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ZERO-DIVISOR GRAPH OF THE RINGS OF REAL MEASURABLE FUNCTIONS WITH THE MEASURES

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ABSTRACT. Let $M(X, \mathcal{A}, \mu)$ be the ring of real-valued measurable functions on a measurable space $(X, \mathcal{A})$ with measure μ. In this paper, we study the zero-divisor graph of $M(X, \mathcal{A}, \mu)$, denoted by $\Gamma(M(X, \mathcal{A}, \mu))$. We give the relationships among graph properties of $\Gamma(M(X, \mathcal{A}, \mu))$, ring properties of $M(X, \mathcal{A}, \mu)$ and measure properties of $(X, \mathcal{A}, \mu)$. Finally, we investigate the continuity properties of $\Gamma(M(X, \mathcal{A}, \mu))$.

1. Introduction

A *σ-algebra* on a set X is a collection $\mathcal{A}$ of subsets of X that includes the empty subset, is closed under complement, and is closed under countable unions. If $\mathcal{A}$ is a σ-algebra on X, then $(X, \mathcal{A})$ is called a *measurable space* and the members of $\mathcal{A}$ are called the *measurable sets* in X. A function μ from a σ-algebra $\mathcal{A}$ to the extended real number line is called a *measure* if for all countable collections $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint sets in $\mathcal{A}$, $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$. To avoid trivialities, we shall also assume that $\mu(A) < \infty$ for at least one $A \in \mathcal{A}$. A *measure space* is a triple $(X, \mathcal{A}, \mu)$, where X is a set, $\mathcal{A}$ a σ-algebra on X, and μ a measure on $\mathcal{A}$. A *complete measure* (or, more precisely, a *complete measure space*) is a measure space in which every subset of every set of measure zero is measurable. The statement "P

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holds *almost everywhere on* $(X, \mathcal{A}, \mu)$" (abbreviated to "P holds a.e. on $(X, \mathcal{A}, \mu)$") means that

$$\mu(\{x \in X : \text{P does not hold on } x\}) = 0.$$

If Y is a topological space and $f : X \longrightarrow Y$ is a function, then f is said to be *measurable* provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y. The *characteristic function* is the function $\chi_A : X \longrightarrow \{0, 1\}$, which for a given measurable set A, has value 1 at elements of A and 0 at elements of $X \backslash A$. For every measurable function f, the *zero set* and the *cozero set* of f are $Z_f := \{x \in X : f(x) = 0\}$ and $\mathrm{co}Z_f := X \setminus Z_f$, respectively.

The space of real measurable functions with pointwise addition and multiplication is a commutative ring with identity. Rings of real-valued measurable functions have been studied in many ways for a long time by many mathematicians (see [2, 3, 15, 16, 27, 28]). In recent years, significant researches have been done by some mathematicians like Momtahan and Henriksen (see [4, 7, 21]). In [18], Hejazipour and Naghipour by valuing the measures on measurable spaces studied the hereditary rings in the rings of real measurable functions. For notational convenience, we assume that $M(X, \mathcal{A}, \mu)$ is the space of measurable functions from X to $\mathbb{R}$ with arbitrary σ-algebra $\mathcal{A}$ on X and arbitrary measure μ on $\mathcal{A}$. For more information about this ring, see [4, 10, 13, 17, 19, 23, 26].

The concept of the zero-divisor graph of a commutative ring was introduced by Beck in 1988 [9]. However, he let all elements of the ring be vertices of the graph and was mainly interested in colorings. Anderson et al. [5] associated an undirect simple graph to a commutative ring with vertices nonzero zero-divisors and with two distinct vertices a and b are adjacent if $ab = 0$. The zero-divisor graph of a commutative ring also has been studied by several other authors [1, 6, 12, 20, 25]. Azarpanah and Motamedi in [8], studied the zero-divisor graph of $C(X)$, ring of real-valued continuous functions on a completely regular Hausdorff space X. In this paper, we study the zero-divisor graph of the ring of real measurable functions with measures.

