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CONTINUOUS FUNCTIONS ON LG-SPACES

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Abstract. By an l-generalized topological space, briefly an LG-space, we mean the ordered pair (F,τ) in which F is a frame and τ is a subframe of F. This notion has been first introduced by A.R. Aliabad and A. Sheykhmiri in [LG-topology, *Bull. Iran. Math. Soc.*, **41** (1), (2015), 239-258]. In this article, we define continuous functions on LG-spaces and determine conditions under which the continuous image of a compact element of an LG-space is compact. Moreover, we introduce the concept of connectedness for LG-spaces and determine conditions under which the continuous image of a connected element of an LG-space is connected. In fact, we show that LG-spaces are models for topological spaces as well as frames are models for topologies.

1. Introduction

A complete lattice L is a lattice in which every subset has a supremum. Clearly, a complete lattice is a bounded lattice, i.e., it has the largest element 1 and the smallest element 0. A frame F is a complete lattice in which the distributive law $a \wedge (\bigvee S) = \bigvee_{s\in S}(a \wedge s)$ holds for every $a \in F$ and $S \subseteq F$. A pseudocomplement of an element a of a bounded lattice L is defined by $\max\{x \in L : \ x \wedge a = 0\}$, if it exists, and denoted by a^*. Obviously, if F is a frame, then $a^* = \bigvee\{x \in L : \ x \wedge a = 0\}$. Let F be a frame. Then a subset G of F which is closed under finite meets and arbitrary joins is called a subframe of F. Let (X,τ) be any topological space. Then it is

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clear that τ is a frame, and if $\mathcal{U} \subseteq \tau$, then $\bigvee_{U\in\mathcal{U}} U = \bigcup_{U\in\mathcal{U}} U$ and $\bigwedge_{U\in\mathcal{U}} U = int_X(\bigcap_{U\in\mathcal{U}} U)$. In fact, this example is a basic model that inspires topologists to study a frame as a pointfree topology.

In [8] and later in [9], Guo-Jun Wang constructed a model for topological spaces on a completely distributive lattice. In [1], this view was followed and generalized to topological spaces known as LG-spaces. This article, in fact, is a continuation of the paper [1]. There are two viewpoints for introducing continuous functions on this structure. In [3], continuous functions are introduced from viewpoint of the locale. In the present paper, in frame viewpoint, we define continuous functions in the context of LG-spaces with a little difference from what is defined in [9]. So, in the following, we will recall some of the material from [1] that are needed to understand the issue better. The reader is referred to [6] and [5] for more details concerning frames. Also, see [4], [7] and [2] for more information about general lattice theory and general topology, respectively

Note that a topological space (X, τ) can be considered as $(P(X), \tau)$. Clearly, τ is a subframe of $P(X)$ and via this viewpoint, the following definition is natural.

Throughout this paper, all lattices considered to be frames, unless otherwise stated explicitly.

The following definition has been proposed for the first time in [1] and for bounded pseudocomplemented distributive lattices, but in this article we will present it according to our purpose for the frames.

Definition 1.1. Let F be a frame and τ be a subframe of F. Then τ is called an l-generalized topology on F and (F, τ) (briefly, F) is called an l-generalized topological space. Every element of τ is said to be open and any element of $\tau^* = \{t^* : \ t \in \tau\}$ is said to be a closed element. Clearly, the set of closed elements is a $\wedge$-structure, since $(\bigvee_{\lambda\in\Lambda} t_\lambda)^* = \bigwedge_{\lambda\in\Lambda} t_\lambda^*$. Furthermore, if τ^* is a sublattice of F, then we say τ is an l-topology on F and (F, τ) (briefly, F) is an l-topological space; for convenience, we denote an l-generalized topological space (resp., l-topological space) by LG-space (resp., L-space). Assuming that τ is an l-generalized topology on a frame F and $a \in F$, we define $a^\circ = \bigvee\{t \in \tau : \ t \leq a\}$ and $\overline{a} = \bigwedge\{x \in \tau^* : \ a \leq x\}$. Sometimes, we use $int_\tau a$ and $cl_\tau a$ instead of a° and $\overline{a}$, respectively.

Obviously, in Definition 1.1, if each element of τ has a complement in F, then $\tau^* = \tau^c = \{t^c : \ t \in \tau\}$, and we have a structure named topoframe which is introduced and studied in [3].

Note that an LG-space need not be an L-space, see [1].

Remark 1.2. Let L be a pseudocomplemented lattice. Then the following statements hold concerning the mapping $* : L \to L$.

(i) The mapping "$*$" is decreasing and $a \leq a^{**}$ for every $a \in L$.

(ii) The mapping "$**$" is the identity mapping on L^*, i.e., $a^{***} = a^*$ for all $a \in L$ (so the mapping"$**$" on L is a closure operator).

(iii) For every $a, b \in L$, we have

$$a \wedge b = 0 \;\Leftrightarrow\; a \leq b^* \;\Leftrightarrow\; b \leq a^* \;\Leftrightarrow\; a^{**} \leq b^* \;\Leftrightarrow\; a^{**} \wedge b = 0.$$

(iv) If L is a frame and $S \subseteq L$, then $(\bigvee_{s \in S} s)^* = \bigwedge_{s \in S} s^*$.

Definition 1.3. Suppose that F is a frame and $S \subseteq F$. We denote the set of finite meets of elements of S by $Fm(S)$. Set $< S >= \{\bigvee D : \; D \subseteq Fm(S)\}$. Clearly, $< S >$ is the smallest subframe of F containing S. If (F, τ) is an LG-space and $\tau =< S >$ for some $S \subseteq F$, then S is said to be a subbase for the topology τ. A set $B \subseteq \tau$ is called a base for an LG-topology τ if for every $t \in \tau$ there exists $D \subseteq B$ such that $t = \bigvee D$. Moreover, assuming that F is a frame and $B \subseteq F$, we say B is a base for a topology on F if $1 = \bigvee B$, and for every $b_1, b_2 \in B$ there exists $D \subseteq B$ such that $b_1 \wedge b_2 = \bigvee D$.

Proposition 1.4. *Let (F, τ) be an LG-space. The following statements hold:*

(a) $0^\circ = 0$ and $1^\circ = 1$.

(b) $a^\circ \leq a$ for every $a \in F$.

(c) If $a, b \in F$ and $a \leq b$, then $a^\circ \leq b^\circ$.

(d) For each $a \in F$, $a^\circ = \bigvee\{t \in B : \; t \leq a\}$ where B is a base for τ.

(e) For each $a \in F$, $a^\circ \in \tau$.

(f) $a = a^\circ$ if and only if $a \in \tau$.

(g) For each $a \in F$, $(a^\circ)^\circ = a^\circ$.

(h) a° is the greatest element of τ that is less than or equal to a.

(i) If $a_1, ..., a_n \in F$, then $(\bigwedge_{i=1}^n a_i)^\circ = \bigwedge_{i=1}^n a_i^\circ$.

Let $\varphi : F \to F$ be a mapping that satisfying (a), (b), (c), (g) and (i). If we define $\tau = \{a \in F : \; \varphi(a) = a\}$, then τ is an LG-topology on F, and the interior operator induced by τ coincides with φ.

Proposition 1.5. *Let (F, τ) be an LG-space. The following statements hold:*

(a) $\overline{0} = 0$ and $\overline{1} = 1$.

(b) For each $a \in F$, $a \leq \overline{a}$.

(c) If $a, b \in F$ and $a \leq b$, then $\overline{a} \leq \overline{b}$.

(d) $\overline{a} \in \tau^*$ for every $a \in F$.

(e) $a = \overline{a}$ if and only if $a \in \tau^*$.

(f) $\overline{\overline{a}} = \overline{a}$ *for all* $a \in F$.

