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THE ANNIHILATOR GRAPH FOR MODULES OVER COMMUTATIVE RINGS

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ABSTRACT. Let R be a commutative ring and M be an R-module. The annihilator graph of M, denoted by $AG(M)$ is a simple undirected graph associated to M whose the set of vertices is $Z_R(M) \setminus \text{Ann}_R(M)$ and two distinct vertices x and y are adjacent if and only if $\text{Ann}_M(xy) \neq \text{Ann}_M(x) \cup \text{Ann}_M(y)$. In this paper, we study the diameter and the girth of $AG(M)$ and we characterize all modules whose annihilator graph is complete. Furthermore, we look for the relationship between the annihilator graph of M and its zero-divisor graph.

1. Introduction

Let R be a commutative ring. The zero-divisor graph of R, denoted by $\Gamma(R)$ is a simple undirected graph whose vertices are the nonzero zero-divisors of R and two distinct vertices x and y are adjacent if and only if $xy = 0$, see [1, 2, 6]. The concept of the zero-divisor graph of a ring, has been generalized for modules in many papers, see [7, 9]. Variations of the zero-divisor graph are created by changing the vertex set, the edge condition, or both. The annihilator graph of R introduced in [5] and studied in some literatures, see [8, 10, 14]. It is a variation of the zero-divisor graph that changes the edge condition. This graph,

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denoted by $AG(R)$ is a graph whose vertices are the nonzero zero-divisors of R and two distinct vertices x and y are adjacent if and only if $\mathrm{Ann}_R(xy) \neq \mathrm{Ann}_R(x) \cup \mathrm{Ann}_R(y)$.

By relying this fact we introduce the annihilator graph for a module. Let M be an R-module. The annihilator graph of M, denoted by $AG(M)$ is a simple undirected graph associated to M whose vertices are the elements of $Z_R(M) \setminus \mathrm{Ann}_R(M)$ and two distinct vertices x and y are adjacent if and only if $\mathrm{Ann}_M(xy) \neq \mathrm{Ann}_M(x) \cup \mathrm{Ann}_M(y)$. We investigate the interplay between the graph theoretic properties of $AG(M)$ and some algebraic properties of M.

Let $G = (V(G), E(G))$ be a simple undirected graph, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. Let $x, y \in V(G)$. We write $x \sim y$, whenever x and y are adjacent. A universal vertex is a vertex that is adjacent to all other vertices of the graph. We say that G is connected if there is a path between any two distinct vertices. For vertices x and y of G, we define $\mathrm{d}(x, y)$ to be the length of a shortest path between x and y (if there is no path, then $\mathrm{d}(x, y) = \infty$). The open neighborhood of a vertex x is defined to be the set $N(x) = \{y \in V(G) : \mathrm{d}(x, y) = 1\}$. The diameter of G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(x, y) : x \text{ and } y \text{ are vertices of } G\}$. The graph G is complete if any two distinct vertices are adjacent and a complete graph with n vertices is denoted by K_n. A complete bipartite graph G is a graph whose vertices can be partitioned into two disjoint nonempty sets A and B such that two distinct vertices are adjacent if and only if they are in distinct sets and it is denoted by $K_{|A|,|B|}$. The girth of G, denoted by $\mathrm{gr}(G)$ is the length of a shortest cycle in G ($\mathrm{gr}(G) = \infty$ if G contains no cycle).

Throughout this paper, R denotes a commutative ring with nonzero identity and M is an R-module. Recall that $\mathrm{Ann}_R(M) = \{r \in R : rM = 0\}$, $Z_R(M) = \{r \in R : rm = 0 \text{ for some nonzero } m \in M\}$ and $\mathrm{Ass}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} = \mathrm{Ann}_R(m) \text{ for some nonzero } m \in M\}$. For $x \in R$, $\mathrm{Ann}_M(x) = \{m \in M : xm = 0\}$. The reader is referred to [15], for notations and terminologies not given in this paper.

2. The annihilator graph for modules

In this section we define a simple undirected graph $AG(M)$ and we study the relations between graph theoretic properties of $AG(M)$ and module theoretic properties of M.

Definition 2.1. Let M be an R-module. The annihilator graph of M, denoted by $AG(M)$ is a simple undirected graph associated to M whose the set of vertices is $Z_R(M) \setminus \mathrm{Ann}_R(M)$ and two distinct vertices x and y are adjacent if and only if $\mathrm{Ann}_M(xy) \neq \mathrm{Ann}_M(x) \cup \mathrm{Ann}_M(y)$.

Lemma 2.2. *Let M be an R-module and x, y be distinct vertices of $AG(M)$. Then the following statements are true:*

(i) *If $\mathrm{Ann}_M(x) \nsubseteq \mathrm{Ann}_M(y)$ and $\mathrm{Ann}_M(y) \nsubseteq \mathrm{Ann}_M(x)$, then x, y are adjacent in $AG(M)$.*
(ii) *If x, y are not adjacent in $AG(M)$, then either $\mathrm{Ann}_M(x) \subseteq \mathrm{Ann}_M(y)$ or $\mathrm{Ann}_M(y) \subseteq \mathrm{Ann}_M(x)$.*
(iii) *If x, y are not adjacent in $AG(M)$, then either $\mathrm{Ann}_R(xM) \subseteq \mathrm{Ann}_R(yM)$ or $\mathrm{Ann}_R(yM) \subseteq \mathrm{Ann}_R(xM)$.*
(iv) *x, y are not adjacent in $AG(M)$ if and only if either $\mathrm{Ann}_M(xy) = \mathrm{Ann}_M(x)$ or $\mathrm{Ann}_M(xy) = \mathrm{Ann}_M(y)$.*

Proof. (i) Suppose that x, y are not adjacent in $AG(M)$. Thus $\mathrm{Ann}_M(x) \cup \mathrm{Ann}_M(y) = \mathrm{Ann}_M(xy)$. So $\mathrm{Ann}_M(xy) = \mathrm{Ann}_M(x)$ or $\mathrm{Ann}_M(xy) = \mathrm{Ann}_M(y)$. Hence, $\mathrm{Ann}_M(x) \subseteq \mathrm{Ann}_M(y)$ or $\mathrm{Ann}_M(y) \subseteq \mathrm{Ann}_M(x)$ which is a contradiction.

