanian
Department of Pure Mathematics and Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, Mashhad, Iran.
Email: erfanian@math.um.ac.ir

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THE $(\triangle, \square)$-EDGE GRAPH $G_{\triangle,\square}$ OF A GRAPH G

GH. NASIRIBOROUJENI, M. MIRZAVAZIRI AND A. ERFANIAN

گراف $(\triangle, \square)$-یال $G_{\triangle,\square}$ از یک گراف G

قربانعلی نصیری بروجنی، مجید میرزاوزیری و احمد عرفانیان
دانشکده علوم ریاضی، دانشگاه فردوسی مشهد، مشهد، ایران

چکیده

به هر گراف ساده مانند $G = (V, E)$، یک گراف ساده $G_{\triangle,\square}$ نظیر می‌کنیم که مجموعه رئوس آن عبارت است از $V_{\triangle,\square} = \{\{x, y\} : x, y \in V\}$ و دو راس دلخواه مانند $\{x, y\}$ و $\{z, w\}$ در این گراف مجاورند فقط و فقط وقتی که $\{x, z\}$، $\{x, w\}$، $\{y, z\}$ و $\{y, w\}$ عضو E یا متناظر با یک عضو از V باشند. گراف $G_{\triangle,\square}$، گراف $(\triangle, \square)$-یال از گراف G نامیده می‌شود. در این مقاله ارتباط بین همبندی گراف G و گراف $G_{\triangle,\square}$ بررسی می‌شود.

کلمات کلیدی: گراف $(\triangle, \square)$-یال؛ شمارش در گراف؛ ترکیبیات شمارشی.