(g) $\overline{a}$ *is the smallest closed element that is greater than or equal to* a.

(h) If (F, τ) *is an* L*-space and* $a_1, ..., a_n \in F$, *then we have* $\overline{(\bigvee_{i=1}^{n} a_i)} = \bigvee_{i=1}^{n} \overline{a_i}$.

Definition 1.6. Suppose that (F, τ) is an LG-space and $a \in F$. If we take $F_a = \downarrow a$ and $\tau_a = \{t \wedge a : t \in \tau\}$, then clearly (F_a, τ_a) is an LG-space. We call (F_a, τ_a) as a subspace of (F, τ) (briefly, we say that F_a is a subspace of F).

In the following, the basic properties of subspaces in LG-spaces are given.

Proposition 1.7. *Suppose that* (F, τ) *is an LG-space and* $a \in F$. *The following statements hold.*

(a) If S *is a subbase for* τ, *then* $S_a = \{s \wedge a : \ s \in S\}$ *is a subbase for* τ_a.

(b) If B *is a base for* τ, *then* $B_a = \{t \wedge a : \ t \in B\}$ *is a base for* τ_a.

Proposition 1.8. *Suppose that* (F, τ) *is an LG-space and* $a \in F$. *Then, the following statements hold.*

(a) $\{(t \wedge a)^* \wedge a : \ t \in \tau\}$ *is the set of closed elements of* F_a. *In particular, if* $a \in F^* = \{x^* : \ x \in F\}$, *then* $\{t^* \wedge a : \ t \in \tau\}$ *is the set of closed elements of* F_a.

(b) If $x \leq a$, *then* $cl_{\tau_a} x = (int_\tau x^* \wedge a)^* \wedge a$.

(c) If $a \in F^*$ *and* $x \leq a$, *then* $cl_{\tau_a} x = cl_\tau x \wedge a$. *In particular, if* a *is a closed element in* F, *then* $cl_{\tau_a} x = cl_\tau x$.

(d) If $x \leq a$, *then* $int_\tau x \leq int_{\tau_a} x$. *The converse of this fact is not necessarily true.*

(e) If a *is an open element in* F *and* $x \leq a$, *then* $int_\tau x = int_{\tau_a} x$.

Definition 1.9. Suppose that (F, τ) is an LG-space. We say $a \in F$ is τ-compact (briefly, compact) if whenever $S \subseteq \tau$ and $a \leq \bigvee S$, then there exists a finite subset D of S such that $a \leq \bigvee D$. We can similarly define Lindelöf, countably compact, etc. Whenever 1 (i.e., the top element of F) is a compact element in (F, τ), we say (F, τ) (briefly, F) is a compact space.

Definition 1.10. Suppose that (F_i, τ_i) is an LG-space, for every $i \in I$. Clearly, $F = \prod_{i \in I} F_i$ with ordinary order is a frame. Two topologies can be defined on F as follows:

(i) $\tau = \{t = (t_i)_{i \in I} : \ t_i \in \tau_i$, and $t_i = 1$ for all except finitely many $i \in I\} \cup \{0\}$. This topology is called product topology on F. When we deal with $\prod_{i \in I} F_i$ as an LG-space, we have in view this topology.

(ii) $\tau_b = \{t = (t_i)_{i\in I} : \ \forall i \in I, \ t_i \in \tau\}$. This topology is called the box topology on F.

Clearly, if $F = \prod_{i\in I} F_i$ and π_j is the projection mapping from F to F_j, then for every $S \subseteq F$, we have $\bigvee S = (\bigvee_{s\in S} \pi_i(s))_{i\in I}$.

Proposition 1.11. *Suppose that (F_i, τ_i) is an LG-space, for every $i \in I$, $F = \prod_{i\in I} F_i$, and τ and τ_b are the product topology and box topology on F, respectively. Then the following statements hold.*

(a) $\tau^* = \{x = (x_i)_{i\in I} \in F : \ x_i \in \tau_i^*, \ and \ x_i = 0, \ for \ all \ except \ finitely \ many \ i \in I\} \cup \{1\}$.

(b) $\tau_b^* = \{x = (x_i)_{i\in I} \in F : \ x_i \in \tau_i^*, \ \forall i \in I\}$.

(c) For every $x \in F$, if I is infinite, then we have $int_\tau x \neq 0$ if and only if $x_i = 1$ for all except finitely many $i \in I$. Also, if I is finite, then we have $int_\tau x \neq 0$ if and only if there exists $i \in I$ such that $x_i^\circ \neq 0$.

(d) For every $x \in F$, if I is finite or $int_\tau x \neq 0$, then $int_\tau x = (x_i^\circ)_{i\in I}$.

(e) For every $x \in F$, if I is infinite, then we have $cl_\tau x \neq 1$ if and only if $x_i = 0$ for all except finitely many $i \in I$, and if I is finite, then we have $cl_\tau x \neq 1$ if and only if there exists $i \in I$ such that $cl_\tau x_i \neq 1$.

(f) For every $x \in F$, if I is finite or $cl_\tau x \neq 1$, then $cl_\tau x = (\overline{x_i})_{i\in I}$.

(g) If $x \in F$, then $int_{\tau_b} x = (x_i^\circ)_{i\in I}$.

(h) If $x \in F$, then $cl_{\tau_b} x = (\overline{x_i})_{i\in I}$.

2. Continuous functions on *LG*-spaces

We introduce the notion of continuity for *LG*-spaces by means of adjoint mappings as follows.

Definition 2.1. Suppose that X and Y are posets and $f : X \to Y$ and $g : Y \to X$ are order-preserving mappings. We say that f is the left adjoint of g (or g is the right adjoint of f) whenever

$$\forall x \in X, \forall y \in Y \ x \leq g(y) \ \Leftrightarrow \ f(x) \leq y.$$

We denote by f_* the right adjoint of f, if exists.

Some basic properties of adjoint mappings are given in the next remark.

Remark 2.2. Suppose that X and Y are posets. It is easy to see that if $f : X \to Y$ and $g : Y \to X$ are two order-preserving mappings, then the following statements hold.

(a) The right adjoint of f (left adjoint of g), if it exists, is unique.

(b) If f is left adjoint and X and Y are bounded, then $f(x) = 0$ if and only if $x \leq f_*(0)$. Consequently, $f(0) = 0$.

(c) If f is left adjoint and X and Y are bounded, then $f_*(y) = 1$ if and only if $f(1) \leq y$. Consequently, $f_*(1) = 1$.

(d) f is the left adjoint of g if and only if $fg \leq I_Y$ and $gf \geq I_X$.

(e) If f and g are adjoint, then $fgf = f$ and $gfg = g$.

(f) If f is left adjoint, then ff_* (f_*f) is an interior (closure) operator.

(g) If f is left adjoint, then f is one to one if and only if $f_*f(x) = x$ for every $x \in X$.

(h) If f is left adjoint, then $ff_*(y) = y$ if and only if $y \in f(X)$. So, f is onto if and only if $ff_*(y) = y$ for every $y \in Y$.

(i) If X and Y are complete lattices, then f is left adjoint (g is right adjoint) if and only if f preserves arbitrary suprema (g preserves arbitrary infima).

(j) If F_1 and F_2 are frames and $f : F_1 \to F_2$ preserves arbitrary suprema, then f is left adjoint, $f_*(y) = \bigvee\{x \in F_1 : \ f(x) \leq y\}$, the mapping f_* preserves arbitrary infima and $f(x) = \bigwedge\{y \in F_2 : \ x \leq f_*(y)\}$ for every $x \in F_1$.

