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# $m$-TOPOLOGY ON THE RING OF REAL-MEASURABLE FUNCTIONS

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ABSTRACT. In this article we consider the $m$-topology on $M(X,\mathscr{A})$, the ring of all real measurable functions on a measurable space $(X,\mathscr{A})$, and we denote it by $M_m(X,\mathscr{A})$. We show that $M_m(X,\mathscr{A})$ is a Hausdorff regular topological ring, moreover we prove that if $(X,\mathscr{A})$ is a $T$-measurable space and $X$ is a finite set with $|X| = n$, then $M_m(X,\mathscr{A}) \cong \mathbb{R}^n$ as topological rings. Also, we show that $M_m(X,\mathscr{A})$ is never a pseudocompact space and it is also never a countably compact space. We prove that $(X,\mathscr{A})$ is a pseudocompact measurable space, if and only if $M_m(X,\mathscr{A}) = M_u(X,\mathscr{A})$, if and only if $M_m(X,\mathscr{A})$ is a first countable topological space, if and only if $M_m(X,\mathscr{A})$ is a connected space, if and only if $M_m(X,\mathscr{A})$ is a locally connected space, if and only if $M^*(X,\mathscr{A})$ is a connected subset of $M_m(X,\mathscr{A})$.

## 1. Introduction

The reader is presumed to have some background in measure theory, abstract algebra and general topology. Let $\mathbb{R}^X$ be the collection of all real-valued functions on a non-empty set $X$. It is known that $\mathbb{R}^X$ with the (pointwise) addition and multiplication is a reduced commutative ring with identity. Let $(X,\mathscr{A})$ be measurable space and let $M(X,\mathscr{A})$ be the set of all real measurable functions on $X$. Then $M(X,\mathscr{A})$ is a

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subring of $\mathbb{R}^X$. Many people have studied the rings of real measurable functions on a measurable space with different aspects. Hager in [12] shows that $M(X,\mathscr{A})$ is a regular ring in the sense of Von Neumann (i.e., for every $f \in M(X,\mathscr{A})$, there is an element $g$ in $M(X,\mathscr{A})$ with $f^2g = f$). In [21], Viertl studied the real maximal ideals and the fixed maximal ideals of $M(X,\mathscr{A})$, where $\mathscr{A}$ is the set of all Borel sets of $X$. In [1], Amini et al. generalized, simultaneously, the ring of real-valued continuous functions and $M(X,\mathscr{A})$. In [17], Momtahan studied the Goldie dimension of $M(X,\mathscr{A})$. In [2], Azadi et al. proved that $M(X,\mathscr{A})$ is an $\aleph_0$-self-injective ring. In [9], Estaji et al. have given several characterizations of maximal ideals of $M(X,\mathscr{A})$, mostly in terms of certain lattice-theoretic properties of $\mathscr{A}$. In [7], Estaji and Mahmoudi Darghadam investigated rings of real measurable functions vanishing at infinity on a measurable space and in [8], they introduced realcompact subrings of $M(X,\mathscr{A})$, and showed that $M(X,\mathscr{A})^*$ is a realcompact subring of $M(X,\mathscr{A})$, and also $M(X,\mathscr{A})$ is realcompact if and only if $(X,\mathscr{A})$ is a compact measurable space, i.e., $\mathscr{A}$ is a compact lattice.

In this article we are going to define a topology on $M(X,\mathscr{A})$, namely the $m$-topology and to study the space $M(X,\mathscr{A})$ with this topology. In [13, pp. 48-51, 73-74], Hewitt defined the $m$-topology on $C(X)$, the ring of all real valued continuous functions on a completely regular space $X$, by taking the sets of the form

$$\{f \in C(X)\colon |g - f| \leq u\},$$

as a base for the neighborhood system at $g$, where $u$ is a positive unit of $C(X)$, see [10]. He showed that $X$ is pseudocompact if and only if $C_m(X)$, the space $C(X)$ with the $m$-topology, is first countable. In [3], Azarpanah et al. studied compactness in $C(X)$ with the $m$-topology and they proved that every compact subset of $C_m(X)$ has an empty interior. In [5], Azarpanah et al. proved that $q(X)$ with the $m$-topology is connected if and only if $X$ is a pseudocompact almost $P$-space, if and only if $C(X)$ with $r$-topology is connected, where $q(X)$ is the classical ring of quotients of $C(X)$ and the $r$-topology is defined the same as the $m$-topology if we consider positive regular functions instead of positive units and the inequality holds on the cozero-set of the regular function, see [6] and [14]. For any ideal $I \subseteq C_\psi(X)$, Azarpanah et al. in [4] defined a topology on $C(X)$ namely the $m^I$-topology, finer than the $m$-topology in which the component of 0 is exactly the ideal $I$ and $C(X)$ with this topology becomes a topological ring. They showed that compact sets in $C(X)$ with the $m^I$-topology have empty interior if and only if $X \setminus \bigcap Z[I]$ is infinite. For every two subsets $A, B \subseteq X$

such that $A \cup B = X$, Manshoor in [16] defined a topology on $C(X)$ namely the $m_{(A,B)}$-topology, finer than the $m$-topology and $C(X)$ with this topology becomes a topological ring. Connectedness in this space is studied and it is shown that if $A$ and $B$ are closed realcompact subsets of X, then the component of the zero function in $C(X)$ with $m_{(A,B)}$-topology is the ideal $C_K(X)$. In [15], Di Maio et al. analyzed the position of the Krikorian topology with respect to the topology of uniform convergence, and stated necessary and sufficient conditions which force inclusions and coincidence. In [20], Douwen et al. and in [11] Gómez-Pérez and McGovern investigated topological properties of $X$ via properties of $C_m(X)$.

The paper is organized as follows:

In the second section we provide some necessary preliminaries about the $m$-topology on $M(X, \mathscr{A})$.

In Section 3, we show that $M_m(X, \mathscr{A})$ is a completely metrizable space and is a topological ring. We have shown in this section that no point of $M_m(X, \mathscr{A})$ is an almost $P$-point and $M_m(X, \mathscr{A})$ is never a pseudocompact and also it is never countably compact. Moreover we prove that if $(X, \mathscr{A})$ be a $T$-measurable space and $X$ is a finite set with $|X| = n$, then $M_m(X, \mathscr{A}) \cong \mathbb{R}^n$ as topological rings, and if $X$ is infinite then every compact subset of $M_m(X, \mathscr{A})$ has empty interior.

In Section 4, we study important open and closed subsets of $M_m(X, \mathscr{A})$, and we prove that an ideal $I$ of $M(X, \mathscr{A})$ is bounded if and only if $I$ is connected subset of $M_m(X, \mathscr{A})$, if and only if $I \subseteq M_\psi(X, \mathscr{A})$. Also we show $f \in M_\psi(X, \mathscr{A})$ if and only if $\varphi_f$ is continuous, where $\varphi_f \colon \mathbb{R} \longrightarrow M_m(X, \mathscr{A})$ is given by $\varphi_f(r) = \mathbf{r}f$ for every $f \in M(X, \mathscr{A})$.

Finally, in Section 5 we prove that the space $(X, \mathscr{A})$ is a pseudocompact measurable space if and only if $M_m(X, \mathscr{A})$ coincides with $M_u(X, \mathscr{A})$ and this is equivalent to the first countability, connectedness or locally connectedness of $M_m(X, \mathscr{A})$. Moreover we show that all of them are equivalent to saying that $M^*(X, \mathscr{A})$ is a connected subset of $M_m(X, \mathscr{A})$.

## 2. Preliminaries

The $m$-topology on $C(X)$ is defined by taking the sets of the form

$$B(f, u) = \{g \in C(X) \colon |f - g| < u\}$$

as a base for the neighborhood system at $f$, for each $f \in C(X)$, where $u$ runs through the set of all positive units of $C(X)$. Also, for every $f \in \mathbb{R}^X$, $Z(f) := \{x \in X : f(x) = 0\}$ is called the zero-set of $f$.

Let us recall some general notation from [19]. A collection $\mathscr{A}$ of subsets of a nonempty set $X$ is said to be a *measurable sets* in $X$ if $\mathscr{A}$ has the following three properties:

(1) $X \in \mathscr{A}$.
( 2) If $A \in \mathscr{A}$, then $A^c \in \mathscr{A}$, where $A^c$ is the complement of $A$ relative to $X$.
(3) If $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{A}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{A}$.

Also, $(X, \mathscr{A})$ is called a *measurable space*. If $X$ is a measurable space, $Y$ is a topological space, and $f$ is a mapping of $X$ into $Y$, then $f$ is said to be *measurable* provided that $f^{-1}(V)$ is a measurable set in $X$ for every open set $V$ in $Y$. If $X$ is a measurable space, then the set of all measurable maps from $X$ into $\mathbb{R}$ is denoted $M(X, \mathscr{A})$, and the members of $M(X, \mathscr{A})$ are called the *real measurable functions* on $X$, where $\mathbb{R}$ denotes the set of all real numbers with the ordinary topology.

*Remark* 2.1. Let $(X, \mathscr{A})$ be a measurable space, we set

$$U(X, \mathscr{A}) := \{g\colon\ g \text{ is a unit element of } M(X, \mathscr{A})\},$$

and

$$U^+(X, \mathscr{A}) := \{g \in U(X, \mathscr{A})\colon\ g \text{ is a positive element of } M(X, \mathscr{A})\}.$$

Suppose that

$$B(f, u) := \{g \in M(X, \mathscr{A})\colon |f - g| < u\}$$

and

$$B_f := \{B(f, u)\colon u \in U^+(X, \mathscr{A})\}$$

for every $f \in M(X, \mathscr{A})$ and every $u \in U^+(X, \mathscr{A})$. Then the following statements hold for every $f, g \in M(X, \mathscr{A})\}$ and every $u, v \in U^+(X, \mathscr{A})$.