This paper has two main purposes. Firstly, we study the relationships among graph properties of the graph $\Gamma(M(X, \mathcal{A}, \mu))$, ring properties of the ring $M(X, \mathcal{A}, \mu)$ and measure properties of the measure space $(X, \mathcal{A}, \mu)$. Secondly, we investigate the relationship between vertices and edges of $\Gamma(M(X, \mathcal{A}, \mu))$ and continuous functions. The organization of the paper is as follows: In Section 2, we determine the distance between vertices, radius, diameter and the girth of $\Gamma(M(X, \mathcal{A}, \mu))$ by the properties of measure spaces. In Section 3, we investigate cycles in $\Gamma(M(X,\mathcal{A},\mu))$. In Section 4, we study continuity properties of $\Gamma(M(X,\mathcal{A},\mu))$. As the main result of this section, we approximate vertices of $\Gamma(M(X,\mathcal{A},\mu))$ by the vertices of $\Gamma(C_C(X))$, the zero-divisor graph of $C_C(X)$.

2. Basic properties of $\Gamma(M(X,\mathcal{A},\mu))$

Naturally, the rings of real measurable functions are studied without paying attention to the measures (see [4, 7, 15, 16, 21, 27, 28]). But the measures played such a prominent role in the study of the spaces of measurable functions. In [18], we studied the rings of real measurable functions with measures. Since this article intends to examine the zero-divisor graph of the rings of real measurable functions with measures, we redefine the definition of the zero-divisor graph.

Definition 2.1. A function $f \in M(X,\mathcal{A},\mu)$ is called a *zero-divisor* of $M(X,\mathcal{A},\mu)$, if there exists a function $g \in M(X,\mathcal{A},\mu)$ such that

$$\mu(\{x \in X : g(x) \neq 0\}) \neq 0 \quad \text{and} \quad \mu(\{x \in X : f(x)g(x) \neq 0\}) = 0.$$

Let $Z(M(X,\mathcal{A},\mu))$ denote the set of zero-divisors of $M(X,\mathcal{A},\mu)$.

Definition 2.2. The *zero-divisor graph* of $M(X,\mathcal{A},\mu)$, denoted by $\Gamma(M(X,\mathcal{A},\mu))$, is the graph with vertices

$$Z(M(X,\mathcal{A},\mu))\backslash\{f \in M(X,\mathcal{A},\mu) : f = 0 \text{ a.e. on } (X,\mathcal{A},\mu)\}$$

and two distinct vertices f and g are adjacent if $fg = 0$ a.e. on $(X,\mathcal{A},\mu)$.

To enter the discussion, we need the following important lemma.

Lemma 2.3. *Let $(X,\mathcal{A},\mu)$ be a measure space and $f \in M(X,\mathcal{A},\mu)$. Then $f \in \Gamma(M(X,\mathcal{A},\mu))$ if and only if $\mu(Z_f)$ and $\mu(coZ_f)$ are nonzero.*

Proof. Suppose that $f \in \Gamma(M(X,\mathcal{A},\mu))$. Then there exists a measurable function g such that $g \neq 0$ a.e. on $(X,\mathcal{A},\mu)$ and g is adjacent to f. If $\mu(Z_f) = 0$, then $\mu(\text{co}Z_g) \leq \mu(Z_f) = 0$, which is a contradiction. Also, since $f \neq 0$ a.e. on $(X,\mathcal{A},\mu)$, we have $\mu(\text{co}Z_f) \neq 0$. Conversely, assume that $\mu(Z_f)$ and $\mu(\text{co}Z_f)$ are nonzero. Obviously, $f \neq 0$ a.e. on $(X,\mathcal{A},\mu)$. Moreover the measurable function $g := \chi_{Z_f}$ is a nonzero function a.e. on $(X,\mathcal{A},\mu)$ and $\mu(\{x \in X : f(x)g(x) \neq 0\}) = 0$. □

According to the above lemma, the set that is presented in the next notation has an important role in the study of $\Gamma(M(X,\mathcal{A},\mu))$.

Notation 2.4. Let $\mathcal{A}$ be a $\sigma -$ algebra on X. We set:

$$M_\mu := \{A \in \mathcal{A} : \ \mu(A) \text{ and } \mu(A^c) \text{ are nonzero}\}.$$

Recall that for two vertices f and g of $\Gamma(M(X, \mathcal{A}, \mu))$, $d(f, g)$ is the length of the shortest path from f to g. The following theorem characterizes the concept of distance in $\Gamma(M(X, \mathcal{A}, \mu))$.

Theorem 2.5. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then the graph $\Gamma(M(X, \mathcal{A}, \mu))$ is a connected graph and for every $f, g \in \Gamma(M(X, \mathcal{A}, \mu))$, we have:*

(a) $d(f, g) = 1$ if and only if $\mu(coZ_f \cap coZ_g) = 0$.