Suppose that $h : X \to Y$ is a function. Then the set function $f : P(X) \to P(Y)$ defined by $f(A) = \{h(a) : \ a \in A\}$ is a left adjoint mapping and $f_*(B) = h^{-1}(B) = \{x \in X : \ h(x) \in B\}$ for every $B \in P(Y)$. It seems that this example is the basic model on which one could extend the concept of adjoint mapping. We will find out later in Proposition 2.11 that a set function $f : P(X) \to P(Y)$ is induced by a function $h : X \to Y$ if and only if f has a right adjoint mapping which is also left adjoint.
As we mentioned in part (b) of the above remark, for every left adjoint mapping f we have $f(0) = 0$. However, this is not true for the right adjoint mappings. For example, let $f : F_1 \to F_2$ be such that $f(x) = 0$ for every $x \in F_1$. Then f is a left adjoint mapping and $f_*(y) = 1$ for every $y \in F_2$. On the other hand, if f is a set function $f : P(X) \to P(Y)$ induced by $h : X \to Y$, then $f_*(B^*) = h^{-1}(B^c) = (h^{-1}(B))^c = (f_*(B))^*$ for every $B \in P(Y)$. The following proposition shows that even the inequality $f_*(y^*) \leq (f_*(y))^*$ does not hold, in general.

Proposition 2.3. *Let F_1 and F_2 be two frames. Suppose that $f : F_1 \to F_2$ is a left adjoint mapping. Then the following statements are equivalent.*

(a) For every $x \in F_1$, if $f(x) = 0$, then $x = 0$.

(b) For every $y \in F_2$, if $y \wedge f(1) = 0$, then $f_*(y) = 0$.

(c) $f_*(0) = 0$.

(d) For all $y \in F_2$, we have $f_*(y^*) \leq (f_*(y))^*$.

(e) There exists $y \in F_2$ such that $f_*(y^*) \leq (f_*(y))^*$.

Proof. (a) $\Rightarrow$ (b). Let $y \in F_2$ and $y \wedge f(1) = 0$. Thus, $ff_*(y) \leq y \wedge f(1) = 0$ and so $f_*(y) = 0$.

(b) $\Rightarrow$ (c). If we take $y = 0$, then we are done.

(c) $\Rightarrow$ (d). Let $y \in F_2$, then $y \wedge y^* = 0$ and so $0 = f_*(y \wedge y^*) = f_*(y^*) \wedge f_*(y)$. Therefore, $f_*(y^*) \leq (f_*(y))^*$.

(d) $\Rightarrow$ (e). It is clear.

(e) $\Rightarrow$ (a). Suppose that $f(x) = 0$ and $y \in F_2$ such that $f_*(y^*) \leq (f_*(y))^*$. Hence, we can write

$$0 = f_*(y^*) \wedge f_*(y) = f_*(y^* \wedge y) = f_*(0)$$
$$\Rightarrow\ x \leq f_* f(x) = f_*(0) = 0 \quad \therefore\ x = 0.$$

□

Definition 2.4. Let (F_1, τ_1) and (F_2, τ_2) be LG-spaces and $f : F_1 \to F_2$ be a left adjoint mapping. Then

(i) f is said to be weakly continuous at $a \in F_1$, if whenever $f(a) \leq t$, then $a \leq int(f_*(t))$ where $t \in \tau_2$.

(ii) f is said to be weakly continuous if it is weakly continuous at each element $a \in F_1$.

One can easily see that if $f : F_1 \to F_2$ is left adjoint, then f is weakly continuous at $a \in F_1$ if and only if for every open element $t \geq f(a)$, there is $s \in \tau_1$ such that $a \leq s \leq f_*(t)$ or equivalently, $a \leq s$ and $f(s) \leq t$.

Proposition 2.5. *A left adjoint mapping $f : F_1 \to F_2$ is weakly continuous if and only if for any $t \in \tau_2$ we have $f_*(t) \in \tau_1$.*

Proof. ($\Rightarrow$). Let $t \in \tau_2$. We show that $f_*(t) \in \tau_1$. Assuming that $A = \{x \in F_1 :\ f(x) \leq t\}$, we have $f_*(t) = \bigvee A$. Let $x \in A$. Then $f(x) \leq t$ and by the hypothesis, there exists $v_x \in \tau_1$ such that $x \leq v_x \leq f_*(t)$. Therefore, $f_*(t) = \bigvee_{x\in A} x \leq \bigvee_{x\in A} v_x \leq f_*(t)$ and so $f_*(t) = \bigvee_{x\in A} v_x \in \tau_1$.

($\Leftarrow$). It is clear. □

As an immediate consequence of Proposition 2.5 is the fact that, if X and Y are topological spaces, then a function $f : X \to Y$ is continuous if and only if, for every $A \subseteq X$ and every open subset W of Y containing $f(A)$, we have $A \subseteq int(f^{-1}(W))$.
Assume that $f : X \to Y$ is a function, $A \subseteq X$ and $f(A) \subseteq B \subseteq Y$. We denote the restriction of f from A to B by f_A^B. If $A = X$ $(B = Y)$, then, for convenience, we use f^B (f_A) instead of f_X^B (f_A^Y). Also, if X and Y are posets, then we use "a" instead of "$\downarrow a$".

Proposition 2.6. *Suppose that a left adjoint mapping $f : (F_1, \tau_1) \to (F_2, \tau_2)$ is weakly continuous and $\downarrow a$ is an LG-subspace of F_1 and $\downarrow b$ is an LG-subspace of F_2 that contains the range of f. Then $g = f_a^b$ is weakly continuous.*

Proof. Clearly, g is a left adjoint mapping. Assume that $t \wedge b \in \tau_b$ where $t \in \tau_2$ and τ_b is the LG-topology of $\downarrow b$ as a subspace of (F_2, τ_2). We show that $g_*(t \wedge b) \in \tau_a$. First, note that $\{x \wedge a : x \in F_1, f(x \wedge a) \leq t\} = \{x \wedge a : x \in F_1, f(x) \leq t\}$ and $g_*(b) = \bigvee\{x \wedge a : g(x \wedge a) \leq b\} = a$. Therefore, we can write

$$g_*(t \wedge b) = g_*(t) \wedge g_*(b) = (\bigvee\{x \wedge a : x \in F_1, g(x \wedge a) \leq t\}) \wedge a$$

$$= (\bigvee\{x \wedge a : x \in F_1, f(x \wedge a) \leq t\}) \wedge a$$

$$= (\bigvee\{x \wedge a : x \in F_1, f(x) \leq t\}) \wedge a$$

$$= (\bigvee\{x \in F_1 : f(x) \leq t\} \wedge a) \wedge a = f_*(t) \wedge a \in \tau_a.$$

□

Let (F_1, τ_1) and (F_2, τ_2) be LG-spaces, $f : F_1 \to F_2$ be a function and $\downarrow a$ be an arbitrary LG-subspace of F_1. Then, as mentioned before, we denote by f_a the restriction of f on $\downarrow a$.

Proposition 2.7. *Let (F_1, τ_1) and (F_2, τ_2) be LG-spaces, $f : F_1 \to F_2$ be a left adjoint mapping and $\bigvee_{i \in I} x_i = 1$ where $x_i \in \tau_1$ for every $i \in I$. Then f is weakly continuous if and only if f_{x_i} is weakly continuous for every $i \in I$.*

Proof. ($\Rightarrow$). It is clear by Proposition 2.6.

($\Leftarrow$). Assume that $t \in \tau_2$. As we see in the proof of Proposition 2.6, $(f_{x_i})_*(t) = f_*(t) \wedge x_i$ is open in F_1 for every $i \in I$. So, we can write

$$f_*(t) = f_*(t) \wedge (\vee_{i \in I} x_i) = \bigvee_{i \in I} (f_*(t) \wedge x_i) = \bigvee_{i \in I} (f_{x_i})_*(t) \in \tau_1.$$

□

It is well-known that two continuous functions $f, g : X \to Y$, where X and Y are topological spaces with Y to be Hausdorff, are identical if and only if $f_D = g_D$ for some dense subset D of X. Now, we generalize this fact for LG-spaces.