(1) $f \in B(f, u)$.
(2) $B(f, u \wedge v) \subseteq B(f, u) \cap B(f, v)$.
(3) If $g \in B(f, u)$ and $v := u - |f - g|$, then $B(f, v) \subseteq B(f, u)$.
(4) For every $g \in B(f, u)$, there exists an element $\nu$ in $U^+(X, \mathscr{A})$ such that $B(g, \nu) \subseteq B(f, u)$.
(5) If $U \in B_f$, then $f \in U$.
(6) If $U, V \in B_f$, then $U \cap V \in B_f$.
(7) If $U \in B_f$, then there is a $V \in B_f$ such that $U \in B_g$ for each $g \in V$.
(8) If $U \in B_f$ and $U \subseteq V \subseteq M(X, \mathscr{A})$, then $V \in B_f$.

If

$$\tau_m := \{\, G \subseteq M(X, \mathscr{A})\colon \forall f \in G\ \ \exists U \in B_f;\ U \subseteq G\,\},$$

then, by [23, Theorem 4.2], $\tau_m$ is a topology on $M(X, \mathscr{A})$ such that the neighborhood system at each $f \in M(X, \mathscr{A})$ is precisely $B_f$. The topology $\tau_m$ on $M(X,\mathscr{A})$ is in fact the $m$-topology on $M(X,\mathscr{A})$, and the notation $M_m(X,\mathscr{A})$ will be used when referring to $M(X,\mathscr{A})$ under the $m$-topology. Also, if

$$\tau_u := \{\, G \subseteq M(X,\mathscr{A})\colon \forall f \in G \;\exists \varepsilon \in \mathbb{R};\; \varepsilon > 0 \text{ and } B(f,\varepsilon) \subseteq G \,\},$$

then, by [23, Theorem 4.2], $\tau_u$ is a topology on $M(X,\mathscr{A})$ which is called the uniform topology (or the $u$-topology) on $M(X,\mathscr{A})$. Throughout this article, the notation $M_u(X,\mathscr{A})$ will be used when referring to $M(X,\mathscr{A})$ under the $u$-topology. $M^*_m(X,\mathscr{A})$ and $M^*_u(X,\mathscr{A})$ are defined similarly.

## 3. Topological properties of space $M_m(X,\mathscr{A})$

If $(X;\rho)$ is a metric space, then by the ball (or spheroid) centered at $x \in X$ and having radius $\delta$ we mean the set

$$S_\rho(x,\delta) := \{y \in X\colon \rho(x,y) < \delta\}.$$

The constant function, on any set, whose constant value is the real number $r$, is denoted by $\mathbf{r}$.

**Proposition 3.1.** *Let $(X,\mathscr{A})$ be a measurable space. Then the following statements hold.*

(1) *$M_u(X,\mathscr{A})$ is a completely metrizable space.*
(2) *$M^*_u(X,\mathscr{A})$ is a completely metrizable space.*
(3) *$M^*_u(X,\mathscr{A})$ is a Banach space.*

*Proof.* (1). Clearly $\rho\colon M(X,\mathscr{A}) \times M(X,\mathscr{A}) \to [0,+\infty)$ defined by

$$\rho(f,g) = \sup_{x\in X} \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}$$

is a metric on $M(X,\mathscr{A})$.

Let $f \in M(X,\mathscr{A})$ and $\varepsilon \in (0,+\infty)$ be given. Then

$$S_\rho\left(f, \frac{\varepsilon}{\mathbf{1}+\varepsilon}\right) \subseteq B(f,\varepsilon),$$

and there exists an element $\varepsilon_0$ in $(0,1)$ with $\varepsilon_0 < \varepsilon$ such that

$$B\left(f, \frac{\boldsymbol{\varepsilon_0}}{\mathbf{1}-\boldsymbol{\varepsilon_0}}\right) \subseteq S_\rho(f,\boldsymbol{\varepsilon_0}) \subseteq S_\rho(f,\varepsilon).$$

Therefore, if $\tau$ is the metric topology induced by $\rho$, then $\tau_u = \tau$.

Suppose $\{f_n\}_{n\in\mathbb{N}} \subseteq M(X,\mathscr{A})$ be a cauchy sequence, so that for every $\varepsilon > 0$, there exists an element $N$ in $\mathbb{N}$ such that

$$\rho(f_n,f_m) = \sup_{x\in X} \frac{|f_n(x)-f_m(x)|}{1+|f_n(x)-f_m(x)|} < \frac{\varepsilon}{1+\varepsilon},$$

for every $m, n > N$, which implies that $|f_m(x) - f_n(x)| < \varepsilon$ for every $x \in X$ whenever $m, n > N$. Then, by [18, Theorem 7.8], the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$, defined on $X$, converges uniformly on $X$. We define $f\colon X \to \mathbb{R}$ by $f(x) = \lim_{n\to\infty} f_n(x)$. Then $f \in M(X, \mathscr{A})$ and also, $|f_n - f| < \varepsilon$ whenever $n > N$, which implies that $\rho(f_n, f) < \dfrac{\varepsilon}{\mathbf{1}+\varepsilon}$ for every $n > N$. Therefore, $\{f_n\}_{n\in\mathbb{N}}$ converges to $f$.

(2). If we define $\rho\colon M^*(X, \mathscr{A}) \times M^*(X, \mathscr{A}) \to [0, +\infty)$ given by

$$\rho(f, g) = \sup_{x\in X} |f(x) - g(x)|,$$

then $\rho$ is a metric on $M^*(X, \mathscr{A})$. The rest is similar to the proof of the first statement.

(3). Let $\{f_n\} \subseteq M^*_u(X, \mathscr{A})$ be a cauchy sequence.Then for every $\varepsilon > 0$, there exists an element $k$ in $\mathbb{N}$ such that $\sup_{x\in X} |f_n(x)-f_m(x)| < \varepsilon$, for every $m, n \geq k$ and every $x \in X$. Hence $\{f_n(x)\}_{n\in\mathbb{N}}$ is a cauchy sequence in $\mathbb{R}$, and so $\lim_{n\longrightarrow\infty} f_n(x) = f(x) \in \mathbb{R}$ for every $x \in X$. Consider $n$ be fixed and $m \longrightarrow \infty$. Then we have $\sup_{x\in X} |f_n(x) - f(x)| \leq \varepsilon$. Since $f$ is a bounded measurable function, we conclude that $M^*_u(X, \mathscr{A})$ is a Banach space. □

Recall that a topological ring is a ring $R$ which is also a Hausdorff topological space such that both the addition and the multiplication operations are continuous.

**Proposition 3.2.** *Let $(X, \mathscr{A})$ be a measurable space. Then the ring $M(X, \mathscr{A})$ with the m-topology is a topological ring.*

*Proof.* Let $f, g \in M(X, \mathscr{A})$ and $G \in \tau_m$ with $f + g \in G$ be given. Then there exists an element $u$ in $U^+(X, \mathscr{A})$ such that $B(f + g, u) \subseteq G$. It is clear that for every $(t, k) \in B\left(f, \dfrac{u}{\mathbf{3}}\right) \times B\left(g, \dfrac{u}{\mathbf{3}}\right)$,

$$|f + g - (t + k)| \leq |f - t| + |g - k| < \frac{u}{\mathbf{3}} + \frac{u}{\mathbf{3}} < u.$$

Therefore, the addition operation is continuous .

Let $f, g \in M(X, \mathscr{A})$ and $u \in U^+(X, \mathscr{A})$ with $fg \in B(f + g, u)$ be given. We set $v := \dfrac{u}{\mathbf{2}(\mathbf{1} + |g| + |f| + u)}$. Then $v \in U^+(X, \mathscr{A})$ and for every $(t, k) \in B(f, v) \times B(g, v)$,

$$\begin{aligned} |fg - tk| &\leq |f - t||g| + |f - t||g - k| + |f||g - k| \\ &< (\mathbf{1} + |g| + |f| + u)v \\ &< u. \end{aligned}$$

Hence, the multiplication operation is continuous. The proof is now complete. □

*Remark* 3.3. Let $(X, \mathscr{A})$ be a measurable space. Then

$$\left\{\frac{\mathbf{1}}{\mathbf{n}} \in M(X, \mathscr{A})\colon n \in \mathbb{N}\right\}$$

is a discrete subspace of $M_m(X, \mathscr{A})$.

It is well known that for every $T_1$-space $X$, $X$ is a regular space if and only if for every $x \in X$ and every neighbourhood $U$ of $x$ there exists a neighbourhood $V$ of $x$ such that $\mathrm{cl}_X V \subseteq U$.

**Proposition 3.4.** *$M_m(X, \mathscr{A})$ is a Hausdorff regular space.*

*Proof.* let $f, g \in M(X, \mathscr{A})$ with $f \neq g$ be given. Then there exists an element $x_0$ in $X$ such that $f(x_0) \neq g(x_0)$. Consider $u := \dfrac{1}{2}|f(x_0) - g(x_0)|$. If $h \in B(f, u) \cap B(g, u)$, then

$$\begin{aligned} |f(x_0) - g(x_0)| &\leq |f(x_0) - h(x_0)| + |h(x_0) - g(x_0)| < 2u \\ &= |f(x_0) - g(x_0)|, \end{aligned}$$

which is a contradiction. Therefore, $M_m(X, \mathscr{A})$ is a Hausdorff space.