(b) $d(f, g) = 2$ if and only if $\mu(coZ_f \cap coZ_g)$ and $\mu(Z_f \cap Z_g)$ are nonzero.

(c) $d(f, g) = 3$ if and only if $\mu(coZ_f \cap coZ_g) \neq 0$ and $\mu(Z_f \cap Z_g) = 0$.

Proof. (a) By the definition, f is adjacent to g if and only if $\mu(\{x : f(x)g(x) \neq 0\}) = 0$ if and only if $\mu(\{x : f(x) \neq 0 \text{ and } g(x) \neq 0\}) = 0$ if and only if $\mu(\text{co}Z_f \cap \text{co}Z_g) = 0$.

(b) Assume that $d(f, g) = 2$. Then $\mu(\text{co}Z_f \cap \text{co}Z_g) \neq 0$ and there exists $h \in \Gamma(M(X, \mathcal{A}, \mu))$ such that h is adjacent to both f and g. Therefore $\mu(\text{co}Z_h \cap \text{co}Z_f) = \mu(\text{co}Z_h \cap \text{co}Z_g) = 0$ and so $\text{co}Z_h \subseteq (Z_f \cap Z_g)$ a.e. on $(X, \mathcal{A}, \mu)$. Now if $\mu(Z_f \cap Z_g) = 0$, then $\mu(\text{co}Z_h) = 0$, which is a contradiction. Conversely, let $\mu(\text{co}Z_f \cap \text{co}Z_g) \neq 0$ and $\mu(Z_f \cap Z_g) \neq 0$. Then $d(f, g) > 1$ and $\chi_{Z_f \cap Z_g}$ is a vertex of $\Gamma(M(X, \mathcal{A}, \mu))$. It is easy to check that $fh = gh = 0$ a.e. on $(X, \mathcal{A}, \mu)$.

(c) Assume that $\mu(\text{co}Z_f \cap \text{co}Z_g) \neq 0$ and $\mu(Z_f \cap Z_g) = 0$. Then $d(f, g) > 2$ and $\text{co}Z_f \cup \text{co}Z_g = X$ a.e. on $(X, \mathcal{A}, \mu)$. If $\mu(Z_f \setminus Z_g) = 0$, then $\text{co}Z_f \subseteq \text{co}Z_g$ and so $\text{co}Z_g = X$ a.e. on $(X, \mathcal{A}, \mu)$, which is a contradiction. Therefore $Z_g \setminus Z_f, Z_f \setminus Z_g \in M_\mu$ and $f\chi_{Z_f \setminus Z_g} = \chi_{Z_g \setminus Z_f}\chi_{Z_f \setminus Z_g} = g\chi_{Z_g \setminus Z_f} = 0$ a.e. on $(X, \mathcal{A}, \mu)$. Conversely, suppose that $d(f, g) = 3$. Then $\mu(\text{co}Z_f \cap \text{co}Z_g) \neq 0$ and $\mu(Z_f \cap Z_g) = 0$, by parts (a) and (b).

The connectivity of $\Gamma(M(X, \mathcal{A}, \mu))$ is a consequence of parts (a), (b) and (c). □

In the following, we recall an important definition for studying the rings of real measurable functions $M(X, \mathcal{A}, \mu)$, (see [18], Definition 2.5).

Definition 2.6. Suppose that $E \in \mathcal{A}$ and $\mu(E) \neq 0$. Then the set E is called *near-zero* if for every subset $A \subseteq E$ such that $\mu(A) \neq 0$, $A = E$ a.e. on $(X, \mathcal{A}, \mu)$.

The *associated number* of a vertex f, denoted by $e(f)$, is

$$e(f) := \max\{d(f, g) : g \in \Gamma(M(X, \mathcal{A}, \mu)) \text{ and } f \neq g \text{ a.e. on } (X, \mathcal{A}, \mu)\}.$$

The *radius* of $\Gamma(M(X, \mathcal{A}, \mu))$ is the smallest associated number and denoted by $\rho\Gamma(M(X, \mathcal{A}, \mu))$.

Theorem 2.7. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $f \in \Gamma(M(X, \mathcal{A}, \mu))$. Then the following properties hold:*

(a) If $|M_\mu| = 2$, then $e(f) = 1$.

(b) If $|M_\mu| \neq 2$ and coZ_f is a near-zero set, then $e(f) = 2$.