(ii) It is contrapositive of part (i).

(iii) Suppose that x, y are not adjacent in $AG(M)$. It follows that either $\mathrm{Ann}_M(x) \subseteq \mathrm{Ann}_M(y)$ or $\mathrm{Ann}_M(y) \subseteq \mathrm{Ann}_M(x)$, by (ii). Let $\mathrm{Ann}_M(x) \subseteq \mathrm{Ann}_M(y)$ and $r \in \mathrm{Ann}_R(xM)$. Then $rxM = 0$ and so $rM \subseteq \mathrm{Ann}_M(x)$. Hence, $rM \subseteq \mathrm{Ann}_M(y)$ and then $ryM = 0$. Therefore, $r \in \mathrm{Ann}_R(yM)$. So $\mathrm{Ann}_R(xM) \subseteq \mathrm{Ann}_R(yM)$.

(iv) It is obvious by the proof of part (i). □

Lemma 2.3. *Let M be an R-module and x, y be distinct vertices of $AG(M)$. Let $x \notin r(\mathrm{Ann}_R(M)) = \{x \in R : x^t \in \mathrm{Ann}_R(M)$ for some $t \in \mathbb{N}\}$ and $\mathrm{Ann}_M(x)$ be a prime submodule of M. Then x, y are adjacent in $AG(M)$ if and only if $\mathrm{Ann}_M(y) \nsubseteq \mathrm{Ann}_M(x)$.*

Proof. Assume that $\mathrm{Ann}_M(y) \nsubseteq \mathrm{Ann}_M(x)$ and $m \in \mathrm{Ann}_M(y) \backslash \mathrm{Ann}_M(x)$. Then $ym = 0 \in \mathrm{Ann}_M(x)$. Since $\mathrm{Ann}_M(x)$ is a prime submodule of M, $xyM = 0$. So $\mathrm{Ann}_M(x) \cup \mathrm{Ann}_M(y) \neq \mathrm{Ann}_M(xy)$. Conversely, suppose that $\mathrm{Ann}_M(x) \cup \mathrm{Ann}_M(y) \neq \mathrm{Ann}_M(xy)$. Thus there exists $m \in M$ such that $xym = 0$ but $xm \neq 0 \neq ym$. If $\mathrm{Ann}_M(y) \subseteq \mathrm{Ann}_M(x)$, then $xm \in \mathrm{Ann}_M(x)$ and $m \notin \mathrm{Ann}_M(x)$ which implies that $x^2M = 0$ and it is a contradiction. Hence, $\mathrm{Ann}_M(y) \nsubseteq \mathrm{Ann}_M(x)$. □

Theorem 2.4. *Let M be an R-module and x, y be distinct vertices of $AG(M)$. Then the following statements are equivalent:*

(i) *x, y are adjacent in $AG(M)$.*
(ii) *$xM \cap \mathrm{Ann}_M(y) \neq 0$ and $yM \cap \mathrm{Ann}_M(x) \neq 0$.*
(iii) *$x \in Z_R(yM)$ and $y \in Z_R(xM)$.*

Proof. (i) $\Rightarrow$ (ii) Let x, y be distinct vertices of $AG(M)$. Then there exists $m \in M$ such that $xym = 0$ but $xm \neq 0 \neq ym$. So $xM \cap \mathrm{Ann}_M(y) \neq 0$ and $yM \cap \mathrm{Ann}_M(x) \neq 0$.

(ii) $\Rightarrow$ (i) By the hypothesis there exist $m, m' \in M$ such that $xym = xym' = 0$, $xm \neq 0$ and $ym' \neq 0$. If $m = m'$ or $ym \neq 0$ or $xm' \neq 0$, then there is nothing to prove. Now assume that $m \neq m'$, $ym = 0$ and $xm' = 0$. Thus $xy(m + m') = 0$ but $x(m + m') = xm \neq 0$ and $y(m + m') = ym' \neq 0$. So x, y are adjacent in $AG(M)$.

(ii) $\Leftrightarrow$ (iii) It is clear. $\square$

Let M be an R-module. A submodule Q of M is said to be primary submodule of M precisely when $M/Q \neq 0$, and for each $a \in Z_R(M/Q)$, there exists $n \in \mathbb{N}$ such that $a^n(M/Q) = 0$. It is well known that if Q is primary submodule of M, then $\mathrm{Ann}_R(M/Q)$ is a primary ideal of R. In the following we offer a sufficient and necessary condition for completeness of $AG(M)$, whenever M is Noetherian. We begin with the following lemma.