Let $f \in M(X, \mathscr{A})$ and a neighbourhood $G$ of $f$ in $M_m(X, \mathscr{A})$ be given. Then there exists an element $u \in U^+(X, \mathscr{A})$ such that $B(f, u) \subseteq G$. If $g \in \mathrm{cl}_m B\left(f, \dfrac{u}{\mathbf{2}}\right)$, then there exists an element $h$ in $B\left(g, \dfrac{u}{\mathbf{2}}\right) \cap B\left(f, \dfrac{u}{\mathbf{2}}\right)$, which implies that

$$|f - g| \leq |f - h| + |h - g| < \frac{u}{\mathbf{2}} + \frac{u}{\mathbf{2}} = u.$$

Therefore, $f \in B\left(f, \dfrac{u}{\mathbf{2}}\right) \subseteq \mathrm{cl}_m B\left(f, \dfrac{u}{\mathbf{2}}\right) \subseteq G$, and the proof is now complete. □

Recall that a point $x$ of a topological space $X$ is called an almost $P$-point of $X$ if, whenever $x \in Z(f)$ for some $f \in C(X)$, it follows that $x \in \mathrm{cl}_X \mathrm{int}_X Z(f)$. Equivalently, $x \in X$ is an almost $P$-point if and only if every $G_\delta$-set containing $x$ has nonempty interior.

**Proposition 3.5.** *Let $(X, \mathscr{A})$ be a measurable space. $M_m(X, \mathscr{A})$ does not contain any almost $P$-point.*

*Proof.* Let $f \in M(X, \mathscr{A})$ be given. Then $\bigcap_{n \in \mathbb{N}} B\left(f, \frac{\mathbf{1}}{\mathbf{n}}\right) = \{f\}$ is a $G_\delta$-set in $M_m(X, \mathscr{A})$. Since $\dfrac{u}{\mathbf{2}} + f \in B(f, u)$ for every $u \in U^+(X, \mathscr{A})$, we conclude that $\{f\}$ is not open, which implies that $\mathrm{int}_m\{f\} = \emptyset$. Therefore, $f$ is not an almost $P$-point, and this completes the proof. □

Recall that a topological space $X$ is said to be pseudocompact if every function in $C(X)$ is bounded and $X$ is countably compact if and only if each countable open cover of $X$ has a finite subcover.

**Proposition 3.6.** *Let $(X, \mathscr{A})$ be a measurable space. Then the following statements hold.*

(1) *$M_m(X, \mathscr{A})$ is never a pseudocompact space.*
(2) *$M_m(X, \mathscr{A})$ is never a countably compact space.*

*Proof.* (1). Let $a \in X$ be given. We define $\eta_a \colon M_m(X, \mathscr{A}) \to \mathbb{R}$ by $\eta_a(f) = f(a)$. Since $\eta_a(g) = g(a) \in \big(f(a) - \varepsilon, f(a) + \varepsilon\big)$ for every $\varepsilon \in (0, \infty)$ and every $g \in B(f, \varepsilon)$, we conclude that $\eta_a$ is a continuous function. Now using $\eta_a(\mathbf{n}) = n$ for every $n \in \mathbb{N}$, we obtain that $\eta_a$ is an unbounded continuous function. Therefore, $M_m(X, \mathscr{A})$ is not a pseudocompact space.

(2). Since every countably compact space is pseudocompact, $M_m(X, \mathscr{A})$ is not countably compact. □

*Remark* 3.7. Let $(X, \mathscr{A})$ be a measurable space. For every $f \in M(X, \mathscr{A})$, there is a unit real-measurable function $u$ in $M(X, \mathscr{A})$ such that $(-\mathbf{1} \vee f) \wedge \mathbf{1} = uf$.

We recall from [22, Theorem 4.5] that the connected component $C$ of zero in a topological ring $A$ is a closed ideal, and $a + C$ is the connected component of $a$ for each $a \in A$. Also, we recall from [9] that a measurable space $(X, \mathscr{A})$ is said to be *$T$-measurable* if whenever $x$ and $y$ are distinct points in $X$, there is a measurable set containing one and not the other.

**Proposition 3.8.** *Let $(X, \mathscr{A})$ be a $T$-measurable space. If $X$ is a finite set with $|X| = n$, then $M_m(X, \mathscr{A}) \cong \mathbb{R}^n$ as topological rings.*

*Proof.* Since $\mathscr{A} = \mathcal{P}(X)$, $M_m(X, \mathscr{A}) = C(X) \cong \mathbb{R}^n$ as topological rings. □

The following example shows that $T$-measurable in proposition 3.8 can not be removed. More generally, whenever $(X, \mathscr{A})$ is not even $T$-measurable and $X$ is finite, then again we have $M_m(X, \mathscr{A}) \cong \mathbb{R}^k$ as topological rings, where $k \leq |X|$.

**Example 3.9.** Let $X = \{1, 2, 3, 4, 5\}$ and

$$\chi_{\{1\}} = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \in \{2, 3, 4, 5\}. \end{cases}$$

be given. Hence, $\mathscr{A}=\{\emptyset,\{1\},\{2,3,4,5\},X\}$ is the smallest $\sigma-$algebra on $X$ such that $\chi_{\{1\}}$ is a measurable function and consequently

$$M(X,\mathscr{A})=\{a\chi_{\{1\}}+b\chi_{\{2,3,4,5\}}\colon a,b\in\mathbb{R}\}\cong\mathbb{R}^2.$$

For the proof of the following proposition we apply the same technique which is used in that of Theorem 4.1 in [3].

**Proposition 3.10.** *Let $(X,\mathscr{A})$ be a measurable space, and let $X$ be an infinite set. If $F$ is a compact subset of $M_m(X,\mathscr{A})$, then* $\mathrm{int}_m F=\emptyset$.

*Proof.* We argue by contradiction. Let us assume that $f\in\mathrm{int}_m F$. Then there exists a positive unit function $u\in M(X,\mathscr{A})$ such that $B(f,u)\subseteq F$. By hypothesis, there exist $g_1,g_2,...,g_m\in F$ such that $F\subseteq\bigcup_{m=1}^{n}B\left(g_m,\frac{u}{\mathbf{4}}\right)$. Let $A:=\{x_1,x_2,...,x_n,x_{n+1}\}\subseteq X$ with $x_i\neq x_j$ for every $1\le i\neq j\le n+1$, be given. We set $t_i:=\frac{u}{2}\chi_{\{x_i\}}$ and $h_i:=f+t_i$ for every $1\le i\le n+1$, then $h_i\in B(f,u)\subseteq\mathrm{int}_m F$, and so, by the "pigeon-hole" principle, there exist $1\le i\neq j\le n+1$ and $1\le s\le n$ such that $h_i,h_j\in B(g_s,\frac{u}{\mathbf{4}})$, which implies that

$$|t_i-t_j|=|h_i-h_j|\le|h_i-g_s|+|h_j-g_s|<\frac{u}{\mathbf{4}}+\frac{u}{\mathbf{4}}=\frac{u}{\mathbf{2}}.$$

Therefore,

$$\frac{u(x_i)}{2}=|t_i(x_i)-t_j(x_i)|<\frac{u}{2}(x_i)$$

and this is a contradiction. □

## 4. Open and closed subsets of $M_m(X,\mathscr{A})$

In this section, we are going to find the largest bounded ideal of $M(X,\mathscr{A})$ which is at the same time the largest connected ideal. Using this find the component of 0 in $M_m(X,\mathscr{A})$.

Let $(X,\mathscr{A})$ be a measurable space. We set

$$M^+(X,\mathscr{A}):=\{f\in M(X,\mathscr{A})\colon f(x)\ge 0 \text{ for all } x\in X\},$$

and

$$M^-(X,\mathscr{A}):=\{f\in M(X,\mathscr{A})\colon f(x)\le 0 \text{ for all } x\in X\}.$$

**Proposition 4.1.** *Let $(X,\mathscr{A})$ be a measurable space. Then the following statements hold.*

1. *The interior of a proper ideal $I$ of $M(X,\mathscr{A})$ in $M_m(X,\mathscr{A})$ is the empty set.*
2. *The set $M^*(X,\mathscr{A})$ is a clopen subset of $M_m(X,\mathscr{A})$.*
3. *The set $U(X,\mathscr{A})$ is a dense open subset of $M_m(X,\mathscr{A})$.*
4. *If $D$ is the set of all zero divisors of $M(X,\mathscr{A})$, then $D=M(X,\mathscr{A})\setminus U(X,\mathscr{A})$.*

(5) *The set* $U^+(X,\mathscr{A})$ *is an open subset of* $M_m(X,\mathscr{A})$.
(6) *The set* $M^+(X,\mathscr{A})$ *coincides with the closure of* $U^+(X,\mathscr{A})$ *in* $M_m(X,\mathscr{A})$.
(7) *The set* $M^-(X,\mathscr{A})$ *coincides with the closure of* $U^-(X,\mathscr{A})$ *in* $M_m(X,\mathscr{A})$.
(8) *The set* $M^+(X,\mathscr{A})\cup M^-(X,\mathscr{A})$ *is contained in the closure of* $U(X,\mathscr{A})$.
(9) *The set* $M^+(X,\mathscr{A})$ *is not an open subset of* $M_m(X,\mathscr{A})$.