(c) If $|M_\mu| \neq 2$ and coZ_f is not a near-zero set, then $e(f) = 3$.

In respect to the above three properties, we have the following statements about the radius of $\Gamma(M(X, \mathcal{A}, \mu))$:

(a') If $|M_\mu| = 2$, then $\rho\Gamma(M(X, \mathcal{A}, \mu)) = 1$.

(b') If $|M_\mu| \neq 2$ and M_μ has a near-zero set, then $\rho\Gamma(M(X, \mathcal{A}, \mu)) = 2$.

(c') If $|M_\mu| \neq 2$ and M_μ has not any near-zero set, then $\rho\Gamma(M(X, \mathcal{A}, \mu)) = 3$.

Proof. (a) Suppose that $|M_\mu| = 2$. Thus $M_\mu = \{\text{co}Z_f, Z_f\}$ a.e. on $(X, \mathcal{A}, \mu)$ and hence $\Gamma(M(X, \mathcal{A}, \mu))$ is a collection of segments. This means that $e(f) = 1$.

(b) Suppose that $|M_\mu| \neq 2$ and coZ_f is a near-zero set. For every $g \in \Gamma(M(X, \mathcal{A}, \mu))\backslash\{f\}$, we consider two following cases:

Case 1: co$Z_g \subseteq Z_f$ a.e. on $(X, \mathcal{A}, \mu)$. Then $d(f, g) = 1$, by Theorem 2.5(a).

Case 2: co$Z_f \subseteq$ coZ_g a.e on $(X, \mathcal{A}, \mu)$. By using Theorem 2.5(b), $d(f, g) = 2$.

Now for $A \in M_\mu\backslash\{Z_f, \text{co}Z_f\}$, $d(f, \chi_{A\cup \text{co}Z_f}) = 2$ and according to the above cases $e(f) = 2$.

(c) Assume that $|M_\mu| \neq 2$ and coZ_f is not a near-zero set. Then there exists $A \in M_\mu$ such that $A \subseteq$ coZ_f and $\mu(A) \neq \mu(\text{co}Z_f)$. Set $B := Z_f \cup A$ and $g := \chi_B$. Since $\mu(\text{co}Z_f \cap \text{co}Z_g) = \mu(A) \neq 0$ and $\mu(Z_f \cap Z_g) = 0$, $d(f, g) = 3$ and therefore $e(f) = 3$.

(a') Suppose that $M_\mu = \{A, B\}$. Then for every $g \in \Gamma(M(X, \mathcal{A}, \mu))$, co$Z_g = A$ or co$Z_g = B$. By using part (a), $e(g) = 1$ and so $\rho\Gamma(M(X, \mathcal{A}, \mu)) = 1$.

(b') Assume that $|M_\mu| > 2$ and $A \in M_\mu$ is a near-zero set. Then the function $g := \chi_A$ satisfies in the part (b) and so $e(g) = 2$. If $h \in \Gamma(M(X, \mathcal{A}, \mu))$ and $e(h) = 1$, then Z_h and coZ_h are near-zero sets, which is a contradiction. Therefore $\rho\Gamma(M(X, \mathcal{A}, \mu)) = 2$.

(c') Let $g \in \Gamma(M(X, \mathcal{A}, \mu))$. Since $|M_\mu| > 2$ and M_μ has not any near-zero set, there exists a measurable set A such that $A \subseteq$ coZ_g, $\mu(A) \neq 0$ and $\mu(A) \neq \mu(\text{co}Z_g)$. Set $B := Z_g \cup A$ and $h := \chi_B$. Therefore $d(g, h) = 3$, by Theorem 2.5(c). Thus $e(g) = 3$ and hence $\rho\Gamma(M(X, \mathcal{A}, \mu)) = 3$. □

Remark 2.8. Regarding to the above theorem, it does not occur that M_μ is a singleton. If M_μ has only two members, then $\Gamma(M(X, \mathcal{A}, \mu))$ is a collection of segments.