Lemma 2.5. *Let M be a Noetherian R-module and let $0 = \cap_{i=1}^{n} Q_i$ be a minimal primary decomposition of the zero submodule of M with $r(\mathrm{Ann}_R(M/Q_i)) = \mathfrak{p}_i$, for each $i = 1, \cdots, n$. Suppose that $\mathfrak{p}_j$ is a minimal member of $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_n\} = \mathrm{Ass}_R(M)$ with respect to inclusion. Then there exists $a_j \in R$ such that $Q_j = \mathrm{Ann}_M(a_j)$.*

Proof. Let $0 = \cap_{i=1}^{n} Q_i$ be a minimal primary decomposition of the zero submodule of M with $r(\mathrm{Ann}_R(M/Q_i)) = \mathfrak{p}_i$, for each $i = 1, \cdots, n$. Suppose that $\mathfrak{p}_j = r(\mathrm{Ann}_R(M/Q_j))$ is a minimal element of $\mathrm{Ass}_R(M)$, for some j with $1 \leq j \leq n$. Then $\cap_{i=1, i\neq j}^{n} \mathrm{Ann}_R(M/Q_i) \nsubseteq \mathfrak{p}_j$. Suppose that $a_j \in \cap_{i=1, i\neq j}^{n} \mathrm{Ann}_R(M/Q_i) \setminus \mathfrak{p}_j$. We show that $\mathrm{Ann}_M(a_j) = Q_j$. We have $\mathrm{Ann}_M(a_j) = (0 :_M a_j) = (\cap_{i=1}^{n} Q_i :_M a_j) = \cap_{i=1}^{n} (Q_i :_M a_j) = (Q_j :_M a_j)$. It is clear that $Q_j \subseteq (Q_j :_M a_j)$. If there exists $m \in (Q_j :_M a_j)$ with $m \notin Q_j$, then $a_j^t M \subseteq Q_j$ for some $t \in \mathbb{N}$ and so $a_j \in \mathfrak{p}_j$ which is a contradiction. Hence, $Q_j = \mathrm{Ann}_M(a_j)$. $\square$

Let M be an R-module. Then the zero submodule is a primary submodule of M if and only if $Z_R(M) = r(\mathrm{Ann}_R(M))$.

Theorem 2.6. *Let M be a Noetherian R-module. Then $AG(M)$ is a complete graph if and only if $Z_R(M) = r(\mathrm{Ann}_R(M))$.*

Proof. $\Rightarrow$ Let $0 = \cap_{j=1}^{n} Q_i$ be a minimal primary decomposition of the zero submodule of M with $r(\mathrm{Ann}_R(M/Q_i)) = \mathfrak{p}_i$, for each $i = 1, \cdots, n$. Let $\mathfrak{p}_j$ be a minimal element of $\mathrm{Ass}_R(M)$, for some $1 \leq j \leq n$. Then by Lemma 2.5, there exists $a_j \in \cap_{i=1, i\neq j}^{n} \mathrm{Ann}_R(M/Q_i) \setminus \mathfrak{p}_j$ such that $Q_j = \mathrm{Ann}_M(a_j)$. Suppose that $c \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ and $c \neq a_j$. By the hypothesis c, a_j are adjacent in $AG(M)$. So $\mathrm{Ann}_M(a_j) \cup \mathrm{Ann}_M(c) \neq \mathrm{Ann}_M(a_j c)$. Thus there exists $m \in M$ such that $a_j cm = 0$ but $a_j m \neq 0$. Hence, $c^t M \subseteq Q_j$ for some $t \in \mathbb{N}$ so $c^t \in \mathrm{Ann}_R(M/Q_j) \subseteq \mathfrak{p}_j$. Therefore,

$Z_R(M) = \mathfrak{p}_j \cup \{a_j\}$. Let $\mathfrak{p}_j \subset \mathfrak{p}_k$, for some $1 \leq k \leq n$. Since $\mathfrak{p}_k \subseteq Z_R(M) = \mathfrak{p}_j \cup \{a_j\}$, $\mathfrak{p}_k = \mathfrak{p}_j \cup \{a_j\}$ which is a contradiction. Hence, $n = 1$ and so 0 is a primary submodule of M. So $\mathrm{Ass}_R(M) = \{\mathfrak{p}_j\}$ and consequently $Z_R(M) = r(\mathrm{Ann}_R(M))$.

$\Leftarrow$ Let $Z_R(M) = r(\mathrm{Ann}_R(M))$ and let $x, y \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ be two distinct vertices of $AG(M)$. Then $\mathrm{Ann}_M(x)$ and $\mathrm{Ann}_M(y)$ are essential submodules of M by [3, Theorem 5]. So $xM \cap \mathrm{Ann}_M(y) \neq 0$ and $yM \cap \mathrm{Ann}_M(x) \neq 0$. Hence, x, y are adjacent in $AG(M)$ by Theorem 2.4. □

The following example has been presented to show that the property of being Noetherian is a necessary condition in Theorem 2.6.

Example 2.7. Consider $M = \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module, where p is a prime integer. It is easy to see that $AG(M)$ is a complete graph but $Z_\mathbb{Z}(M) = p\mathbb{Z}$ and $r(\mathrm{Ann}_\mathbb{Z}(M)) = 0$.

Proposition 2.8. *Let M be an R-module and x, y be distinct vertices of $AG(M)$. If $\mathrm{Ann}_M(x) = \mathrm{Ann}_M(y)$, then $N_{AG(M)}(x) = N_{AG(M)}(y)$.*

Proof. Let $z \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ and $z \in N_{AG(M)}(x)$. Then there exists $m \in M$ such that $xzm = 0$ but $xm \neq 0 \neq zm$. So $zm \in \mathrm{Ann}_M(y)$ and $ym \neq 0 \neq zm$. It means that y, z are adjacent in $AG(M)$. Hence, $z \in N_{AG(M)}(y)$. The reverse inclusion can be proved similarly. □

3. Relation between the zero-divisor graph and the annihilator graph

Let M be an R-module. The zero-divisor graph of M, denoted by $\Gamma(M)$ is a simple undirected graph associated to M whose vertices are the elements of $Z_R(M) \setminus \mathrm{Ann}_R(M)$ and two distinct vertices x and y are adjacent if and only if $xyM = 0$, see [11].