*Proof.* (1). We argue by contradiction. Let us assume that $I$ is a proper ideal of $M(X,\mathscr{A})$, and suppose that $f\in M(X,\mathscr{A})$ is in the interior of $I$. Then there exists an element $u$ in $U^+(X,\mathscr{A})$ such that $f+\frac{u}{2}\in B(f,u)\subseteq I$, which implies that $\frac{u}{2}\in I$ and this is a contradiction to the fact that $I$ is a proper ideal of $M(X,\mathscr{A})$.

(2). Since for every $f\in M^*(X,\mathscr{A})$, $f\in B(f,\mathbf{1})\subseteq M^*(X,\mathscr{A})$, we conclude that $M^*(X,\mathscr{A})$ is an open subset of $M_m(X,\mathscr{A})$. Let $f\in M(X,\mathscr{A})$ be in the closure of $M^*(X,\mathscr{A})$. Then there is an element $g$ in $B(f,1)\cap M^*(X,\mathscr{A})$, which implies that there is an element $n$ in $\mathbb{N}$ such that $|f|\leq|g|+\mathbf{1}\leq\mathbf{n}$, and so $f\in M^*(X,\mathscr{A})$. Therefore, $M^*(X,\mathscr{A})$ is a clopen subset of $M_m(X,\mathscr{A})$.

(3). Since for every $u\in U(X,\mathscr{A})$, $B\left(u,\dfrac{|u|}{\mathbf{2}}\right)\subseteq U(X,\mathscr{A})$, we conclude that $U(X,\mathscr{A})$ is an open subset of $M_m(X,\mathscr{A})$.

Let $f\in M(X,\mathscr{A})$ and $u\in U^+(X,\mathscr{A})$ be given. We define $h\colon X\to\mathbb{R}$ by

$$h(x):=\begin{cases} f(x)+\frac{u(x)}{2} & \text{if } f(x)\geq 0\\ f(x)-\frac{u(x)}{2} & \text{if } f(x)<0.\end{cases}$$

Then $h\in U(X,\mathscr{A})\cap B(f,u)\neq\emptyset$. Therefore, $U(X,\mathscr{A})$ is a dense subset of $M_m(X,\mathscr{A})$.

(4). Let $\mathbf{0}\neq f\in M(X,\mathscr{A})\setminus U(X,\mathscr{A})$ be given. Since $\chi_{Z(f)}f=\mathbf{0}$ and $\mathbf{0}\neq\chi_{Z(f)}\in M(X,\mathscr{A})$, we conclude that $f\in D$. Therefore, $D=M(X,\mathscr{A})\setminus U(X,\mathscr{A})$, because $D\subseteq M(X,\mathscr{A})\setminus U(X,\mathscr{A})$.

(5). If $u\in U^+(X,\mathscr{A})$, then $B\left(u,\frac{u}{\mathbf{2}}\right)\subseteq U^+(X,\mathscr{A})$, which implies that $U^+(X,\mathscr{A})$ is an open subset of $M_m(X,\mathscr{A})$.

(6). If $f\in M^+(X,\mathscr{A})$ and $u\in U^+(X,\mathscr{A})$, then

$$f+\frac{u}{\mathbf{2}}\in U^+(X,\mathscr{A})\cap B(f,u),$$

which implies that $M^+(X,\mathscr{A})\subseteq \mathrm{cl}_m U^+(X,\mathscr{A})$. Let $g\in M(X,\mathscr{A})$ and $a\in X$ with $g(a)<0$ be given. We put $r=\frac{1}{2}|g(a)|>0$. If

$h \in B(g, \mathbf{r}) \cap U^+(X, \mathscr{A})$, then

$$|g(a) - h(a)| \geq |g(a)| = 2r > r,$$

which is a contradiction. Therefore, $M^+(X, \mathscr{A}) = \mathrm{cl}_m U^+(X, \mathscr{A})$.

(7). The proof is similar to the proof of part (6).

(8). It is clear.

(9). Let $f \in M^+(X, \mathscr{A}) \setminus U^+(X, \mathscr{A})$ be given. Then $B(f, u) \not\subseteq M^+(X, \mathscr{A})$ for every $u \in U^+(X, \mathscr{A})$. Therefore, $M^+(X, \mathscr{A})$ is not an open subset of $M_m(X, \mathscr{A})$. □

Recall that a space $X$ is said to be extremally disconnected if every open set has an open closure.

As an immediate consequence we now have the following proposition.

**Proposition 4.2.** *Let $(X, \mathscr{A})$ be a measurable space. Then $M_m(X, \mathscr{A})$ is not an extremally disconnected space.*

Recall that an element $c$ of a lattice $L$ is said to be compact if for any $S \subseteq L$, $c \leq \bigvee S$ implies $c \leq \bigvee T$ for some finite $T \subseteq S$.

**Definition 4.3.** Let $(X, \mathscr{A})$ be a measurable space. We denote by $M_\infty(X, \mathscr{A})$ the family of all functions $f \in M(X, \mathscr{A})$ for which the set $\left\{x \in X \colon |f(x)| \geq \frac{1}{n}\right\}$ is a compact element of $\mathscr{A}$ for every $n \in \mathbb{N}$.

**Proposition 4.4.** *Let $(X, \mathscr{A})$ be a measurable space. Then $M_\infty(X, \mathscr{A})$ is a closed subset of $M_m(X, \mathscr{A})$.*

*Proof.* Let $f \in \mathrm{cl}_m M_\infty(X, \mathscr{A})$ and $n \in \mathbb{N}$ be given. Then there exists an element $g$ in $B\left(f, \frac{1}{\mathbf{2}^n}\right) \cap M_\infty(X, \mathscr{A})$, which implies that for every $x \in X$,

$$|f(x)| \geq \frac{1}{n} \Rightarrow |g(x)| \geq |f(x)| - \frac{1}{2^n} \geq \frac{1}{n} - \frac{1}{2^n} \geq \frac{1}{2^n},$$

that is

$$\left\{x \in X \colon |f(x)| \geq \frac{1}{n}\right\} \subseteq \left\{x \in X \colon |g(x)| \geq \frac{1}{2^n}\right\},$$

and since $g \in M_\infty(X, \mathscr{A})$, we conclude that $\{x \in X \colon |f(x)| \geq \frac{1}{n}\}$ is a compact element of the lattice $\mathscr{A}$. Therefore, $f \in M_\infty(X, \mathscr{A})$ and the proof is now complete. □

To prove the main results of this section, we need the following lemma.

**Lemma 4.5.** *If $g$ is an element of $M(X,\mathscr{A})$, then the set*

$$A_g := \{f \in M(X,\mathscr{A})\colon\ fg \text{ is a bounded element of } M(X,\mathscr{A})\ \}$$

*is a clopen subset of $M_m(X,\mathscr{A})$.*

*Proof.* Let $f \in A_g$ be given. Since for every $h \in B\left(f, \frac{\mathbf{1}}{\mathbf{1}+|g|}\right)$, $|hg| < |fg| + \frac{|g|}{\mathbf{1}+|g|}$, we conclude that $B\left(f, \frac{\mathbf{1}}{\mathbf{1}+|g|}\right) \subseteq A_g$. Therefore, $A_g$ is an open subset of $M_m(X,\mathscr{A})$.

Let $f \in \mathrm{cl}_m A_g$ be given. Then there exists an element $h$ in $B\left(f, \frac{\mathbf{1}}{\mathbf{1}+|g|}\right) \cap A_g$, and since $|fg| < |hg| + \frac{|g|}{\mathbf{1}+|g|}$, we conclude that $f \in A_g$. Therefore, $A_g$ is a clopen subset of $M_m(X,\mathscr{A})$. □

Throughout this article we assume that

$$U_{\{x_n\}} := \{u \in U^+(X,\mathscr{A})\colon\ \lim_{n\to\infty} u(x_n) = 0\}$$

and

$$A_{\{x_n\}} := \{f \in M(X,\mathscr{A})\colon |f| < u \text{ for some } u \in U_{\{x_n\}}\ \}$$

for every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$. It is obvious that for every $f, g \in U_{\{x_n\}}$, $f+g \in U_{\{x_n\}}$ and $fg \in U_{\{x_n\}}$.

To prove Proposition 4.13, we need the following lemma.

**Lemma 4.6.** *For every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, $A_{\{x_n\}}$ is a clopen subset of $M_m(X,\mathscr{A})$.*

*Proof.* Let $f \in A_{\{x_n\}}$ be given. Then there exists an element $u$ in $U_{\{x_n\}}$ such that $|f| < u$, which implies that for every $v \in U_{\{x_n\}}$,

$$h \in B(f,v) \Rightarrow |h| < |f| + v < u + v \in U_{\{x_n\}} \Rightarrow h \in A_{\{x_n\}}.$$

Hence $A_{\{x_n\}}$ is an open subset of $M_m(X,\mathscr{A})$.