Proposition 2.8. *Let (F_1, τ_1) and (F_2, τ_2) be two LG-spaces, $f, g : (F_1, \tau_1) \to (F_2, \tau_2)$ be two weakly continuous and $f_*(0) = 0 = g_*(0)$ such that $f_d = g_d$ for some τ_1-dense element d of F_1 (i.e., we have $t \wedge d \neq 0$ for each $t \in \tau_1 \setminus \{0\}$). Also, suppose that for every $x \in F_1$ we have $f(x) = g(x)$ whenever $f^{-1}(\downarrow(f(x) \wedge t_1)) \cap g^{-1}(\downarrow(g(x) \wedge t_2)) \subseteq \{0\}$ for every disjoint elements $t_1, t_2 \in \tau_2$. Then, we have $f = g$.*

Proof. Assume on the contrary that $f(x) \neq g(x)$ for some $x \in F_1$. By our hypothesis, there exist $0 \neq a \in F_1$ and $t_1, t_2 \in \tau_2$ such that $f(a) \leq f(x) \wedge t_1$, $g(a) \leq g(x) \wedge t_2$ and $t_1 \wedge t_2 = 0$. Clearly, $a \leq f_*(t_1) \wedge g_*(t_2)$ which implies that $x_0 = f_*(t_1) \wedge g_*(t_2) \wedge d \neq 0$. Therefore, $f(x_0) \leq t_1$, $g(x_0) \leq t_2$ and since $t_1 \wedge t_2 = 0$, it follows that $f(x_0) \wedge g(x_0) = 0$. Since $x_0 \leq d$, it follows that $f(x_0) = g(x_0)$ and consequently, $f(x_0) = g(x_0) = 0$. On the other, since $f_*(0) = 0 = g_*(0)$ and $x_0 \neq 0$, we deduce, by Proposition 2.3, that $f(x_0) \neq 0 \neq g(x_0)$, and this is a contradiction. □

Proposition 2.9. *Suppose that $f : F_1 \to F_2$ and $g : F_2 \to F_3$ are left adjoint mappings. Then the following statements hold.*

(a) gf is a left adjoint mapping and $(gf)_* = f_* g_*$.

(b) If f is weakly continuous at $a \in F_1$ and g is weakly continuous at $f(a)$, then gf is weakly continuous at a.

(c) If f and g are weakly continuous, then gf is also weakly continuous.

Proof. (a). It is clear.

(b). Assume that t is an open element in F_3 such that $(gf)(a) \leq t$. Then, by the hypothesis, there exists an open element s in F_2 such that $f(a) \leq s \leq g_*(t)$. Also, since f is weakly continuous at a, there exists an open element r in F_1 such that $a \leq r \leq f_*(s)$. Therefore, $a \leq r \leq f_*(s) \leq f_* g_*(t) = (gf)_*(t)$. Thus, gf is weakly continuous at a.

(c). By (b), it is clear. □

Definition 2.10. Let F_1 and F_2 be two frames and $f : F_1 \to F_2$ be a left adjoint mapping. We say that f is perfect provided that for every $y \in F_2$, we have $f_*(y) = 0$ if and only if $y \wedge f(1) = 0$. Also, f is said to be semi-perfect if $f_*(0) = 0$. In addition, a left adjoint mapping f is called an RL-adjoint mapping, if f_* preserves arbitrary suprema; i.e., the right adjoint of f is a left adjoint mapping. Note that the notion of RL-adoint mappings has been first introduced in [9] as GOH.

Example. (a) Let $f : P(X) \to P(Y)$ be a set function induced by $h : X \to Y$. It is easy to see that f is a perfect RL-adjoint mapping. In addition, f is weakly continuous if and only if h is continuous.

(b) Suppose that $\mathbf{2} = \{0, 1\}$, F is a frame and $1 \neq a \in F$. Define $f : F \to \mathbf{2}$ with $f(x) = 0$ whenever $x \leq a$ and $f(x) = 1$ whenever $x \nleq a$. It is easy to see that

(i) f is left adjoint;

(ii) f is an RL-adjoint mapping if and only if $a = 0$;

(iii) if F is an LG-space, then f as a function from (F, τ) to $(\mathbf{2}, \mathbf{2})$, is weakly continuous if and only if a is an open element in F.

Clearly, if $f : F_1 \to F_2$ is an RL-adjoint mapping, then since f_* preserves arbitrary suprema, it follows that $f_*(0) = 0$ and hence f is semi-perfect.

Proposition 2.11. *A set function $f : P(X) \to P(Y)$ is induced by a function $h : X \to Y$ if and only if f is an RL-adjoint mapping.*

Proof. Assume that $f : P(X) \to P(Y)$ is an RL-adjoint mapping. It suffices to show that, for every $x \in X$ there exists a point $y \in Y$ such that $f(\{x\}) = \{y\}$. To see this, suppose that $x \in X$ and $f(\{x\}) = B$ for some $B \in P(Y)$. By the hypothesis, we can write

$$\{x\} \subseteq f_* f(\{x\}) = f_*(B) = f_*(\cup_{y \in B}\{y\}) = \cup_{y \in B} f_*(\{y\})$$
$$\Rightarrow\ \exists y \in B\ x \in f_*(\{y\})\ \Rightarrow\ \{x\} \subseteq f_*(\{y\})\ \Rightarrow\ f(\{x\}) \subseteq \{y\}.$$

On the other hand, since f is an RL-adjoint mapping, it is semi-perfect. Therefore, by Proposition 2.3, $f(\{x\}) \neq \varnothing$ and so $f(\{x\}) = \{y\}$. $\square$

Definition 2.12. Let (F_1, τ_1) and (F_2, τ_2) be LG-spaces and $f : F_1 \to F_2$ be a left adjoint mapping. Then f is said to be continuous if it is weakly continuous and RL-adjoint.

Clearly, propositions 2.5, 2.6, 2.7 and 2.8 are also true for continuous functions.
Let $f : F_1 \to F_2$ be a continuous function. It is a natural question whether $f_*(y)$ is closed in F_1 for every closed element $y \in F_2$. We need the following lemma, which is probably well-known, to answer this question and some others.

Lemma 2.13. *Assume that F_1 and F_2 are frames, $h : F_1 \to F_2$ and $f : F_1 \times F_1 \to F_2 \times F_2$ with $f(a, b) = (h(a), h(b))$. Then the following statements hold.*

(i) f is a frame homomorphism if and only if h is so.

(ii) f is a left adjoint function if and only if h is so, and in this case $f_*(c, d) = (h_*(c), h_*(d))$ for every $(c, d) \in F_2 \times F_2$.

(iii) If F_1 is a finite chain, then f is a frame homomorphism if and only if h is a $\{0, 1\}$-order homomorphism (i.e., h is an order-preserving mapping such that $h(0) = 0$ and $h(1) = 1$).

The next example shows that an RL-adjoint mappings need not be perfect, in general. Also, it shows that the continuous preimage of a closed element is not necessarily closed.

Example 2.14. (a) Let $F_1 = \{0, a, 1\}$ be a chain, $h : F_1 \to F_1$ with $h(0) = 0$ and $h(a) = h(1) = 1$. Define $F = F_1 \times F_1$ and $f : F \to F$ with

$f(a,b) = (h(a), h(b))$. By Lemma 2.13, f is an RL-adjoint mapping. Again from Lemma 2.13, it follows that $f_*(a,a) = (h_*(a), h_*(a)) = (0,0) = 0_F$. Therefore, f is not perfect.