Lemma 3.1. *Let M be an R-module and x, y be distinct vertices of $AG(M)$. Then the following statements are true:*

(i) *If x, y are adjacent in $\Gamma(M)$, then x, y are adjacent in $AG(M)$. In particular, if P is a path in $\Gamma(M)$, then P is a path in $AG(M)$.*
(ii) *If $d_{\Gamma(M)}(x, y) = 3$, then x, y are adjacent in $AG(M)$.*

Proof. (i) Suppose that x, y are adjacent in $\Gamma(M)$. Thus $xyM = 0$ and so $\mathrm{Ann}_M(xy) = M$; but $\mathrm{Ann}_M(x) \neq M$ and $\mathrm{Ann}_M(y) \neq M$. Hence, $\mathrm{Ann}_M(xy) \neq \mathrm{Ann}_M(x) \cup \mathrm{Ann}_M(y)$ and x, y are adjacent in $AG(M)$.

(ii) Suppose that $d_{\Gamma(M)}(x, y) = 3$. Thus $xyM \neq 0$ and there exist $a, b \in Z_R(M) \setminus \mathrm{Ann}_R(M) \cup \{x, y\}$ such that $axM = 0$, $abM = 0$ and

$byM = 0$. If $\mathrm{Ann}_M(x) \subseteq \mathrm{Ann}_M(y)$, then in view of $axM = 0$ it follows that $aM \subseteq \mathrm{Ann}_M(x) \subseteq \mathrm{Ann}_M(y)$. Thus $ayM = 0$ which contradicts to the hypothesis. Hence, $\mathrm{Ann}_M(x) \nsubseteq \mathrm{Ann}_M(y)$. By a similar argument one can show that $\mathrm{Ann}_M(y) \nsubseteq \mathrm{Ann}_M(x)$. Therefore, x, y are adjacent in $AG(M)$ by Lemma 2.2(i). □

Lemma 3.2. *Let M be an R-module and x, y be distinct vertices of $AG(M)$. If $\mathrm{Ann}_M(x)$ and $\mathrm{Ann}_M(y)$ are distinct prime submodules of M, then x, y are adjacent in $\Gamma(M)$ and so are adjacent in $AG(M)$.*

Proof. Assume that $P_1 = \mathrm{Ann}_M(x), P_2 = \mathrm{Ann}_M(y)$ are two distinct prime submodules of M and $m \in P_1 \setminus P_2$. Thus $xm = 0 \in P_2$ which implies that $xM \subseteq P_2 = \mathrm{Ann}_M(y)$. Hence, $xyM = 0$ and so x, y are adjacent in $\Gamma(M)$. The second assertion follows by Lemma 3.1(i). □

Let M be an R-module and $\mathrm{Spec}_R(M)$ denote the set of prime submodules of M. Then $m - \mathrm{Ass}_R(M) = \{P \in \mathrm{Spec}_R(M) : P = \mathrm{Ann}_M(a)$ for some $0 \neq a \in R\}$.

Corollary 3.3. *Let M be an R-module such that for every edge of $AG(M)$, $x \sim y$ say, either $\mathrm{Ann}_M(x) \in m - \mathrm{Ass}_R(M)$ or $\mathrm{Ann}_M(y) \in m - \mathrm{Ass}_R(M)$. Then $\Gamma(M) = AG(M)$.*

Proof. In view of Lemma 3.1(i), $\Gamma(M)$ is a subgraph of $AG(M)$. Let x, y be distinct adjacent vertices of $AG(M)$ and let either $\mathrm{Ann}_M(x) \in m - \mathrm{Ass}_R(M)$ or $\mathrm{Ann}_M(y) \in m - \mathrm{Ass}_R(M)$. Without loss of generality we may assume that $\mathrm{Ann}_M(x) \in m - \mathrm{Ass}_R(M)$. Thus $\mathrm{Ann}_M(xy) \neq \mathrm{Ann}_M(x) \cup \mathrm{Ann}_M(y)$. Hence, there is $m \in M$ such that $xym = 0$ but $xm \neq 0 \neq ym$. Therefore, $ym \in \mathrm{Ann}_M(x)$ and $m \notin \mathrm{Ann}_M(x)$. So $xyM = 0$ since $\mathrm{Ann}_M(x)$ is a prime submodule of M and x and y are adjacent in $\Gamma(M)$. □

Theorem 3.4. *Let M be an R-module and $\Gamma(M)$ be a connected graph. Then $AG(M)$ is a connected graph and $\mathrm{diam}(AG(M)) \leqslant 2$.*

Proof. Suppose that x, y are distinct non-adjacent vertices of $AG(M)$. Thus by Lemma 2.2(ii), either $\mathrm{Ann}_M(x) \subseteq \mathrm{Ann}_M(y)$ or $\mathrm{Ann}_M(y) \subseteq \mathrm{Ann}_M(x)$. Without loss of generality we may assume that $\mathrm{Ann}_M(x) \subseteq \mathrm{Ann}_M(y)$. Thus $\mathrm{Ann}_R(xM) \subseteq \mathrm{Ann}_R(yM)$, by Lemma 2.2(iii). Since x is not an isolated vertex of $\Gamma(M)$, thus there exists $z \in \mathrm{Ann}_R(xM) \setminus \mathrm{Ann}_R(M)$ such that $xzM = 0$. So $yzM = 0$. Hence, $x \sim z \sim y$ is a path in $\Gamma(M)$ and so is a path in $AG(M)$. □

Theorem 3.5. *Let M be a Noetherian R-module and $\Gamma(M)$ be a connected graph. Then $\mathrm{gr}(AG(M)) \in \{3, 4, \infty\}$.*