Let $f \in \mathrm{cl}_m A_{\{x_n\}}$ and $v \in U_{\{x_n\}}$ be given. Then there exists an element $h$ in $B(f,v) \cap A_{\{x_n\}}$, which implies that there exists an element $u \in U_{\{x_n\}}$ such that $|h| < u$, and so $|f| < |h| + v < u + v \in U_{\{x_n\}}$, i.e., $f \in A_{\{x_n\}}$. Therefore, $A_{\{x_n\}}$ is a clopen subset of $M_m(X,\mathscr{A})$. □

*Remark* 4.7. Let $(X,\mathscr{A})$ be a measurable space. Since $M_m(X,\mathscr{A})$ is a topological ring, then:

(1) If $A$ is an open subset of $M_m(X,\mathscr{A})$, then $f + A$ is an open subset of $M_m(X,\mathscr{A})$ for every $f \in M(X,\mathscr{A})$.
(2) If $A$ is a connected subset of $M_m(X,\mathscr{A})$ with $\mathbf{0} \in A$, then $f + A$ is a connected subset of $M_m(X,\mathscr{A})$ for every $f \in M(X,\mathscr{A})$.

**Proposition 4.8.** *Let $(X, \mathscr{A})$ be a measurable space. If $f \in M(X, \mathscr{A})$ is an unbounded measurable function, then there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that for every function $g\colon \{x_n\}_{n\in\mathbb{N}} \to \mathbb{R}$, there exists an element $\bar{g}$ in $M(X, \mathscr{A})$ with $\bar{g}(x_n) = g(x_n)$ and $f(x_n) > n$ for every $n \in \mathbb{N}$.*

*Proof.* By induction, define a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ and a family $\{A_n\}_{n\in\mathbb{N}} \in \mathscr{A}$ such that

$$A_1 \supsetneqq A_2 \supsetneqq A_3 \supsetneqq \cdots \supsetneqq A_k$$

and

$$x_k \in A_k \setminus \bigcup_{j=1}^{k-1} A_j$$

for every $k \in \mathbb{N}$ as follows. By hypothesis, there exists an element $x_1$ in $X$ such that $|f(x_1)| > 1$. We set $A_1 := f^{-1}\big((-\infty, -1) \cup (1, \infty)\big)$. If $n > 1$, then, by hypothesis, there exists an element $x_n$ in $X$ such that $|f(x_n)| > \max\{n, |f(x_{n-1})|\}$. We set

$$A_n := f^{-1}\big((-\infty, -\max\{n, |f(x_{n-1})|\}) \cup (\max\{n, |f(x_{n-1})|\}, \infty)\big).$$

Let $g\colon \{x_n\}_{n\in\mathbb{N}} \to \mathbb{R}$ be a function. We set $B_n := A_n \setminus \bigcup_{j=n+1}^{\infty} A_j$ for every $n \in \mathbb{N}$. It is clear that $A_1 = \bigcup_{n=0}^{\infty} B_n$ and $B_n \cap B_m = \emptyset$ for every $n, m \in \mathbb{N}$ with $m \neq n$. We define $\bar{g}\colon X \longrightarrow \mathbb{R}$ by

$$\bar{g}(x) = \begin{cases} g(x_n) & \text{if } x \in B_n \text{ for some } n \in \mathbb{N} \\ 1 & x \in X \setminus A_1 \end{cases}$$

Then $\bar{g} \in M(X, \mathscr{A})$ and $\bar{g}(x_n) = g(x_n)$ for every $n \in \mathbb{N}$. □

For a measurable space $(X, \mathscr{A})$, the subset $M^*(X, \mathscr{A})$ of $M(X, \mathscr{A})$, consisting of all bounded functions in $M(X, \mathscr{A})$, is also closed under the algebraic and order operations on $M(X, \mathscr{A})$. A measurable space $(X, \mathscr{A})$ is said to be pseudocompact if every function in $M(X, \mathscr{A})$ is bounded, i.e., $M^*(X, \mathscr{A}) = M(X, \mathscr{A})$.

For a measurable space $(X, \mathscr{A})$ and a non-empty subset $Y$ of $X$, we set

$$\mathscr{A}_Y := \{V \cap Y : V \in \mathscr{A}\},$$

and $(Y, \mathscr{A}_Y)$ is a measurable space.

**Proposition 4.9.** *If a measurable space $(X, \mathscr{A})$ is not pseudocompact, then there exists an infinite countable subset $A$ of $X$ such that*

1. *for every function $f\colon A \to \mathbb{R}$, there exists an element $\bar{f}$ in $M(X, \mathscr{A})$ such that $\bar{f}(x) = f(x)$ for every $x \in A$, and*
2. *if $f\colon A \to \mathbb{R}$ is a function with $Z(f) = \emptyset$, then there exists an element $\bar{f}$ in $U(X, \mathscr{A})$ such that $\bar{f}(x)f(x) = 1$ for every $x \in A$.*

*Proof.* The proof is similar to the proof of Proposition 4.8. □

**Definition 4.10.** Let $(X,\mathscr{A})$ be a measurable space. We denote by $M_\psi(X,\mathscr{A})$ the family of all functions $f \in M(X,\mathscr{A})$ for which $(\operatorname{coz}(f), \mathscr{A}_{\operatorname{coz}(f)})$ is a pseudocompact measurable space. It is clear that $M_\psi(X,\mathscr{A}) \subseteq M^*(X,\mathscr{A})$.

As in Proposition 3.11 in [3], we define $\varphi_f\colon \mathbb{R} \longrightarrow M_m(X,\mathscr{A})$ given by $\varphi_f(r) = \mathbf{r}f$ for every $f \in M(X,\mathscr{A})$.

The following lemma is the counterpart of Proposition 3.11 in [3].

**Lemma 4.11.** *Let $(X,\mathscr{A})$ be a measurable space. Then $f \in M_\psi(X,\mathscr{A})$ if and only if $\varphi_f$ is continuous.*

*Proof. Necessity.* Let $u \in U^+(X,\mathscr{A})$ be given. By hypothesis, there exists an element $\alpha$ in $\mathbb{R}$ such that $u(x) > \alpha > 0$ for every $x \in \operatorname{coz}(f)$, and also, there exists an element $t$ in $\mathbb{N}$ such that $|f| < t$. We claim that

$$(r - \frac{\alpha}{t}, r + \frac{\alpha}{t}) \subseteq \varphi_f^{-1}\big(B(\mathbf{r}f, u)\big)$$

Let $s \in (r - \frac{\alpha}{t}, r + \frac{\alpha}{t})$ be given. If $x \in \operatorname{coz}(f)$, then

$$|\mathbf{r}f - \varphi_f(s)|(x) = |\mathbf{r}f - \mathbf{s}f|(x) = |r-s||f(x)| < \frac{\alpha}{t}|f(x)| < \alpha < u(x),$$

and if $x \in Z(f)$, then

$$|\mathbf{r}f - \varphi_f(s)|(x) = |\mathbf{r}f - \mathbf{s}f|(x) = 0 < u(x).$$

This proves the claim. Therefore, $\varphi_f$ is continuous.

*Sufficiency.* We argue by contradiction. Let us assume that $\varphi_f$ is continuous and $f \notin M_\psi(X,\mathscr{A})$ for some $f \in M(X,\mathscr{A})$. Then, by Proposition 4.9, there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \operatorname{coz}(f)$ such that for every function $f\colon \{x_n\}_{n\in\mathbb{N}} \to \mathbb{R}$, there exists an element $\bar{f}$ in $M(\operatorname{coz}(f), \mathscr{A}_{\operatorname{coz}(f)})$ such that $\bar{f}(x_n) = f(x_n)$ for every $n \in \mathbb{N}$. To get a contradiction, we need to consider two cases:

Case 1: Suppose that there exists a positive real number $\alpha$ such that $|f(x_n)| > \alpha$ for every $n \in \mathbb{N}$. We define $g\colon \{x_n\}_{n\in\mathbb{N}} \to \mathbb{R}$ by $g(x_n) = n$. Then there exists an element $\bar{g}$ in $M(\operatorname{coz}(f), \mathscr{A}_{\operatorname{coz}(f)})$ such that $\bar{g}(x_n) = g(x_n) = n$, by Proposition 4.8.

Now, we define $h\colon X \to \mathbb{R}$ by

$$h(x) = \begin{cases} \bar{g}(x) & \text{if } x \in \operatorname{coz}(f) \\ 0 & \text{if } x \in Z(f). \end{cases}$$

Then $h \in M(X,\mathscr{A})$ and $h(x_n) = n$ for every $n \in \mathbb{N}$. Let $s \in (0,+\infty)$ such that $\varphi_f\big((-s,s)\big) \subseteq B(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{1}+|\mathbf{h}|})$. Then $|t|\alpha < |tf(x_n)| \lneqq$

$\frac{1}{1+|h(x_n)|} = \frac{1}{1+n}$ for every $t \in (-s,s)$ and every $n \in \mathbb{N}$, which implies that $\lim_{n\to\infty} \frac{1}{1+n} = 0 \geq |t|\alpha > 0$ and this is a contradiction.