(b) Using the example of part (a) and Lemma 2.13, if we put $\tau = F$, then it follows that $g = f_*$ is a continuous function (since $f_*(a,b) = (h_*(a), h_*(b))$). Also, clearly, $g_*(0_F) = (h_{**}(0), h_{**}(0)) = (a,a)$ and (a,a) is not a closed element in F.

Let F_i be a frame for every $i \in I$ and $F = \prod_{i\in I} F_i$. Suppose that $j \in I$ and $x \in F$ are such that $x_i = 1$ (resp., $x_i = 0$) for every $i \neq j$, then for convenience, some times, we denote x by $\widetilde{x_j}$ (resp., $\widehat{x_j}$). Clearly, if $\pi_j : F \to F_j$ is the projection mapping, then we have $(\pi_j)_*(x_j) = \widetilde{x_j}$ for every $x_j \in F_j$.

Proposition 2.15. *Suppose that (F_i, τ_i) is an LG-space for every $i \in I$ and $\prod_{i\in I} F_i$ equipped with the product topology (resp., the box topology). For every $j \in I$, consider the projection mapping $\pi_j : \prod_{i\in I} F_i \to F_j$. Then the following statements hold.*

(a) π_j is an open mapping (i.e., $\pi_j(t) \in \tau_j$ for every open element t in $\prod_{i\in I} F_i$).

(b) π_j is a closed mapping (i.e., $\pi_j(t^*) \in \tau_j^*$ for every closed element t^* in $\prod_{i\in I} F_i$).

(c) π_j is weakly continuous.

Proof. The proof is routine. □

Note that, in Proposition 2.15, π_j distributes over any arbitrary join of nonempty family. To see this, suppose that $x_\alpha \in F_j$ for every $\alpha \in A$. Then we can write

$$(\pi_j)_*(\vee_{\alpha\in A} x_\alpha) = \widetilde{\vee_{\alpha\in A} x_\alpha} = \vee_{\alpha\in A} \widetilde{x_\alpha} = \vee_{\alpha\in A} (\pi_j)_*(x_\alpha).$$

However, if $|I| \geq 2$, then $(\pi_j)_*(0) = \widetilde{0} \neq 0$ and so it does not preserve arbitrary suprema. Therefore, if $|I| \geq 2$, then π_j is not an RL-adjoint mapping.

Proposition 2.16. *Suppose that (F_1, τ_1) and (F_2, τ_2) are LG-spaces and $f : F_1 \to F_2$ is a left adjoint mapping. Also, suppose that B and S are base and subbase for LG-space F_2, respectively. Then the following statements hold.*

(a) If $Fm(S) = \tau_2$, then f is weakly continuous if and only if $f_*(s) \in \tau_1$ for every $s \in S$.

(b) If f is an RL-adjoint mapping, then the following statements are equivalent.

(i) f is continuous.

(ii) $f_*(b)$ is open in F_1 for every $b \in B$.

(iii) $f_*(s)$ *is open in* F_1 *for every* $s \in S$.

Proof. (a). Since f_* preserves finite meets, it is clear.

(b). Since f_* preserves arbitrary suprema and infima, it is clear. □

Proposition 2.17. *Suppose that* (F_i, τ_i) *for every* $i \in I$, *and* (F, τ) *are LG-spaces,* $\prod_{i \in I} F_i$ *equipped by product topology and* $f : (F, \tau) \to \prod_{i \in I} F_i$. *Then* f *is weakly continuous if and only if* $\pi_j f$ *is so for every* $j \in I$.

Proof. ($\Rightarrow$) By Propositions 2.9 and 2.15, it is clear.

($\Leftarrow$) First, we show that f is a left adjoint mapping. To see this, we prove that f preserves arbitrary suprema. Since π_j and $\pi_j f$ are left adjoint mappings, we can write

$$\begin{aligned} f(\vee_{\lambda \in \Lambda} x_\lambda) &= (\pi_j f(\vee_{\lambda \in \Lambda} x_\lambda))_{j \in I} = (\vee_{\lambda \in \Lambda} \pi_j f(x_\lambda))_{j \in I} \\ &= (\pi_j(\vee_{\lambda \in \Lambda} f(x_\lambda))_{j \in I} = \vee_{\lambda \in \Lambda} f(x_\lambda). \end{aligned}$$

Now, by Proposition 2.16, it is enough to show that $f_*((\pi_j)_*(t_j))$ is open in F, where $j \in I$ and $t_j \in \tau_j$. This is easy, since $\pi_j f$ is weakly continuous and $f_*((\pi_j)_*(t_j)) = (\pi_j f)_*(t_j)$. □

Note that, by the proof of Proposition 2.17, f is a left adjoint mapping if and only if $\pi_j f$ is so for every $j \in I$.

Corollary 2.18. *Let* L *and* F_i $(i \in I)$ *be LG-spaces,* $F = \prod_{i \in I} F_i$ *and* f_i *be a mapping from* L *to* F_i *for each* $i \in I$. *Define* $f : L \to F$ *with* $f(x) = (f_i(x))_{i \in I}$. *Then the following statements hold.*

(a) f *is a left adjoint mapping if and only if* f_i *is such for every* $i \in I$. *Also,* f *is weakly continuous if and only if* f_i *is such for every* $i \in I$.

(b) If f_* *exists, then for every* $y = (y_i)_{i \in I} \in F$ *we have* $f_*(y) = \bigwedge_{i \in I} f_{i_*}(y_i)$.

(c) If there exists $j \in I$ *such that* f_j *is semi-perfect, then* f *is also semi-perfect.*

Proof. (a). Since $\pi_i f = f_i$ for every $i \in I$, it is clear, by Proposition 2.17.

(b). Define $g : F \to L$ with $g(y) = \bigwedge_{i \in I} f_{i*}(y_i)$. For every $x \in L$ and every $y = (y_i)_{i \in I} \in F$, we can write

$$gf(x) = \bigwedge_{i \in I} f_{i_*}(f_i(x)) \geq \bigwedge_{i \in I} x = x,$$

$$\begin{aligned} fg(y) = f(\wedge_{i \in I} f_{i_*}(y_i)) &= \Big(f_j(\wedge_{i \in I} f_{i_*}(y_i))\Big)_{j \in I} \\ &\leq \Big(f_j f_{j_*}(y_j)\Big)_{j \in I} \leq (y_j)_{j \in I} = y. \end{aligned}$$

Therefore, $g = f_*$.

(c). By part (b), it is clear. $\square$

Applying Corollary 2.18, we can find some useful examples as follows. (i) Suppose that $L = M_2 = \{0, \alpha, \beta, 1\}$ and $F_1 = F_2 = \{0, 1\}$. Define $f_1 : L \to F_1$ with $f_1(0) = f_1(\alpha) = 0$, $f_1(\beta) = f_1(1) = 1$ and $f_2 : L \to F_2$ with $f_2(0) = f_2(\beta) = 0$, $f_2(\alpha) = f_2(1) = 1$. If we put $f = (f_1, f_2)$, then, Clearly, f_1 and f_2 are not even semi-perfect whereas f is perfect. To see this, note that $f_{1_*}(0) = \alpha$, $f_{2_*}(0) = \beta$, $f_{1_*}(1) = f_{2_*}(1) = 1$. Thus, by Corollary 2.18, $0 = f_*((c, d)) = f_{1_*}(c) \wedge f_{2_*}(d)$ if and only if $f_{1_*}(c) = \alpha$ and $f_{2_*}(d) = \beta$, if and only if $(c, d) = (0, 0) = 0_{F \times F}$.