The *diameter* of $\Gamma(M(X, \mathcal{A}, \mu))$ is

$$\mathrm{diam}\Gamma(M(X, \mathcal{A}, \mu)) := \sup\{d(f, g) : f, g \in \Gamma(M(X, \mathcal{A}, \mu))\}.$$

The *girth* of $\Gamma(M(X, \mathcal{A}, \mu))$ is the length of the shortest cycle in $\Gamma(M(X, \mathcal{A}, \mu))$, denoted by $\mathrm{gr}\Gamma(M(X, \mathcal{A}, \mu))$, and we set $\mathrm{gr}\Gamma(M(X, \mathcal{A}, \mu)) = \infty$ if $\Gamma(M(X, \mathcal{A}, \mu))$ contains no cycle. It should be noted by Theorems 2.5 and 2.7 that $\mathrm{diam}\Gamma(M(X, \mathcal{A}, \mu)) \leq 3$.

Theorem 2.9. *Let $(X, \mathcal{A}, \mu)$ be a measure space and M_μ has at least three members. Then*

$$\mathrm{diam}\,\Gamma(M(X, \mathcal{A}, \mu)) = \mathrm{gr}\,\Gamma(M(X, \mathcal{A}, \mu)) = 3.$$

Proof. Suppose that M_μ has at least three disjoint members A, B and C. Consider the following six cases:

Case 1: Assume that A, B and C are pairwise disjoint members in M_μ. Set $K := A^c \cap B^c$, $L := A \cup K$ and $M := B \cup K$. Then by Lemma 2.3, the measurable functions χ_A, χ_B, χ_K, χ_L and χ_M are in $\Gamma(M(X, \mathcal{A}, \mu))$. Since $\mu(L \cap M) = \mu(K) \neq 0$, $d(\chi_L, \chi_M) \neq 1$, by Theorem 2.5(a). On the other hand, if $f \in \Gamma(M(X, \mathcal{A}, \mu)$ is adjacent to both χ_L and χ_M, then $\mathrm{co}Z_f \subseteq A \cap B$, which is a contradiction. Therefore by Theorem 2.5(c), $d(\chi_K, \chi_L) = 3$ and so $\mathrm{diam}\,\Gamma(M(X, \mathcal{A}, \mu)) = 3$. It is easy to check that $\chi_A\chi_B = \chi_A\chi_C = \chi_B\chi_C = 0$ a.e. on $(X, \mathcal{A}, \mu)$ and so $\mathrm{gr}\,\Gamma(M(X, \mathcal{A}, \mu)) = 3$.

Case 2: Assume that $A \subseteq B \subseteq C$. If $\mu(C \setminus B) = 0$, then $\mu(C) = \mu(B)$ and therefore $B = C$ a.e. on $(X, \mathcal{A}, \mu)$, which is a contradiction. On the other hand, $\mu((C\setminus B)^c) = \mu(C^c \cup B) \geq \mu(B) \neq 0$ and so $C\setminus B \in M_\mu$. Similarly, it can be show that $B \setminus A \in M_\mu$. Now $C \setminus B$, $B \setminus A$ and A are in M_μ and satisfy in Case 1.

Case 3: Assume that $A \subseteq B$ and $C \cap B = \emptyset$. As the proof of of Case 2, we can be shown that $B \setminus A \in M_\mu$. Therefore $B \setminus A$, A and C satisfy in Case 1.

Case 4: Assume that $A \cap B$, $A\setminus B$ and $B\setminus A$ are not empty sets and $C \cap (A \cup B) = \emptyset$. Then $A \setminus B$, $B \setminus A$ and C satisfy in Case 1.

Case 5: Assume that $A \subseteq B \cup C$. If $\mu(C \setminus (A \cup B)) = 0$, then $C \subseteq A \subseteq B$ or $A \subseteq C \subseteq B$. This means that the sets A, B and C satisfy in Case 2. If $\mu(C \setminus (A \cup B)) \neq 0$, then $\mu((C \setminus (A \cup B))^c) \geq \mu(A \cup B) \neq 0$ and so $C \setminus (A \cup B) \in M_\mu$. In the same way, it can be shown that if $\mu(B\setminus(A\cup C)) \neq 0$, $B\setminus(A\cup C) \in M_\mu$. Therefore $C\setminus(A\cup B)$, $B\setminus(A\cup C)$ and A satisfy in Case 1.