Case 2: Suppose that for every $i \in \mathbb{N}$, there is $x_{n_i} \in \{x_n\}$ such that $|f(x_{n_i})| < \frac{1}{i+1}$, then $\lim_{i\to\infty} f(x_{n_i}) = 0$. We define $g\colon \{x_n\} \to \mathbb{R}$ by

$$g(x_n) = \begin{cases} \frac{1-f^2(x_n)}{f^2(x_n)} & \text{if } n \in \{n_i\}_{i\in\mathbb{N}} \\ 0 & \text{if } n \notin \{n_i\}_{i\in\mathbb{N}} \end{cases}$$

Then there exists an element $\bar{g}$ in $M(\operatorname{coz}(f), \mathscr{A}_{\operatorname{coz}(f)})$ such that $\bar{g}(x_n) = g(x_n)$. Now, we define $h\colon X \to \mathbb{R}$ by

$$h(x) = \begin{cases} \bar{g}(x) & \text{if } x \in \operatorname{coz}(f) \\ 0 & \text{if } x \in z(f). \end{cases}$$

Then $h \in M(X, \mathscr{A})$ and

$$\frac{1}{1+|h(x_{n_i})|} = \frac{1}{1+|\frac{1-f^2(x_{n_i})}{f^2(x_{n_i})}|} = \frac{f^2(x_{n_i})}{f^2(x_{n_i}) + (1-f^2(x_{n_i}))} = f^2(x_{n_i})$$

for every $i \in \mathbb{N}$. Let $s \in (0,+\infty)$ such that $\varphi_f\big((-s,s)\big) \subseteq B\left(\mathbf{0}, \frac{1}{1+|h|}\right)$, then $|tf(x_{n_i})| \lneqq \frac{1}{1+|h(x_{n_i})|} = f^2(x_{n_i})$ for every $t \in (-s,s)$ and every $i \in \mathbb{N}$, which implies that $|t| \leq \lim_{i\to\infty} |f(x_{n_i})| = 0$, i.e., $(-s,s) = \{0\}$, which is a contradiction. □

**Definition 4.12.** Let $(X, \mathscr{A})$ be a measurable space. An ideal $I$ of $M(X, \mathscr{A})$ is called bounded if every element of $I$ is bounded, i.e., $I \subseteq M^*(X, \mathscr{A})$.

By the following result, $M(X, \mathscr{A})$ is the largest bounded and connected ideal of $M(X, \mathscr{A})$.

**Proposition 4.13.** *Let $I$ be an ideal of $M(X, \mathscr{A})$. Then the following statements are equivalent.*

1. *$I$ is a connected subset of $M_m(X, \mathscr{A})$.*
2. *$I$ is bounded.*
3. *$I \subseteq M_\psi(X, \mathscr{A})$.*

*Proof.* (1) ⇒ (2). We argue by contradiction. Let us assume that $f \in I$ is an unbounded function, then there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \operatorname{coz}(f)$ such that $\lim_{n\to\infty} f(x_n) = \infty$, which implies that $f \in I \setminus A_{\{x_n\}}$. Since $\mathbf{0} \in I \cap A_{\{x_n\}}$, we conclude from Lemma 4.6 that $I$ is a disconnected subset of $M_m(X, \mathscr{A})$, which is a contradiction.

(2) ⇒ (3). By way of contradiction assume that $f \in I \setminus M_\psi(X, \mathscr{A})$, then, by Proposition 4.9, there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \operatorname{coz}(f)$ such

that $\lim_{n\to\infty} f(x_n) = \infty$ and there exists an element $\bar{g}$ in $M(X, \mathscr{A})$ such that $\bar{g}(x_n)f(x_n) = 1$ for every $n \in \mathbb{N}$. It is clear that $\bar{g}f^2$ is an unbounded function and $\bar{g}f^2 \in I$, which is a contradiction.

(3) $\Rightarrow$ (1). Since the continuous image of a connected space is connected, we conclude from Lemma 4.11 that $\varphi_f(\mathbb{R})$ is a connected subset of $M_m(X, \mathscr{A})$ for every $f \in I$. It is clear that $\mathbf{0} \in \bigcap_{f\in I} \varphi_f(\mathbb{R})$ and $I = \bigcup_{f\in I} \varphi_f(\mathbb{R})$, then $I$ is a connected subset of $M_m(X, \mathscr{A})$. □

As an immediate consequence from Proposition 4.13 we now have the following corollary.

**Corollary 4.14.** *Let $(X, \mathscr{A})$ be a measurable space. Then $M_\psi(X, \mathscr{A})$ is the largest bounded ideal in $M(X, \mathscr{A})$.*

**Proposition 4.15.** *Let $(X, \mathscr{A})$ be a measurable space, and $f \in M(X, \mathscr{A})$. Then $f \in M_\psi(X, \mathscr{A})$ if and only if $fg$ is a bounded function for every $g \in M(X, \mathscr{A})$.*

*Proof.* If $f \in M_\psi(X, \mathscr{A})$, then $fg \in M_\psi(X, \mathscr{A})$ for all $f \in M(X, \mathscr{A})$, so $fg$ is bounded. Conversely $f = f.1$ is bounded, hence $f \in M_\psi(X, \mathscr{A})$. □

**Corollary 4.16.** *A measurable space $(X, \mathscr{A})$ is pseudocompact if and only if every ideal of $M(X, \mathscr{A})$ is a connected subset of $M_m(X, \mathscr{A})$.*

*Proof.* By Proposition 4.13, it is clear. □

Also, the following proposition that was proved in [7] is needed in this paper.

**Proposition 4.17.** *The following statements are equivalent.*

(1) *The measurable space $(X, \mathscr{A})$ is compact .*
(2) *The set $X$ is a finite set and $\mathscr{A} = \mathcal{P}(X)$.*
(3) *The measurable space $(X, \mathscr{A})$ is pseudocompact .*

As an immediate consequence from Proposition 4.17 we now have the following corollary.

**Corollary 4.18.** *For measurable space $(X, \mathscr{A})$, $f \in M_\psi(X, \mathscr{A})$ if and only if $\operatorname{coz}(f)$ is a compact element of $\mathscr{A}$ for every $f \in M(X, \mathscr{A})$.*

We recall from [9] that an ideal $I$ of $M(X, \mathscr{A})$ is called fixed if the set $\bigcap_{f\in I} Z(f)$ is nonempty; otherwise, $I$ is called free.

The follwing results which is the counterpart of Corollary 3.9 in [3] show the connection between free maximal ideals of $M(X, \mathscr{A})$ and connected ideals of $M_m(X, \mathscr{A})$.

**Proposition 4.19.** *Let $(X, \mathscr{A})$ be not a pseudocompact measurable space, and assume that $I$ is a proper ideal of $M(X, \mathscr{A})$. Then $I$ is a connected subset of $M_m(X, \mathscr{A})$ if and only if $I \subseteq M$ for every free maximal ideal $M$ of $M(X, \mathscr{A})$.*

*Proof. Necessity.* Let $J$ be a free ideal of $M(X, \mathscr{A})$, and let $f \in I$ be given. Then for every $x \in \operatorname{coz}(f)$, there exists an element $f_x$ in $J$ such that $x \in \operatorname{coz}(f_x)$, which implies that $\operatorname{coz}(f) \subseteq \bigcup_{x \in \operatorname{coz}(f)} \operatorname{coz}(f_x)$. Since, by Proposition 4.13 and Corollary 4.18, $\operatorname{coz}(f)$ is a compact element of $\mathscr{A}$, there are $x_1, \ldots, x_n \in \operatorname{coz}(f)$ such that

$$\operatorname{coz}(f) \subseteq \bigcup_{i=1}^{n} \operatorname{coz}(f_{x_i}) = \operatorname{coz}\left(\sum_{i=1}^{n} f_{x_i}^2\right),$$

and so there exists an element $h$ in $M(X, \mathscr{A})$ such that

$$f = h\left(\sum_{i=1}^{n} f_{x_i}^2\right) \in J.$$

*Sufficiency.* Let $f \in I$ be given. Hence, by Proposition 4.13 and Corollary 4.18, it suffices to prove that $\operatorname{coz}(f)$ is a compact element of $\mathscr{A}$. If not, then there exists an element $g$ in $M(\operatorname{coz}(f), \mathscr{A}_{\operatorname{coz}(f)})$ such that $g \notin M^*(\operatorname{coz}(f), \mathscr{A}_{\operatorname{coz}(f)})$, and so the map $h\colon X \to \mathbb{R}$ by

$$h(x) = \begin{cases} |g(x)| + 1 & \text{if } x \in \operatorname{coz}(f) \\ 0 & \text{if } x \in X \setminus \operatorname{coz}(f) \end{cases}$$

belongs to $M(X, \mathscr{A}) \setminus M^*(X, \mathscr{A})$. We put $V_n := \{x \in X\colon |h(x)| \geq n\}$ for every $n \in \mathbb{N}$. Since $\{V_n\colon n \in \mathbb{N}\} \subseteq \mathscr{A}$ has the finite intersection property, we conclude that there exists a free ultrafilter $\mathcal{F}$ of $\mathscr{A}$ such that $\{V_n\colon n \in \mathbb{N}\} \subseteq \mathcal{F}$, which implies from [9, Proposition 3.6] that

$$M := \{f \in M(X, \mathscr{A})\colon Z(f) \in \mathcal{F}\}$$

is a free maximal ideal of $M(X, \mathscr{A})$, and by our hypothesis, we infer that $f \in M$. Therefore, $\emptyset = Z(f) \cap \operatorname{coz}(f) = Z(f) \cap V_1 \in \mathcal{F}$ and this is a contradiction to the fact that $\emptyset \notin \mathcal{F}$. □

**Proposition 4.20.** *Let $(X, \mathscr{A})$ be a measurable space, and let $I$ be an ideal of $M(X, \mathscr{A})$. Then the following statements hold.*

(1) *If $I$ is a Lindelöf subspace of $M_m(X, \mathscr{A})$, then $I \subseteq M_\psi(X, \mathscr{A})$.*
(2) *If $I \neq \{\mathbf{0}\}$, then $I$ is not a compact subset of $M_m(X, \mathscr{A})$.*

*Proof.* (1). By hypothesis, there exists a family $\{B(f_n, u_n)\}_{n\in\mathbb{N}}$ such that

$$I \subseteq \bigcup_{n=1}^{\infty} B(f_n, u_n).$$

We argue by contradiction. Let us assume that there exists an element $f$ in $I$ such that $f \notin M_\psi(X, \mathscr{A})$, which implies from Lemma 4.8 that there is a sequence $A := \{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that for every function $g\colon \{x_n\}_{n\in\mathbb{N}} \to \mathbb{R}$, there exists an element $\bar{g}$ in $M(X, \mathscr{A})$ such that $\bar{g}(x_n) = g(x_n)$ and $f(x_n) > n$ for every $n \in \mathbb{N}$. We define $g\colon \{x_n\}_{n\in\mathbb{N}} \to \mathbb{R}$ by $g(x_n) = |f_n(x_n)| + u_n(x_n)$, and so there exists a $\bar{g} \in M(X, \mathscr{A})$ such that $\bar{g}|_A = g$. We can assume that there exists an element $m$ in $\mathbb{N}$ such that $\bar{g}f \in B(f_m, u_m)$, which implies that $|\bar{g}f| < |f_m| + u_m$, and so

$$|f_m(x_m)| + u_m(x_m) = |\bar{g}(x_m)| < |f(x_m)\bar{g}|(x_m) < |f_m(x_m)| + u_m(x_m),$$

which is a contradiction.