(ii) Define $g : [0, 1] \to [0, 1]$ with $g(x) = x^{1/n}$. Clearly, g is an order-isomorphism and so g is a perfect RL-adjoint mapping. Now, we show that $f = (g, g)$ is not RL-adjoint. Take $t \in [0, 1)$, then by Corollary 2.18, we have $f_*((1, t)) \vee f_*((t, 1)) = (g_*(1) \wedge g_*(t)) \vee (g_*(t) \wedge g_*(1)) = t^n \vee t^n \neq 1 = f_*((1, 1)) = f_*((1, t) \vee (t, 1))$. Therefore, f is not an LR-adjoint mapping.

Definition 2.19. Let (F_1, τ_1) and (F_2, τ_2) be LG-spaces and $f : F_1 \to F_2$ be a left adjoint mapping. Then f is said to be a homeomorphism if it is one-to-one, onto, continuous and, in addition, f_* is continuous.

The next proposition gives some equivalent conditions for a mapping between LG-spaces to be a homeomorphism which has a straightforward proof.

Proposition 2.20. *Suppose that (F_1, τ_1) and (F_2, τ_2) are LG-spaces, $f : F_1 \to F_2$ is a one-to-one and onto left adjoint mapping. Then the following statements are equivalent.*

(a) f is homeomorphism.

(b) $t \in \tau_2$ if and only if $f_*(t) \in \tau_1$.

(c) f is continuous and open (i.e., $f(t) \in \tau_2$ for every $t \in \tau_1$).

(d) f and f_* are weakly continuous.

Proposition 2.21. *Let $f : (F_1, \tau_1) \to (F_2, \tau_2)$ be a left adjoint mapping. Then $f_*(k)$ is closed in F_1 for each closed element k in F_2 if and only if $f(cl_{F_1} a) \leq cl_{F_2} f(a)$ for each $a \in F_1$.*

Proof. $\Rightarrow$) For $a \in F_1$, we can write

$$f(a) \leq cl_{F_2} f(a) \;\Rightarrow\; a \leq f_*(f(a)) \leq f_*(cl_{F_2} f(a)) \in \tau_1^*$$
$$\Rightarrow\; cl_{F_1} a \leq f_*(cl_{F_2} f(a)) \;\Rightarrow\; f(cl_{F_1} a) \leq cl_{F_2} f(a).$$

$\Leftarrow$) Suppose that $k \in \tau_2^*$ and $a = f_*(k)$, then we can write

$$f(cl_{F_1} a) \leq cl_{F_2} f(a) \leq k \;\Rightarrow\; cl_{F_1} a \leq f_*(k) = a$$

$$\Rightarrow \; cl_{F_1} a = a \; \Rightarrow \; f_*(k) = cl_{F_1} a \in \tau_1^*.$$

□

We need the following lemma to show that a continuous image of a compact element is compact.

Lemma 2.22. *Let F_1 and F_2 be frames and $f : F_1 \to F_2$ be a left adjoint mapping. Then f is an RL-adjoint mapping if and only if for every $a \in F_1$ and every $R \subseteq F_2$, we have*

$$f(a) \leq \bigvee_{r \in R} r \; \Leftrightarrow \; a \leq \bigvee_{r \in R} f_*(r).$$

Proof. $\Rightarrow$) Assume that $f(a) \leq \bigvee_{r \in R} r$ where $R \subseteq F_2$. By the hypothesis, f_* preserves arbitrary suprema and so we can write

$$a \leq f_* f(a) \leq f_*(\vee_{r \in R} r) = \vee_{r \in R} f_*(r).$$

For the reverse inequality, suppose that $a \leq \vee_{r \in R} f_*(r)$. Since f is left adjoint, by previous facts, f preserves arbitrary suprema, and so we can write

$$a \leq \vee_{r \in R} f_*(r) \quad \Rightarrow \quad f(a) \leq f(\vee_{r \in R} f_*(r)) = \vee_{r \in R} f f_*(r) \leq \vee_{r \in R} r.$$

$\Leftarrow$) Assume that $y_i \in F_2$ for every $i \in I$. Clearly, $\bigvee_{i \in I} f_*(y_i) \leq f_*(\bigvee_{i \in I} y_i)$. On the other hand, $f f_*(\bigvee_{i \in I} y_i) \leq \bigvee_{i \in I} y_i$, and thus, it follows from (b) that $f_*(\bigvee_{i \in I} y_i) \leq \bigvee_{i \in I} f_*(y_i)$. □

Proposition 2.23. *Let (F_1, τ_1) and (F_2, τ_2) be LG-spaces, $f : F_1 \to F_2$ be continuous and a be a compact element in F_1. Then $f(a)$ is a compact element in F_2.*

Proof. Assume that $f(a) \leq \bigvee S$ where $S \subseteq \tau_2$. By Lemma 2.22, we have $a \leq \bigvee_{s \in S} f_*(s)$. Since a is compact, it follows that there exists a finite subset $F \subseteq S$ such that $a \leq \bigvee_{s \in F} f_*(s)$, and again it follows from Lemma 2.22 that $f(a) \leq \bigvee_{s \in F} s$ □

Clearly, Proposition 2.23 also holds whenever a is countably compact, Lindelöf or other kinds of compactness.

3. Continuity and connectedness

In this section, we are going to find the relations between continuity and connectedness. To this aim, we need some definitions and facts.

Definition 3.1. Let (F, τ) be an LG-space. We denote by τ^c the set $\{x \in F : \; \exists t \in \tau, \; x = t^c\}$, where t^c is the complement of t in F (if it exists). An element $a \in F$ is called nonconnected if there exist $0 \neq r_a, s_a \in \tau_a$ such that $r_a \wedge s_a = 0$ and $r_a \vee s_a = a$, otherwise, we say that a is connected. If 1 is a nonconnected (resp., connected) element, then, some times, we say that F is nonconnected (resp., connected).

It follows from Definition 3.1 that if F is a frame and $a \leq b \in F$, then a is connected as a point of F if and only if it is connected as a point of $\downarrow b$.

Remark 3.2. Let (F, τ) be an LG-space. Inspired by Definition 3.1, we can give the following two definitions.

(a) $a \in F$ is said to be relatively connected in F if whenever $t_1, t_2 \in \tau$, $t_1 \wedge t_2 = 0$ and $a \leq t_1 \vee t_2$, then $a \leq t_1$ or $a \leq t_2$.

(b) $a \in F$ is said to be weakly relatively connected in F if whenever t is complemented in τ, then $a \leq t$ or $a \leq t^c$.

It is easy to see that if $a \in F$ is connected, then a is relatively connected in F, and also if $a \in F$ is relatively connected in F, then a is weakly relatively connected in F. The converses of these two facts are not necessarily true. For example:

(i) Suppose that X is an infinite set with cofinite topology and $a, b \in X$ are two distinct points. Then $A = \{a, b\}$ is not connected whereas it is relatively connected in $P(X)$.

(ii) Suppose that (X, τ) is a connected space and $U, V \in \tau$ are two nonempty disjoint open sets. Then $A = U \cup V$ is not relatively connected in $P(X)$ whereas A is weakly relatively connected in $P(X)$.

In the sequel, consider the lattice $M_2 = \{0, \alpha, \beta, 1\}$. As we will see in the following proposition, the lattice M_2 has an important role in the connectedness of LG-spaces, as well as the role of discrete space $\{0, 1\}$ in the realm of the connectedness of topological spaces.

Proposition 3.3. *Let (F, τ) be an LG-space. The following statements are equivalent.*

(a) F is connected.

(b) For every continuous $f : (F, \tau) \to (M_2, M_2)$ we have $\{\alpha, \beta\} \nsubseteq f(F)$.

(c) $\tau \cap \tau^c = \{0, 1\}$.