Proof. If $\Gamma(M) = AG(M)$, then in view of [11, Teorem 3.3], $\mathrm{gr}(AG(M)) \in \{3, 4, \infty\}$. Now, suppose that $\Gamma(M) \neq AG(M)$ and x, y are two distinct adjacent vertices of $AG(M)$ such that they are non-adjacent in $\Gamma(M)$. Since $\Gamma(M)$ is a connected graph, there exist $a, b \in Z_R(M) \setminus \mathrm{Ann}_R(M) \cup \{x, y\}$ such that $axM = byM = 0$. If $a = b$, then $x \sim a \sim y$ is a path in $\Gamma(M)$ and so $x \sim a \sim y \sim x$ is a cycle in $AG(M)$ of length three. So we may assume that $a \neq b$. If $abM = 0$, then $x \sim a \sim b \sim y$ is a path in $\Gamma(M)$. Thus $x \sim a \sim b \sim y \sim x$ is a cycle in $AG(M)$ of length four. If $abM \neq 0$, then $x \sim ab \sim y$ is a path in $\Gamma(M)$ and so $x \sim ab \sim y \sim x$ is a cycle in $AG(M)$ of length three. Therefore, $\mathrm{gr}(AG(M)) \in \{3, 4, \infty\}$. □

Consider $\mathbb{Z}_8$ as a $\mathbb{Z}_8$-module. It is easy to see that $\mathrm{gr}(AG(\mathbb{Z}_8)) = 3$ and $\mathrm{gr}(\Gamma(\mathbb{Z}_8)) = \infty$.

Theorem 3.6. *Let M be a Noetherian R-module and $AG(M)$ be a complete graph. Then $c \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ is a universal vertex of $\Gamma(M)$ if and only if $\mathrm{Ann}_M(c)$ is a prime submodule of M.*

Proof. Let $c \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ be a universal vertex of $\Gamma(M)$. We show that $\mathrm{Ann}_M(c)$ is a prime submodule of M. Assume that $x \in R, m \in M \setminus \mathrm{Ann}_M(c)$ and $xm \in \mathrm{Ann}_M(c)$. By [11, Theorem 2.1], $Z_R(M) = \mathrm{Ann}_R(cM)$ and $x \in Z_R(M)$ thus $xM \subseteq \mathrm{Ann}_M(c)$ as desired. Hence, $\mathrm{Ann}_M(c)$ is a prime submodule of M.

Suppose that $c \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ and $\mathrm{Ann}_M(c)$ is a prime submodule of M. We show that c is a universal vertex of $\Gamma(M)$. Let $x \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ be a vertex of $\Gamma(M)$ and $x \neq c$. In view of the assumption $AG(M)$ is a complete graph so there exists $m \in \mathrm{Ann}_M(cx)$ such that $xm \neq 0 \neq cm$. Thus $xm \in \mathrm{Ann}_M(c)$ and $cm \neq 0$. Hence, $xcM = 0$ and so c, x are adjacent in $\Gamma(M)$. □

Corollary 3.7. *Let M be a Noetherian R-module and $AG(M)$ be a complete graph with $|Z_R(M) \setminus \mathrm{Ann}_R(M)| \geqslant 3$. If $\Gamma(M)$ is a star graph, then $|m - \mathrm{Ass}_R(M)| = 1$.*

Proof. Let $Z_R(M) \setminus \mathrm{Ann}_R(M) = \{a, b, c, \cdots\}$ and let $\Gamma(M)$ be a star graph. If $P_1 = \mathrm{Ann}_M(a)$ and $P_2 = \mathrm{Ann}_M(b)$ are prime submodules of M, then by Theorem 3.6, a and b are universal vertices of $\Gamma(M)$ which is a contradiction. Thus $|m - \mathrm{Ass}_R(M)| \leq 1$. Since M is Noetherian, $|m - \mathrm{Ass}_R(M)| \geq 1$. □

Consider $\mathbb{Z}_8$ as a $\mathbb{Z}$-module. It is easy to check that $AG(\mathbb{Z}_8)$ is a complete graph and $m - \mathrm{Ass}_{\mathbb{Z}}(\mathbb{Z}_8) = \{2\mathbb{Z}\}$ but $\Gamma(\mathbb{Z}_8)$ is not a star graph. Note that 4 is a universal vertex of $\Gamma(\mathbb{Z}_8)$. Also, $2 \sim 12$ in $\Gamma(\mathbb{Z}_8)$.

Theorem 3.8. *Let M be an R-module and $\Gamma(M)$ be a star graph with the universal vertex c. Then the following statements are true:*

(i) *If $c \notin r(\mathrm{Ann}_R(M))$, then $\Gamma(M) = K_1$.*
(ii) *If $c \in r(\mathrm{Ann}_R(M))$, then $\Gamma(M) = K_{1,1}$ or $Rc = cZ_R(M) \cup \{c\}$.*

Proof. (i) In [11, Theorem 2.1], it has been proved that $Z_R(M) = \mathrm{Ann}_R(cM) \cup \{c\}$ and $c = c^2$. If there exists $a \in R \setminus Z_R(M)$ such that $ac \neq c$, then ac and x are adjacent for all $x \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ which is a contradiction. So $ac = c$ and $\Gamma(M) = K_1$. Let $ac = c$, for all $a \in R \setminus Z_R(M)$. Then $Rc = cZ_R(M) \cup c(R \setminus Z_R(M)) = cZ_R(M) \cup \{c\} = c\mathrm{Ann}_R(cM) \cup \{c\}$. In this case we have $R = \mathbb{Z}_2 \oplus R'$ and $M = \oplus \mathbb{Z}_2 \oplus M'$, where R' is a subring of R and M' is an R-submodule of M. Moreover $c = (1, 0)$ and $\mathrm{Ann}_R(cM) = 0 \times R'$, see [11, Theorem 2.2]. Thus $c\mathrm{Ann}_R(cM) = c(0 \times R') = \{(0, 0)\}$. Hence, $Rc = \{(0, 0), c = (1, 0)\}$.