(2). We proceed by contradiction. Assume that there exist

$$f_1, f_2, ..., f_n \in M(X, \mathscr{A})$$

and $u \in U^+(X, \mathscr{A})$ such that $I \subseteq \bigcup_{i=1}^{n} B(f_i, u)$. Let $a \notin \bigcap_{f\in I} Z(f)$ and $g \in I$ with $g(a) \neq 0$ be given. We set $b := \max\{|f_1(a)| + u(a), \ldots, f_n(a)| + u(a)\}$ and $f := \mathbf{b}\big(\mathbf{g}(\mathbf{a})\big)^{-1} g$, then there is a natural number $k$ such that $f \in B(f_k, u)$, which implies that $b = |f(a)| < |f_k(a)| + u(a)$, and this is a contradiction with choose $b$. □

**Example 4.21.** The following statements hold.

1. The ideal $\{f \in M_m\big(\mathbb{N}, \mathcal{P}(\mathbb{N})\big) \colon f(1) = 0\}$ of $M\big(\mathbb{N}, \mathcal{P}(\mathbb{N})\big)$ is not a Lindelöf subspace of $M_m\big(\mathbb{N}, \mathcal{P}(\mathbb{N})\big)$.
2. The set $\{f \in M\big(\mathbb{N}, \mathcal{P}(\mathbb{N})\big) \colon \mathbb{N} \setminus \{n\} \subseteq Z(f)\}$ is a Lindelöf ideal of $M_m\big(\mathbb{N}, \mathcal{P}(\mathbb{N})\big)$.
3. If $I_n := \{f \in M\big(\mathbb{N}, \mathcal{P}(\mathbb{N})\big) \colon \mathbb{N} \setminus \{n\} \subseteq Z(f)\}$ for every natural number $n$, then $I_{n_1} + \cdots + I_{n_k}$ is a Lindelöf ideal of $M_m\big(\mathbb{N}, \mathcal{P}(\mathbb{N})\big)$ for every natural number $k$.
4. $M_\psi\big(\mathbb{N}, \mathcal{P}(\mathbb{N})\big)$ is a Lindelöf ideal of $M_m\big(\mathbb{N}, \mathcal{P}(\mathbb{N})\big)$.

We recall from [23, 26B] that the quasicomponent of $x$ in a space $X$ is the intersection of all clopen subsets of $X$ which contain $x$, and the component of $x$ is contained in the quasicomponent of $x$.

**Proposition 4.22.** *If $K$ is the quasicomponent of zero in $M_m(X, \mathscr{A})$, then $K$ is a subideal of $M_\psi(X, \mathscr{A})$, especially,*

$$M_\psi(X, \mathscr{A}) = K = \bigcap_{g\in M(X,\mathscr{A})} A_g.$$

*Proof.* Let $C$ be the component of zero in $M_m(X,\mathscr{A})$. Then by Lemma 4.5 and Proposition 4.13,

$$C \subseteq M_\psi(X,\mathscr{A}) \subseteq K \subseteq \bigcap_{g\in M(X,\mathscr{A})} A_g$$

One can easily confirm that $\bigcap_{g\in M(X,\mathscr{A})} A_g$ is a bounded ideal of $M(X,\mathscr{A})$, then, Proposition 4.13, $M_\psi(X,\mathscr{A}) = K = \bigcap_{g\in M(X,\mathscr{A})} A_g$. □

## 5. Pseudocompactness of $(X,\mathscr{A})$ versus topological properties of $M_m(X,\mathscr{A})$

In this section, we shall establish some properties of a pseudocompact measurable space $(X,\mathscr{A})$ which are equivalent to properties of the topological space $M_m(X,\mathscr{A})$.

**Proposition 5.1.** *A measurable space $(X,\mathscr{A})$ is pseudocompact if and only if $M_m(X,\mathscr{A}) = M_u(X,\mathscr{A})$*

*Proof. Necessity.* By Remark 2.1, $\tau_m \subseteq \tau_u$. Let $G \in \tau_m$ and $f \in G$ be given. Then there exists an element $u$ in $U^+(X)$ such that $B(f,u) \subseteq G$. Since $(X,\mathscr{A})$ is pseudocompact, we conclude that $\varepsilon := \inf\{U(X,\mathscr{A})\colon x \in X\}$ is a non-zero real number, which implies that $u(f,\varepsilon) \subseteq B(f,u) \subseteq G$, and so $G \in \tau_u$. Therefore, $\tau_u \subseteq \tau_m$.

*Sufficiency.* We argue by contradiction. Let us assume that $f$ is an unbounded function in $M(X,\mathscr{A})$. Then there exists an element $\varepsilon > 0$ in $\mathbb{R}$ such that $B(\mathbf{0},\varepsilon) \subseteq B\big(\mathbf{0}, \dfrac{\mathbf{1}}{|f| \vee \mathbf{1}}\big)$, which implies that $\dfrac{\varepsilon}{\mathbf{2}} < \dfrac{\mathbf{1}}{|f| \vee \mathbf{1}}$, and so $|f| \vee \mathbf{1} < \dfrac{\mathbf{2}}{\varepsilon}$, that is $f$ is bounded, and this is a contradiction. □

**Proposition 5.2.** *If $M_m(X,\mathscr{A})$ is a Lindelöf space, then $(X,\mathscr{A})$ is a pseudocompact measurable space.*

*Proof.* By hypothesis, there exist $\{f_n\}_{\mathbb{N}} \subseteq M(X,\mathscr{A})$ and $\{u_n\}_{\mathbb{N}} \subseteq U^+(X,\mathscr{A})$ such that $M(X,\mathscr{A}) = \bigcup_{n=1}^{\infty} B(f_n,u_n)$. By way of contradiction assume that $\mathscr{A}$ is not a pseudocompact measurable space. Then there exists a subset $A := \{x_n\colon n \in \mathbb{N}\}$ of $X$ such that for every $f \in \mathbb{R}^A$, there exists an element $\bar{f} \in M(X,\mathscr{A})$ such that $\bar{f}|_A = f$. We define $g\colon A \to \mathbb{R}$ by $g(x) = u_n(x) + f_n(x)$, then there exist $\bar{g} \in M(X,\mathscr{A})$ and $n \in \mathbb{N}$ such that $\bar{g}|_A = g$ and $\bar{g} \in B(f_n,u_n)$, which implies that

$$u_n(x_n) = |g(x_n) - f_n(x_n)| < u_n(x_n),$$

and this is a contradiction. □

We recall that a topological space $X$ is called semi-compact if there exists a family of compact subsets $\{K_n\}_{n\in\mathbb{N}}$ of $X$ such that for every compact subset $K$ of $X$, there exists an element $n$ in $\mathbb{N}$ such that $K \subseteq K_n$.

The following corollary which follows from Proposition 3.10 is the counterpart of Corollary 4.3 in [3].

**Corollary 5.3.** *Let $(X, \mathscr{A})$ be a $T$-measurable space. Then the following statements are equivalent.*

(1) *$M_m(X, \mathscr{A})$ is locally compact.*
(2) *$M_m(X, \mathscr{A})$ is $\sigma$- compact.*
(3) *$M_m(X, \mathscr{A})$ is semi-compact.*
(4) *$M_m(X, \mathscr{A})$ is a second category space.*
(5) *$X$ is finite.*

*Proof.* If $X$ is a finite set, then, by Proposition 3.8, $M_m(X, \mathscr{A})$ is locally compact, $\sigma$- compact and semi-compact.

(1) $\Rightarrow$ (5). By Proposition 3.10, it is clear.

(2) $\Rightarrow$ (4). By hypothesis, there exists a family of compact subsets $\{F_n\}_{n\in\mathbb{N}}$ of $M_m(X, \mathscr{A})$ such that $M_m(X, \mathscr{A}) = \bigcup_{n=1}^{\infty} F_n$. Then $M_m(X, \mathscr{A})$ is a Lindelöf space, which implies from Proposition 5.2 that $(X, \mathscr{A})$ is a pseudocompact space, and so, by Propositions 5.1, 3.1 and Theorem 16.25 in [10], $M_m(X, \mathscr{A})$ is a second category space.