Case 6: Assume that the above five cases are not establish. We claim that $A \setminus (B \cup C)$, $B \setminus (A \cup C)$ and $C \setminus (A \cup B)$ satisfy in Case 1. If $\mu(A \setminus (B \cup C) = 0$, then $A \subseteq B \cup C$ and hence A, B and C satisfy in the Case 5. On the other hand, $\mu((A \setminus (B \cup C))^c) \geq \mu(B \cup C) \neq 0$ and so $A \setminus (B \cup C) \in M_\mu$. Similarly, it can be shown that $B \setminus (A \cup C)$ and $C \setminus (A \cup B)$ are in M_μ. □

3. Cycles in zero-divisor graph of $M(X, \mathcal{A}, \mu)$

In this section, we intend to study the cycles and related issues to the cycles in the zero-divisor graph of the rings of real measurable functions, $\Gamma(M(X, \mathcal{A}, \mu))$.

A graph is called *triangulated* if each vertices is a vertex of a triangle.

Theorem 3.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $|M_\mu| > 2$. The following statements are equivalent:*

(a) The graph $\Gamma(M(X, \mathcal{A}, \mu))$ is a triangulated graph.

(b) M_μ has not any near-zero set.

(c) There is no any maximal ideal in the ring $M(X, \mathcal{A}, \mu)$ generated by an idempotent.

Proof. $(a) \Longrightarrow (b)$. Assume that $\Gamma(M(X, \mathcal{A}, \mu))$ is a triangulated graph and $A \in M_\mu$. By Lemma 2.3, $f := 1 - \chi_A$ is a vertex of $\Gamma(M(X, \mathcal{A}, \mu))$. Thus there exist two vertices g and h such that $fg = gh = hf = 0$ a.e. on$(X, \mathcal{A}, \mu)$ and hence $\mu(\text{co}Z_f \cap \text{co}Z_h) = \mu(\text{co}Z_g \cap \text{co}Z_h) = \mu(\text{co}Z_f \cap \text{co}Z_h) = 0$. This means that $\text{co}Z_g$ and $\text{co}Z_h$ are disjoint subsets of A a.e. on $(X, \mathcal{A}, \mu)$. Since $\mu(\text{co}Z_g) \neq 0$ and $\mu(\text{co}Z_h) \neq 0$, A is not a near-zero set.

$(b) \Longrightarrow (c)$. Assume that M_μ has not any near-zero set and M be a maximal ideal in $M(X, \mathcal{A}, \mu)$ generated by an idempotent. Since every idempotent in the rings of real measurable functions has the form of a characteristic function of a measurable set, there exists $A \in \mathcal{A}$ such that $M = \langle \chi_A \rangle$. Suppose that B is a measurable set in M_μ such that $B \subseteq A$ a.e. on $(X, \mathcal{A}, \mu)$ and $\mu(B) \neq 0$. Therefore $\chi_A \chi_B \in M$ and so $\chi_B \in M$. This means that $B = A$ a.e. on $(X, \mathcal{A}, \mu)$ and so A is a near-zero set.

$(c) \Longrightarrow (a)$. Assume that there is not any maximal ideal in $M(X, \mathcal{A}, \mu)$ generated by an idempotent and f is an arbitrary vertex of $\Gamma(M(X, \mathcal{A}, \mu))$. If Z_f is near-zero, we set $W := \langle \chi_{Z_f} \rangle$. Suppose that U is an ideal in $M(X, \mathcal{A}, \mu)$, $W \subseteq U$ and $h \in U \backslash W$. Therefore $\text{co}Z_h \subseteq Z_f$ a.e. on $(X, \mathcal{A}, \mu)$ and so $\mu(\text{co}Z_h) = 0$ or $h \in W$. This means that W is a maximal ideal in $M(X, \mathcal{A}, \mu)$ generated by an idempotent, which is a contradiction. Now, since Z_f is not a near-zero set, there exists $A \subseteq Z_f$ such that $\mu(A) \neq 0$ and $\mu(A) \neq \mu(Z_f)$. We set $g := \chi_A$ and $h := \chi_{Z_f \setminus A}$. It easy to check that $g, h \in \Gamma(M(X, \mathcal{A}, \mu))$. Therefore $fg = gh = hf = 0$ a.e. on $(X, \mathcal{A}, \mu)$ and so $\Gamma(M(X, \mathcal{A}, \mu))$ is a triangulated graph. □

Corollary 3.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $\Gamma(M(X, \mathcal{A}, \mu))$ be a triangulated graph. Then for every countable set $B \in \mathcal{A}$, $\mu(B) = 0$.*

Proof. Suppose that for $x \in X$, $\mu(\{x\}) \neq 0$. Then $\{x\}$ is a near-zero and by Theorem 3.1, $\Gamma(M(X, \mathcal{A}, \mu))$ is not a triangulated graph, which is a contradiction. Hence for every countable set $B = \{x_1, x_2, ...\} \in \mathcal{A}$, $\mu(B) = \sum_{i=1}^{\infty} \mu(\{x_i\}) = 0$. □

A graph is called *hypertriangulated* if each edge of $\Gamma(M(X, \mathcal{A}, \mu))$ is a edge of a triangle.