(ii) It is easy to see that $c \neq c^2$. If $c^2 \notin \mathrm{Ann}_R(M)$, then $\Gamma(M) = K_1$. Let $c^2M = 0$. If there exists $a \in R \setminus Z_R(M)$ such that $ac \neq c$, then $\Gamma(M) = K_{1,1}$. Suppose that $ac = c$ for each $a \in R \setminus Z_R(M)$. Thus $Rc = cZ_R(M) \cup c(R \setminus Z_R(M)) = cZ_R(M) \cup \{c\}$. □

A proper submodule P of M is said to be a weakly prime submodule whenever $0 \neq rm \in P$ with $r \in R$ and $m \in M$, then either $m \in P$ or $r \in \mathrm{Ann}_R(M/P)$.

Lemma 3.9. *Let M be an R-module and $x \in Z_R(M) \setminus \mathrm{Ann}_R(M)$. Then $\mathrm{Ann}_M(x)$ is a weakly prime submodule of M if and only if $N_{\Gamma(M)}(x) = N_{AG(M)}(x)$.*

Proof. $\Rightarrow$) It is enough to show that $N_{AG(M)}(x) \subseteq N_{\Gamma(M)}(x)$. Suppose that x, y are adjacent in $AG(M)$. Then there exists $m \in \mathrm{Ann}_M(xy)$ such that $m \notin \mathrm{Ann}_M(x) \cup \mathrm{Ann}_M(y)$. So $0 \neq ym \in \mathrm{Ann}_M(x)$ and $m \notin \mathrm{Ann}_M(x)$. Since $\mathrm{Ann}_M(x)$ is a weakly prime submodule of M, thus $xyM = 0$. Hence, x, y are adjacent in $\Gamma(M)$ and the proof is completed.

$\Leftarrow$) Suppose that $x \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ and $N_{\Gamma(M)}(x) = N_{AG(M)}(x)$. We have to show that $\mathrm{Ann}_M(x)$ is a weakly prime submodule of M. Let $0 \neq ym \in \mathrm{Ann}_M(x)$, for some $m \in M$ and $y \in R$ with $x \neq y$. If $xm = 0$ we are done; otherwise $y \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ and $xym = 0$. Thus $m \in \mathrm{Ann}_M(xy) \setminus \mathrm{Ann}_M(x) \cup \mathrm{Ann}_M(y)$. It means that x, y are adjacent in $AG(M)$ and so they are adjacent in $\Gamma(M)$. Hence, $xyM = 0$ and $yM \subseteq \mathrm{Ann}_M(x)$, as desired. Now, assume that $0 \neq xm \in \mathrm{Ann}_M(x)$, for some $m \in M$. Thus $x^2m = 0$ and so $x \neq x^2$. We show that $x^2M = 0$. In this case $(x - x^2)m = xm \neq 0$, so $x - x^2$ is a vertex of $AG(M)$ and let $x \neq x - x^2$. Moreover $x(x - x^2)m = 0$ thus $x, x - x^2$ are adjacent in $AG(M)$ so by the hypotheses $x(x - x^2)M = 0$. Hence, $x^2(1 - x)M = 0$.

If $1-x \notin Z_R(M)$, then $x^2M = 0$ and we are done. Otherwise, $1-x \in Z_R(M)$. Since $(x-x^2)m \neq 0$, $1-x \in Z_R(M)\setminus \mathrm{Ann}_R(M)$. Hence, $1-x$ is a vertex of $AG(M)$; moreover $\mathrm{Ann}_M(x) \cap \mathrm{Ann}_M(1-x) = 0$. Therefore, $\mathrm{Ann}_M(1-x) \nsubseteq \mathrm{Ann}_M(x)$ and $\mathrm{Ann}_M(x) \nsubseteq \mathrm{Ann}_M(1-x)$. So $x, 1-x$ are are adjacent in $AG(M)$, by Lemma 2.2(i). Thus $x(1-x)M = 0$ which implies that $(x-x^2)m = xm = 0$ contrary to the assumption. □

Lemma 3.10. *Let M be an R-module and $x \in r(\mathrm{Ann}_R(M))\setminus \mathrm{Ann}_R(M)$. Then $\mathrm{Ann}_M(x)$ is a prime submodule of M if and only if $N_{\Gamma(M)}(x) = N_{AG(M)}(x)$.*

Proof. $\Rightarrow$ It is clear that a prime submodule of M is a weakly prime submodule so the result follows by Lemma 3.9.

$\Leftarrow$ Let $x \in r(\mathrm{Ann}_R(M)) \setminus \mathrm{Ann}_R(M)$. We show that $\mathrm{Ann}_M(x)$ is a prime submodule of M. Assume that $xm \in \mathrm{Ann}_M(x)$, for some $m \in M$. If $xm = 0$ there is nothing to prove; so suppose that $xm \neq 0$. Thus $x \neq x^2$. We show that $x^2M = 0$. If $x^2M \neq 0$, then $x^2 \in r(\mathrm{Ann}_R(M)) \setminus \mathrm{Ann}_R(M)$ and so x, x^2 are adjacent in $AG(M)$, see [3, Theorem 5] and Theorem 2.4, so x, x^2 are adjacent in $\Gamma(M)$. Hence, $x^3M = 0$. In this case $x - x^2$ is a vertex of $AG(M)$ and $x \neq x - x^2$. Moreover $x, x - x^2$ are adjacent in $AG(M)$ and so $x(x - x^2)M = 0$. Thus $0 = x^2M - x^3M = x^2M$ contrary to the assumption. Therefore, $x^2M = 0$, as desired. Let $0 \neq ym' \in \mathrm{Ann}_M(x)$, for some $m' \in M$ and $y \in R$ with $x \neq y$. If either $xm' = 0$ or $yM = 0$, then there is nothing to prove. Otherwise, $xm' \neq 0$ and $y \in Z_R(M) \setminus \mathrm{Ann}_R(M)$. Thus $m' \in \mathrm{Ann}_M(xy) \setminus \mathrm{Ann}_M(x) \cup \mathrm{Ann}_M(y)$. It means that x, y are adjacent in $AG(M)$ and so x, y are adjacent in $\Gamma(M)$. Hence, $xyM = 0$ and so $yM \subseteq \mathrm{Ann}_M(x)$ as desired. If $ym' = 0$ and $xyM \neq 0$, then $m' \in \mathrm{Ann}_M(y) \setminus \mathrm{Ann}_M(x)$ and there exists $m'' \in M$ such that $xm'' \in \mathrm{Ann}_M(x) \setminus \mathrm{Ann}_M(y)$. By Lemma 2.2(i), x, y are adjacent in $AG(M)$ and so are adjacent in $\Gamma(M)$ which is a contradiction. Hence, $xyM = 0$. □