(4) $\Rightarrow$ (5). Suppose on the contrary, that $X$ is an infinite set. Then, by Proposition 3.10, $M(X, \mathscr{A}) \setminus F_n$ is a dense open subset of $M_m(X, \mathscr{A})$ for every $n \in \mathbb{N}$. But

$$\bigcap_{n=1}^{\infty} \big(M(X, \mathscr{A}) \setminus F_n\big) = M(X, \mathscr{A}) \setminus \bigcup_{n=1}^{\infty} F_n = \emptyset,$$

which contradics the fact that $M_m(X, \mathscr{A})$ is a second category space.

(3) $\Rightarrow$ (5). It is well known that every semi-compact is $\sigma$- compact. □

We say that a space $X$ satisfies the first axiom of countability or is first-countable; this means that at every point $z$ of $X$ there exists a countable base.

**Proposition 5.4.** *$M_m(X, \mathscr{A})$ is a first countable topological space if and only if $(X, \mathscr{A})$ is a pseudocompact measurable space.*

*Proof. Necessity.* We argue by contradiction. Let us assume that $(X, \mathscr{A})$ is not pseudocompact.

We consider constant function $\mathbf{0}$ and we demonstrate for every countable collection of open neighborhood for $\mathbf{0}$. Then there exists any neighborhood of this function such that is not contained any element of this collection. Since $X$ is not a pseudocompact measurable space; there exists an element $f$ in $M(X, \mathscr{A}) \setminus M^*(X, \mathscr{A})$. We set $g := f^2 + \mathbf{1}$. Then $g \in U^+(X, \mathscr{A})$ is unbounded. Thus there exist a family $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ and a family $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{a_n\}_{n\in\mathbb{N}}$ is a ascending sequence with $\lim_{n\to\infty} a_n = \infty$ and $g(x_n) = a_n$ for every $n \in \mathbb{N}$. Now, let $\{B(0, u_n)\}_n \in \mathbb{N}$ be a countable collection of open neighborhood of $\mathbf{0}$. We put $b_n = \dfrac{1}{2} u_n(x_n)$ for every $n \in \mathbb{N}$. Then there exists a positive element $\sigma$ in $C(\mathbb{R})$ such that $\sigma(a_n) = \dfrac{1}{b_n}$ for every $n \in \mathbb{N}$. Consider $v := \dfrac{1}{\sigma \circ g}$. Then $v \in U^+(X, \mathscr{A})$ and $\nu(x_n) = b_n \leq \dfrac{1}{2} u_n(x_n)$ for every $n \in \mathbb{N}$, which implies that $B(0, u_n) \not\subseteq B(0, v)$ for every $n \in \mathbb{N}$, and this is a contradiction to the fact that $M_m(X, \mathscr{A})$ is a first countable topological space.

*Sufficiency.* By Proposition 5.1, $M_m(X, \mathscr{A}) = M_u(X, \mathscr{A})$. Then by Proposition 3.1, $M_m(X, \mathscr{A})$ is a metric space, which implies that $M_m(X, \mathscr{A})$ is a first countable space. □

**Proposition 5.5.** *If $(X, \mathscr{A})$ is not a pseudocompact measurable space, then the following statements hold.*

(1) *The set of all constant functions in $M(X, \mathscr{A})$ is a discrete subspace of $M_m(X, \mathscr{A})$.*
(2) *$M(X, \mathscr{A})$ has an unbounded unit element.*

*Proof.* (1). By hypothesis, $M(X, \mathscr{A})$ must contain an unbounded function $f$. Consider $\varepsilon \in (0, +\infty)$. Then $u := |f| \vee \varepsilon \in U^+(X, \mathscr{A})$ is an unbounded function, which implies that $u^{-1}$ is also a positive unit such that $0 \in cl_{\mathbb{N}}(u^{-1}[X])$. Hence, there exist a family $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ and a family $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{a_n\}_{n\in\mathbb{N}}$ is a ascending sequence with $\lim_{n\to\infty} a_n = \infty$ and $u(x_n) = a_n$ for every $n \in \mathbb{N}$. Let $r, t \in \mathbb{R}$ be given. If $\mathbf{t} \in B(\mathbf{r}, u^{-1})$, then

$$|r - t| = \lim_{n\to\infty} |\mathbf{r}(x_n) - \mathbf{t}(x_n)| \leq \lim_{n\to\infty} u^{-1}(x_n) = 0,$$

which implies that $\mathbf{t} = \mathbf{r}$. Therefore, the set of all constant functions in $M(X, \mathscr{A})$ is a discrete subspace of $M_m(X, \mathscr{A})$.

(2). It is evident. □

**Proposition 5.6.** *Let $(X, \mathscr{A})$ be a measurable space. Then the following statements are equivalent.*

(1) *$(X,\mathscr{A})$ is a pseudocompact measurable space.*
(2) *$M_m(X,\mathscr{A})$ is a connected space.*
(3) *$M_m(X,\mathscr{A})$ is a locally connected space.*
(4) *$M^*(X,\mathscr{A})$ is a connected subset of $M_m(X,\mathscr{A})$.*

*Proof.* (1) ⇒ (2). By Proposition 4.13, it is clear.

(2) ⇒ (3). By Proposition 4.13,

$$M(X,\mathscr{A}) = M^*(X,\mathscr{A}) = M_\psi(X,\mathscr{A}).$$

Let $0 < \varepsilon \in \mathbb{R}$ be given. By Lemma 4.11, for every $f \in M(X,\mathscr{A})$ with $|f| < \varepsilon$, $\varphi_f([-1,1])$ is a connected subset of $M_m(X,\mathscr{A})$ and since $\mathbf{0} \in \bigcap_{|f|<\varepsilon} \varphi_f([-1,1])$, we conclude that $B(\mathbf{0},\varepsilon) = \bigcup_{|f|<\varepsilon} \varphi_f([-1,1])$ is a connected subset of $M_m(X,\mathscr{A})$. Therefore, by Remark 4.7, $f+B(\mathbf{0},\varepsilon)$ is a connected open subset of $M_m(X,\mathscr{A})$ for every $f \in M(X,\mathscr{A})$. The proof is now complete.

(3) ⇒ (1). It is evident that $M_m(X,\mathscr{A}) \subseteq M_\psi(X,\mathscr{A})$ implies every function is bounded.

(3) ⇒ (4). Let $X$ is not a pseudocompact. Then there exists an element $f$ in $M(X,\mathscr{A})$ such that $f \notin M_\psi(X,\mathscr{A})$. Let $C$ be the connected component of zero in $M_m(X,\mathscr{A})$. Then by Lemma 4.13, $C \subseteq M_\psi(X,\mathscr{A})$. By hypothesis, there exists an open connected subset $G$ such that $\mathbf{0} \in G$. Then there exists an element $u$ in $U^+(X,\mathscr{A})$ such that $B(\mathbf{0},u) \subseteq G \subseteq C$. Since $\frac{f}{\mathbf{1}+|f|}u \in B(\mathbf{0},u)$, we conclude that $f = \frac{f}{\mathbf{1}+|f|}u(\mathbf{1}+|f|)u^{-1} \in C \subseteq M_\psi(X,\mathscr{A})$, which is a contradiction.

(4) ⇒ (1). By Proposition 4.13, $\mathbf{1} \in M^*(X,\mathscr{A}) = M_\psi(X,\mathscr{A})$. Then $(X,\mathscr{A})$ is a pseudocompact measurable space, because $\operatorname{coz}(\mathbf{1}) = X$. □

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# $m$–TOPOLOGY ON THE RING OF REAL–MEASURABLE FUNCTIONS

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# $m$– توپولوژی روی حلقه توابع اندازه پذیر حقیقی

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در این مقاله ما $m$–توپولوژی را روی $M(X, \mathscr{A})$، که حلقه توابع اندازه پذیر حقیقی روی فضای اندازه‌پذیر $(X, \mathscr{A})$ است، در نظر می‌گیریم و آن را با نماد $M_m(X, \mathscr{A})$ نشان می‌دهیم. ثابت می‌کنیم که $M_m(X, \mathscr{A})$ حلقه توپولوژیك منظم و هاسدورف است و علاوه بر این، اگر $(X, \mathscr{A})$ فضای $T$– اندازه‌پذیر و $X$ با $|X| = n$ باشد، آن‌گاه به عنوان حلقه‌های توپولوژیکی $\mathbb{R}^n \cong M_m(X, \mathscr{A})$. همچنین ثابت کرده‌ایم که $M_m(X, \mathscr{A})$ هرگز فضای شبه فشرده و فشرده شمارا نیست. نشان داده‌ایم که $(X, \mathscr{A})$ فضای اندازه‌پذیر شبه‌فشرده است اگر و تنها اگر $M_m(X, \mathscr{A}) = M_u(X, \mathscr{A})$ اگر و تنها اگر $M_m(X, \mathscr{A})$ فضای شمارای اول باشد اگر و تنها اگر $M_m(X, \mathscr{A})$ فضای همبند باشد اگر و تنها اگر $M_m(X, \mathscr{A})$ فضای همبند موضعی باشد اگر و تنها اگر $M_m(X, \mathscr{A})$ زیرمجموعه‌ی همبند $M_m(X, \mathscr{A})$ باشد.

کلمات کلیدی: $m$–توپولوژی، فضای اندازه پذیر، فضای اندازه‌پذیر شبه‌فشرده، فضای همبند، فضای شمارای اول.