Proof. (a) $\Rightarrow$ (b). Suppose that $f : (F, \tau) \to (M_2, M_2)$ is continuous and $\alpha \in f(F)$. We show that $\beta \notin f(F)$. Taking $r = f_*(\alpha)$ and $s = f_*(\beta)$, it follows that

$$r, s \in \tau \ , \ r \wedge s = f_*(\alpha) \wedge f_*(\beta) = f_*(\alpha \wedge \beta) = f_*(0) = 0,$$
$$r \vee s = f_*(\alpha) \vee f_*(\beta) = f_*(\alpha \vee \beta) = f_*(1) = 1.$$

Since $\alpha \in f(F)$, it follows that $r \neq 0$. By connectedness of F, we conclude that $0 = s = f_*(\beta) = \bigvee \{x \in F : \ f(x) \leq \beta\}$ and consequently, $\beta \notin f(F)$.

(b) $\Rightarrow$ (c). On the contrary, suppose that there exists $r \in \tau \cap \tau^c \setminus \{0,1\}$. Now, define $f : F \to M_2$ with $f(0) = 0$, $f(x) = \alpha$ whenever $0 \neq x \leq r$, $f(x) = \beta$ whenever $0 \neq x \leq r^c$ and $f(x) = 1$ whenever $x \wedge r \neq 0$ and $x \wedge r^c \neq 0$. To see that f is a left adjoint mapping, it is enough to show that f preserves arbitrary suprema. Suppose $x_i \in F$ for every $i \in I$. We will have four cases:

(i) $\bigvee_{i\in I} x_i = 0$;
(ii) $0 \neq \bigvee_{i\in I} x_i \leq r$;
(iii) $0 \neq \bigvee_{i\in I} x_i \leq r^c$;
(iv) $(\bigvee_{i\in I} x_i) \wedge r \neq 0$ and $(\bigvee_{i\in I} x_i) \wedge r^c \neq 0$.

It is easy to see that in any case, we have $f(\bigvee_{i\in I} x_i) = \bigvee_{i\in I} f(x_i)$. It remains to prove that f is continuous. This is easy, by the fact that $f_*(\alpha) = r$ and $f_*(\beta) = r^c$.

(c) $\Rightarrow$ (a). It is clear. $\square$

We know that if X and Y are topological spaces, X is connected and $h : X \to Y$ is a weakly continuous function (in point set topology weakly continuous and continuous are equivalent), then $h(X)$ is also connected. This fact is not necessarily true in LG-spaces. For example, suppose that (F_1, τ_1) is an LG-space in which 1 is a connected element and (F_2, τ_2) is an LG-space in which 1 is a disconnected element. Define $f : (F_1, \tau_1) \to (F_2, \tau_2)$ with $f(0) = 0$ and $f(x) = 1$ for every $x \neq 0$. Clearly, f is a weakly continuous function and semi-perfect while 1 is connected and $f(1)$ is disconnected.

Definition 3.4. Let X be a partially ordered set and $D, E \subseteq X$. We say D cuts E if for every $0 \neq e \in E$ there exists $0 \neq d \in D$ such that $d \leq e$.

Proposition 3.5. *Suppose that (F_1, τ_1) and (F_2, τ_2) are LG-spaces and $f : (F_1, \tau_1) \to (F_2, \tau_2)$ is a continuous function. If 1 is connected in F_1 and $f(F_1)$ cuts ${\downarrow} f(1)$, then $f(1)$ is connected.*

Proof. Let $g : {\downarrow} f(1) \to M_2$ be continuous. Clearly, $gf : F_1 \to M_2$ is continuous. Thus, $\{\alpha, \beta\} \nsubseteq (gf)(F_1)$. Without loss of generality, suppose that $\beta \notin (gf)(F_1)$. It is enough to show that $\beta \notin g({\downarrow} f(1))$. On the contrary, assume that $\beta = g(y)$ for some $y \in {\downarrow} f(1)$. By the hypothesis, there exists $x \in F_1$ such that $0 \neq f(x) \leq y$. Thus, $0 \neq g(f(x)) \leq g(y) = \beta$. Therefore, $\beta = gf(x)$, consequently, $\beta \in (gf)(F_1)$ and this is a contradiction. $\square$

Lemma 3.6. *Let F_1 and F_2 be two frames and $f : F_1 \to F_2$ be a left adjoint function. Then f is semi-perfect and $f(F_1)$ cuts ${\downarrow} f(1)$ if and only if f is perfect.*

Proof. $\Rightarrow$) By Proposition 2.3, it is enough to show that from $y \wedge f(1) \neq 0$, it follows that $f_*(y) \neq 0$. To see this, by the hypothesis there exists $0 \neq x \in F_1$ such that $0 \neq f(x) \leq y \wedge f(1)$. Therefore, $0 \neq x \leq f_*f(x) \leq f_*(y \wedge f(1)) = f_*(y) \wedge 1 = f_*(y)$ and so $f_*(y) \neq 0$.

$\Leftarrow$) It suffices to show that $f(F_1)$ cuts $\downarrow f(1)$. Assume that $0 \neq y \leq f(1)$. By the hypothesis, $0 \neq f_*(y)$ and so $0 \neq ff_*(y) \leq y$. Now, taking $x = f_*(y)$, we are done. □

By Proposition 3.5 and Lemma 3.6, we have the following result.

Proposition 3.7. *Let (F_1, τ_1) and (F_2, τ_2) be two LG-spaces, $a \in F_1$ be a connected element and $f : F_1 \to F_2$ be a function such that f_a is a perfect continuous function. Then $f(a)$ is also a connected element.*

Here is an open question. Assuming that (F_1, τ_1) and (F_2, τ_2) are two *LG*-spaces, $a \in F_1$ is a connected element and $f : F_1 \to F_2$ is a perfect continuous function, can we conclude that $f(a)$ is a connected element?

Proposition 3.8. *Let a be a connected element of F_1 and $a \leq b \leq \bar{a}$. Then b is also a connected element.*

Proof. Suppose that $r_b = r \wedge b$ and $s_b = s \wedge b$ are two disjoint elements of τ_b such that $b = r_b \vee s_b$. Clearly, if we take $r_a = r \wedge a$ and $s_a = s \wedge a$, then r_a and s_a are disjoint elements of τ_a and $r_a \vee s_a = a$. Thus, $r_a = 0$ or $s_a = 0$, say $r_a = 0$. Hence, $a \leq r^*$ and so $b \leq \bar{a} \leq r^*$. Therefore, $r_b = r \wedge b = 0$. □

Lemma 3.9. *Let F be an LG-space and $f : F \to M_2$ be a continuous function. Then the following statements are equivalent.*

(a) $\{\alpha, \beta\} \subseteq f(F)$.

(b) f *is onto.*

(c) $1 \in f(F)$.

Proof. (a) $\Rightarrow$ (b). It suffices to show that $1 \in f(F)$. Assume that $f(a) = \alpha$ and $f(b) = \beta$. Then, clearly, $f(a \vee b) = f(a) \vee f(b) = \alpha \vee \beta = 1$.

(b) $\Rightarrow$ (c). It is clear.