Proposition 3.3. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then $\Gamma(M(X, \mathcal{A}, \mu))$ is not hypertriangulated.*

Proof. Suppose that $f \in \Gamma(M(X, \mathcal{A}, \mu))$. Then f is adjacent to $g := \chi_{Z_f}$. Since $\mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) = 0$ and $\mu(Z_f \cap Z_g) = 0$, there is not any element in $\Gamma(M(X, \mathcal{A}, \mu))$ such that adjacent to both f and g, by Theorem 2.5(b). □

A graph is called a *tree*, if it is connected and has no cycles. A *star graph* is a tree with one vertex adjacent to all other vertices.

Theorem 3.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $|M_\mu| > 2$. Then $\Gamma(M(X, \mathcal{A}, \mu))$ is not a star graph.*

Proof. (a) Assume that $|M_\mu| > 2$ and $\Gamma(M(X, \mathcal{A}, \mu))$ is a star graph. Then there exists $f \in \Gamma(M(X, \mathcal{A}, \mu))$ such that f is adjacent to other vertices of $\Gamma(M(X, \mathcal{A}, \mu))$. By Lemma 2.3, $Z_f, \mathrm{co}Z_f \in M_\mu$. Since $|M_\mu| > 2$, there exists $A \in M_\mu$ such that A is other than both Z_f and $\mathrm{co}Z_f$. By the assumptions, $g := \chi_A$ and $h := \chi_{A^c}$ are two vertices of $\Gamma(M(X, \mathcal{A}, \mu))$ and adjacent to f. This implies that $\mu(\mathrm{co}Z_f) = \mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) + \mu(\mathrm{co}Z_f \cap \mathrm{co}Z_h) = 0$, which is a contradiction. □

In the following, we present a notation and a definition that are important in the studying of cycles in $\Gamma(M(X, \mathcal{A}, \mu))$.

Notation 3.5. (a) Let $f \in \Gamma(M(X, \mathcal{A}, \mu))$. We set:

$$[f] := \{h \in \Gamma(M(X, \mathcal{A}, \mu)) : \mathrm{co}Z_f = \mathrm{co}Z_h \text{ a.e. on } (X, \mathcal{A}, \mu)\}$$

(b) For $f, g \in \Gamma(M(X, \mathcal{A}, \mu))$, we say that $f \sim g$ if and only if $[f] = [g]$.

As noted in [22], $\sim$ is an equivalence relation. Furthermore, if $h_1 \sim h_2$ and $h_1 g = 0$, then $\mu(\text{co}Z_{h_1} \cap \text{co}Z_g) = \mu(\text{co}Z_{h_2} \cap \text{co}Z_g) = 0$ and hence $h_2 g = 0$. It follows that multiplication is well-defined on the equivalence classes of $\sim$; that is, if $[f]$ denotes the class of f, then the product $[f][g] = [fg]$ makes sense.

Definition 3.6. The graph of equivalence classes $\Gamma(M(X, \mathcal{A}, \mu))$, denoted by $\Gamma_E(M(X, \mathcal{A}, \mu))$, is the graph associated to $\Gamma(M(X, \mathcal{A}, \mu))$ whose vertices are the classes of elements in $\Gamma(M(X, \mathcal{A}, \mu))$, and each pair of distinct classes $[f]$, $[g]$ are adjacent by an edge if and only if $[f][g] = 0$.

Theorem 3.7. *Let $(X, \mathcal{A}, \mu)$ be a measure space. For every $f \in \Gamma(M(X, \mathcal{A}, \mu))$, the following properties hold:*

(a) There exists a $4-$cycle contains f.

(b) If Z_f or coZ_f is not near-zero, then $[f]$ is in a $3-$cycle.

(c) If Z_f and coZ_f are near-zero, then there is no cycle contains $[f]$.

Proof. (a) For every vertex f, Z_f and $\text{co}Z_f$ are in M_μ. Hence the path with vertices f, χ_{Z_f}, $2\chi_{\text{co}Z_f}$ and $2\chi_{Z_f}$ is a cycle with length 4 containing f.