Corollary 3.11. *Let M be an R-module. If $\Gamma(M) = AG(M)$, then $\mathrm{Ann}_M(x) \in m - \mathrm{Ass}_R(M)$, for each $x \in r(\mathrm{Ann}_R(M)) \setminus \mathrm{Ann}_R(M)$.*

4. Two absorbing submodules and the annihilator graph

Let M be an R-module. A proper submodule N of M is called 2-absorbing if whenever $abm \in N$ for $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in \mathrm{Ann}_R(M/N)$. The reader is referred to [12, 13] for more results and examples about 2-absorbing submodules.

Theorem 4.1. *Let M be an R-module. Then $\Gamma(M) = AG(M)$ if and only if 0 is a 2-absorbing submodule of M.*

Proof. $\Rightarrow$) Let $\Gamma(M) = AG(M)$, $x, y \in R$ and $m \in M$ be such that $xym = 0$. First of all assume that $x = y$. In this case $x^2m = 0$. If $xm = 0$ we are done; otherwise $x \in Z_R(M) \setminus \mathrm{Ann}_R(M)$. By Lemma 3.9, $\mathrm{Ann}_M(x)$ is a weakly prime submodule of M. $x^2m = 0$ and $xm \neq 0$ imply that $x^2M = 0$. Hence, 0 is a 2-absorbing submodule of M. Now suppose that $x \neq y$. If either $xm = 0$ or $ym = 0$, we are done. Let $xm \neq 0$ and $ym \neq 0$. Then $x, y \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ and $m \in \mathrm{Ann}_M(xy) \setminus \mathrm{Ann}_M(x) \cup \mathrm{Ann}_M(y)$. It means that x, y are adjacent in $AG(M)$ and so they are adjacent in $\Gamma(M)$. So $xyM = 0$ which implies that 0 is a 2-absorbing submodule of M.

$\Leftarrow$) It is enough to show that an arbitrary edge of $AG(M)$ is an edge of $\Gamma(M)$. Let $x, y \in Z_R(M) \setminus \mathrm{Ann}_R(M)$ be distinct adjacent vertices of $AG(M)$. Then there exists $m \in M$ such that $xym = 0$ but $xm \neq 0 \neq ym$. Hence, $xyM = 0$ since 0 is a 2-absorbing submodule of M. Therefore, x and y are adjacent in $\Gamma(M)$. □

The following corollary is a generalization of [5, Theorem 3.6].

Corollary 4.2. *Let M be an R-module. If $\Gamma(M) = AG(M)$, then $|\mathrm{MinAss}(M)| \leq 2$.*

Proof. It follows easily by Theorem 4.1, [12, Theorem 2.3] and [4, Theorem 2.4]. □

Theorem 4.3. *Let N be a 2-absorbing submodule of a Noetherian R-module M such that $r(N :_R M) = \mathfrak{p} \cap \mathfrak{q}$, where $\mathfrak{p}$ and $\mathfrak{q}$ are distinct prime ideals of R that are minimal over $N :_R M$. Then $\mathrm{Ass}_R(M/N)$ is union of two totally ordered sets.*

Proof. Let $N = \cap_{i=1}^n Q_i$ be a minimal primary decomposition of N with $r(\mathrm{Ann}_R(M/Q_i)) = \mathfrak{p}_i$, for each $1 \leqslant i \leqslant n$. Then $r(N :_R M) = \cap_{i=1}^n r(Q_i :_R M) = \cap_{i=1}^n \mathfrak{p}_i$ and so $\mathfrak{p} \cap \mathfrak{q} = \cap_{i=1}^n \mathfrak{p}_i$. Without loss of generality we may assume that $\mathfrak{p} = \mathfrak{p}_1$ and $\mathfrak{q} = \mathfrak{p}_2$. Suppose that $3 \leqslant k, t \leqslant n$ and $k \neq t$. By the definition of a minimal primary decomposition there exist $m_k \in \cap_{i \neq k} Q_i \setminus Q_k$ and $m_t \in \cap_{i \neq t} Q_i \setminus Q_t$. Thus $r(N :_R m_k) = r(\cap_{i=1}^n Q_i :_R m_k) = \cap_{i=1}^n r(Q_i :_R m_k) = r(Q_k :_R m_k) = r(Q_k :_R M) = \mathfrak{p}_k$ and $r(N :_R m_t) = r(\cap_{i=1}^n Q_i :_R m_t) = r(Q_t :_R m_t) = r(Q_t :_R M) = \mathfrak{p}_t$. Let $\mathfrak{p}_t \nsubseteq \mathfrak{p}_k$; we show that $\mathfrak{p}_k \subseteq \mathfrak{p}_t$. By the hypotheses we may assume that $\mathfrak{p}_1 \subseteq \mathfrak{p}_k$ moreover we can assume that $\mathfrak{p}_t \nsubseteq \mathfrak{p}_k \cup \mathfrak{p}_2$. Suppose that $a \in \mathfrak{p}_k$ and $b \in \mathfrak{p}_t \setminus \mathfrak{p}_k \cup \mathfrak{p}_2$. So there exists $s \in \mathbb{N}$ such that $a^s m_k \in N, b^s m_t \in N$ and $b^s m_k \notin N$. If $a^s(m_k + m_t) \in N$, then $a \in \mathfrak{p}_t$ and the proof is completed. Now, let $a^s(m_k + m_t) \notin N$. Then $a^s b^s \in N :_R M$ since $b^s(m_k + m_t) \notin N$ and $a^s b^s(m_k + m_t) \in N$. From $ab \in \mathfrak{p}_1 \cap \mathfrak{p}_2$ and $b \notin \mathfrak{p}_1 \cup \mathfrak{p}_2$ it follows that $a \in \mathfrak{p}_1 \cap \mathfrak{p}_2$. So $a^s M \subseteq N$ and