(c) $\Rightarrow$ (a). Clearly, $ff_*(\alpha) \leq \alpha$ and $ff_*(\beta) \leq \beta$. Therefore, since f is continuous, it follows that

$$1 = f(1_F) = ff_*(1) = ff_*(\alpha \vee \beta) = ff_*(\alpha) \vee ff_*(\beta).$$

Hence, we conclude that $ff_*(\alpha) = \alpha$ and $ff_*(\beta) = \beta$ and so $\{\alpha, \beta\} \subseteq f(F)$. □

Proposition 3.10. *Let F be a frame and $x_i \in F$ be connected for every $i \in I$ and $0 \neq a = \bigwedge_{i\in I} x_i$. Then $x = \bigvee_{i\in I} x_i$ is also a connected element of F.*

Proof. Suppose that $f : \downarrow x \to M_2$ be a continuous mapping. Clearly, f_{x_i} is a continuous function from $\downarrow x_i$ to M_2 for every $i \in I$. Therefore, by Proposition 3.3 and Lemma 3.9, for every $i \in I$ we have $f(\downarrow x_i) = \{0, \alpha\}$ or $f(\downarrow x_i) = \{0, \beta\}$. On the other hand, f is semi-perfect and so by Proposition 2.3, $f(a) \neq 0$. Since $a \in \bigcap_{i\in I} \downarrow x_i$, it follows that $f(a) = \alpha$ or $f(a) = \beta$. say $f(a) = \alpha$. Then, obviously, $f(\downarrow x_i) = \{0, \alpha\}$ for every $i \in I$. Now, suppose that $c \in \downarrow x$, then we can write

$$c \leq x = \bigvee_{i\in I} x_i \;\Rightarrow\; c = \bigvee_{i\in I} (c \wedge x_i)$$

$$\Rightarrow\; f(c) = f(\vee_{i\in I}(c \wedge x_i)) = \vee_{i\in I} f(c \wedge x_i) \in \{0, \alpha\}.$$

Therefore, $f(\downarrow x) \subseteq \{0, \alpha\}$. Hence, x is connected. $\square$

By Propositions 3.8 and 3.10, the following corollary is immediate.

Corollary 3.11. *Let F be a frame and $x \in F$. If we take $C(x) = \{c \in F :\ c$ is connected and $x \leq c\}$ and $c_x = \bigvee C(x)$, then c_x is closed.*

Proposition 3.12. *Let F be an LG-space and R be a relation on $F \setminus \{0\}$ such as follows. $a\,R\,b$ whenever $a = b$ or $c_a = c_b \neq 0$. Then we have the following statements.*

(a) R is an equivalence relation on $F \setminus \{0\}$.

(b) $c_x \neq 0$ if and only if $C(x)$ contains at least one non zero connected element.

(c) If we denote by $[x]$ the equivalence class of x with respect to the relation R, then $C(x) \subseteq [x]$ for every $x \in F \setminus \{0\}$.

(d) $c_x = 0$ or $\bigvee [x] = c_x \in C(x)$ for every $x \in F \setminus \{0\}$.

(e) $c_a \wedge c_b \neq 0$ if and only if $[a] = [b]$ and each of classes $[a]$ and $[b]$ contains at least one non zero connected element.

(f) Let X be the set of all connected elements of $F \setminus \{0\}$. Then the mapping $x \to c_x$ is a closure operator on X.

Proof. (a). It is clear.

(b). It is obvious, by the definition of c_a.

(c). Assume that $x \in F \setminus \{0\}$ and $a \in C(x)$. Without loss of generality, suppose that $C(x) \neq \varnothing$. We show $c_x = c_a$. Clearly, $C(a) \subseteq C(x)$, so it is enough to prove that $C(a)$ is cofinal with respect to $C(x)$. To see this, let $y \in C(x)$, then by Proposition 3.10, $x \leq y \leq a \vee y \in C(a)$ and we are done.

(d). Assume that $x \in F \setminus \{0\}$ and $c_x \neq 0$. Clearly, by part (c), $C(x) \subseteq [x]$, so it is enough to prove that $C(x)$ is cofinal with respect to $[x]$. To see this, let $y \in [x]$, then $y \leq c_y = c_x \in C(x)$ and we are done.

(e $\Rightarrow$). By the hypothesis, $c_a \neq 0$ and $c_b \neq 0$. Therefore, $\varnothing \neq C(a) \subseteq [a]$ and $\varnothing \neq C(b) \subseteq [b]$. Now, it suffices to prove that $c_a = c_b$. Since c_a and c_b are connected, by Proposition 3.10, $c_a \vee c_b$ is a connected element greater than or equal to a and b and so $c_a = c_a \vee c_b = c_b$. Thus, $[a] = [b]$.

(e $\Leftarrow$). Since $[a]$ and $[b]$ contain at least one non zero connected element, clearly, $c_a \neq 0 \neq c_b$ and so by part (d), $c_a = \bigvee[a] = \bigvee[b] = c_b$. Hence, $c_a \wedge c_b = c_a \neq 0$.

(f). Since $x \in X$ is connected, clearly the mapping $x \to c_x$ is well defined on X. By the previous parts, the remainder of proof is clear (in fact, if $a, b \in X$ and $a \leq b$, then $[a] = [b]$). □

We conclude the paper by generalizing Proposition 3.10.

Proposition 3.13. *Let F be an LG-space, Γ be an ordinal and $\{a_\lambda\}_{\lambda<\Gamma}$ be a family of connected elements of F such that for every $0 < \lambda_1 < \Gamma$ there exists $\lambda_0 < \lambda_1$ with $a_{\lambda_0} \wedge a_{\lambda_1} \neq 0$. Then $x = \bigvee_{\lambda<\Gamma} a_\lambda$ is connected.*

Proof. Suppose that $f : \downarrow x \to M_2$ is a continuous function. Clearly, for every $\lambda < \Gamma$, f_{a_λ} is a continuous function and so, by Proposition 3.3, is not onto. Without loss of generality, assume that $f_0(\downarrow a_0) \subseteq \{0, \alpha\}$. We show, by transfinite induction, that $f_{a_\lambda}(\downarrow a_\lambda) \subseteq \{0, \alpha\}$ for every $\lambda < \Gamma$. Clearly, this is true for $\lambda = 0$. Now, suppose that $\gamma < \Gamma$ and this claim is true for every $\lambda < \gamma$, then we show that this is true for $\lambda = \gamma$. By the hypothesis, there exists $\lambda_0 < \gamma$ such that $x' = a_{\lambda_0} \wedge a_\gamma \neq 0$. Since $f_{\lambda_0}(\downarrow a_{\lambda_0}) \subseteq \{0, \alpha\}$ and f is semi-perfect, it follows that $f(x') \neq 0$ and so $f(x') = \alpha$. Now, because of $x' \in \downarrow x_\gamma$ and the fact that f_γ is not onto, it turns out that $f_\gamma(\downarrow a_\gamma) \subseteq \{0, \alpha\}$. Now, suppose that $c \in \downarrow x$, then $f(c) \leq f(x) = f(\bigvee_{\lambda<\Gamma} a_\lambda) = \bigvee_{\lambda<\Gamma} f(a_\lambda) \subseteq \{0, \alpha\}$. Therefore, $f(\downarrow x) \subseteq \{0, \alpha\}$. □

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CONTINUOUS FUNCTIONS ON LG-SPACES

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توابع پیوسته روی LG- فضاها

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فرض کنیم که F یک چارچوب و τ یک زیرچارچوب آن باشد. در این صورت (F,τ) را یک l-فضای توپولوژی تعمیم‌یافته (مختصراً LG- فضا) می‌نامیم. این مفهوم اول بار در مرجع [۱] تعریف شد و خواص کلی آن مورد مطالعه قرار گرفت. در ادامه این موضوع، در این مقاله توابع پیوسته روی یک LG-فضا معرفی و مورد بررسی قرار می‌گیرند. شرایطی دست‌یافتنی مطرح خواهد شد که تصویر یک عضو فشرده در یک LG-فضا تحت یک تابع پیوسته، فشرده است. به‌علاوه، همبندی در LG-فضاها تعریف شده و مجدداً شرایطی دست‌یافتنی معرفی می‌شود که تصویر یک عضو همبند در یک LG-فضا تحت یک تابع پیوسته، همبند است. در واقع با نتایج به‌دست آمده، به خوبی نشان داده می‌شود که همان‌گونه که چارچوب‌ها تعمیم مناسبی برای توپولوژی‌اند، LG-فضاها نیز تعمیم مناسبی برای فضاهای توپولوژی‌اند.

کلمات کلیدی: چارچوب، LG-فضا، عنصر فشرده، عنصر همبند، نگاشت پیوسته.