$a^s m_t \in N$ which implies that $a \in \mathfrak{p}_t$. Hence, $\mathrm{Ass}_R(M/N)$ is union of two totally ordered sets such as $\mathrm{Ass}_R(M/N) = \{\mathfrak{p} = \mathfrak{p}_1\} \cup \{\mathfrak{p}_2, \mathfrak{p}_3, ..., \mathfrak{p}_n\}$ or $\mathrm{Ass}_R(M/N) = \{\mathfrak{q} = \mathfrak{p}_2\} \cup \{\mathfrak{p}_1, \mathfrak{p}_3, ..., \mathfrak{p}_n\}$. □

In [10, Theorem 2.5], it is shown that $\Gamma(R) = AG(R)$ whenever for every edges of $AG(R)$, $x \sim y$ say, either $\mathrm{Ann}_R(x) \in \mathrm{Ass}(R)$ or $\mathrm{Ann}_R(y) \in \mathrm{Ass}(R)$. Also the following question is posed: Let R be a non-reduced ring and $x \sim y$ be an edge of $AG(R)$. If $\Gamma(R) = AG(R)$, then is it true either $\mathrm{Ann}_R(x) \in \mathrm{Ass}(R)$ or $\mathrm{Ann}_R(y) \in \mathrm{Ass}(R)$?

The following theorem is an affirmative answer to this question.

Theorem 4.4. *Let M be a Noetherian R-module. Then the following statements are equivalent:*

(i) *For each edge of $AG(M)$, $x \sim y$ say,* $\mathrm{Ann}_M(x) \in m - \mathrm{Ass}_R(M)$ *or* $\mathrm{Ann}_M(y) \in m - \mathrm{Ass}_R(M)$.

(ii) $\Gamma(M) = AG(M)$.

(iii) *For each $x \in Z_R(M) \setminus \mathrm{Ann}_R(M)$, $\mathrm{Ann}_M(x)$ is a weakly prime submodule of M.*

Proof. It is enough to prove (ii) ⇒ (i). Let $x \sim y$ be an edge of $AG(M)$. Since $\Gamma(M) = AG(M)$ by Theorem 4.1 the zero submodule of M is 2-absorbing. Thus $r(\mathrm{Ann}_R(M)) = \mathfrak{p}$ or $r(\mathrm{Ann}_R(M)) = \mathfrak{p}_1 \cap \mathfrak{p}_2$, where $\mathfrak{p}_1, \mathfrak{p}_2$ are prime ideals of R that are minimal over $\mathrm{Ann}_R(M)$. If $r(\mathrm{Ann}_R(M)) = \mathfrak{p}$, then by $xyM = 0$ it follows that $xy \in \mathrm{Ann}_R(M) \subseteq \mathfrak{p}$. So $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. Hence, $\mathrm{Ann}_M(x) \in m - \mathrm{Ass}_R(M)$ or $\mathrm{Ann}_M(y) \in m - \mathrm{Ass}_R(M)$. Now, let $r(\mathrm{Ann}_R(M)) = \mathfrak{p}_1 \cap \mathfrak{p}_2$. If either x or y belongs to $r(\mathrm{Ann}_R(M))$, there is nothing to prove. So assume that $x \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$ and $y \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$. Then by using Theorem 4.3 we get either $\mathrm{Ann}_M(x) = Q_2$ or $\mathrm{Ann}_M(y) = Q_1$. Without loss of generality suppose that $\mathrm{Ann}_M(x) = Q_2$. We show that the primary submodule $\mathrm{Ann}_M(x) = Q_2$ is prime. Let $a \in R$, $m \in M \setminus \mathrm{Ann}_M(x)$ and $am \in \mathrm{Ann}_M(x) = Q_2$. Then $a \in \mathfrak{p}_2$ and so $ax \in \mathfrak{p}_1\mathfrak{p}_2 \subseteq \mathrm{Ann}_R(M)$ which implies that $aM \subseteq \mathrm{Ann}_M(x)$. Therefore, $\mathrm{Ann}_M(x)$ is a prime submodule of M. □

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THE ANNIHILATOR GRAPH FOR MODULES OVER COMMUTATIVE RINGS

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گراف پوچساز برای مدول‌ها روی حلقه‌های جابجایی

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فرض کنید R یک حلقه جابجایی و M یک R-مدول باشد. گراف پوچساز M با نماد $AG(M)$ نشان داده می‌شود و گرافی ساده و غیرجهت‌دار است که مجموعه رئوس آن $Z_R(M) \setminus \mathrm{Ann}_R(M)$ است و دو راس x و y از آن مجاورند هرگاه $\mathrm{Ann}_M(xy) \neq \mathrm{Ann}_M(x) \cup \mathrm{Ann}_M(y)$. در این مقاله، قطر و کمر گراف $AG(M)$ را محاسبه می‌کنیم و همه مدول‌هایی که گراف پوچساز آنها کامل است را مشخص می‌کنیم. علاوه‌برآن، رابطه بین گراف پوچساز M و گراف مقسوم علیه صفر آن را بدست می‌آوریم.

کلمات کلیدی: گراف پوچساز، گراف مقسوم علیه صفر، زیرمدول‌های اول.