(b) If Z_f is not a near-zero set, then there exist disjoint members $A, B \in M_\mu$ such that $\mu(A) \neq 0$, $\mu(B) \neq 0$ and $A \cup B \subseteq Z_f$ a.e. on $(X, \mathcal{A}, \mu)$. Therefore $[f] \cap [\chi_A] \cap [\chi_B] = \emptyset$ and so $[f][\chi_A] = [\chi_A][\chi_B] = [\chi_B][f] = 0$. If $\text{co}Z_f$ is not near-zero, then there exists a measurable set $D \in M_\mu$ such that $\mu(D) \neq 0$, $D \subseteq \text{co}Z_f$ a.e. on $(X, \mathcal{A}, \mu)$ and $\mu(D) \neq \mu(\text{co}Z_f)$. Therefore $[f] \cap [\chi_D] \cap [\chi_{Z_f}] = \emptyset$ and so $[f][\chi_D] = [\chi_D][\chi_{Z_f}] = [\chi_{Z_f}][f] = 0$.

(c) Suppose that Z_f and $\text{co}Z_f$ are near-zero. Then every $g \in \Gamma(M(X, \mathcal{A}, \mu)) \backslash [f]$ is in $[\chi_{Z_f}]$ and every $h \in \Gamma(M(X, \mathcal{A}, \mu)) \backslash [\chi_{Z_f}]$ is in $[f]$. Therefore there is no cycle contains $[f]$. □

If f and g are two vertices of $\Gamma(M(X, \mathcal{A}, \mu))$, by $c(f, g)$, we mean the length of the smallest cycle containing f and g. If there is no cycle containing f and g, $c(f, g) = \infty$. For every two vertices f and g, all possible cases for $c(f, g)$ and $c([f], [g])$ are given in the following two theorems.

Theorem 3.8. *Let f and g be two vertices of $\Gamma(M(X, \mathcal{A}, \mu))$. Then the following properties hold:*

(a) $c(f, g) = 3$ if and only if $\mu(\text{co}Z_f \cap \text{co}Z_g) = 0$ and $\mu(Z_f \cap Z_g) \neq 0$.

(b) $c(f, g) = 4$ if and only if one of the following statements hold:

(1) $\mu(\text{co}Z_f \cap \text{co}Z_g) \neq 0$ and $\mu(Z_f \cap Z_g) \neq 0$.

(2) $\mu(\text{co}Z_f \cap \text{co}Z_g) = 0$ and $\mu(Z_f \cap Z_g) = 0$.

(c) $c(f, g) = 6$ if and only if $\mu(\text{co}Z_f \cap \text{co}Z_g) \neq 0$ and $\mu(Z_f \cap Z_g) = 0$.

Proof. (a) Assume that $c(f,g)=3$. Then there exists a vertex h such that $fg=gh=fh=0$ a.e. on $(X,\mathcal{A},\mu)$. Thus

$$\mu(\mathrm{co}Z_f\cap \mathrm{co}Z_g)=\mu(\mathrm{co}Z_h\cap \mathrm{co}Z_f)=\mu(\mathrm{co}Z_h\cap \mathrm{co}Z_g)=0$$

a.e. on $(X,\mathcal{A},\mu)$ and hence $\mathrm{co}Z_h\subseteq Z_f\cap Z_g$ a.e. on $(X,\mathcal{A},\mu)$. Since h is a vertex, $\mu(\mathrm{co}Z_h)\neq 0$ and therefore $\mu(Z_f\cap Z_g)\neq 0$. Conversely, let $\mu(\mathrm{co}Z_f\cap \mathrm{co}Z_g)=0$ and $\mu(Z_f\cap Z_g)\neq 0$. Then f is adjacent to g and $\chi_{Z_f\cap Z_g}$ is a vertex of $\Gamma(M(X,\mathcal{A},\mu))$. Therefore

$$fg=f\chi_{Z_f\cap Z_g}=g\chi_{Z_f\cap Z_g}=0$$

a.e. on $(X,\mathcal{A},\mu)$.

(b) If $\mu(\mathrm{co}Z_f\cap \mathrm{co}Z_g)\neq 0$ and $\mu(Z_f\cap Z_g)\neq 0$, then f is not adjacent to g and $h:=\chi_{Z_f\cap Z_g}$ is a vertex of $\Gamma(M(X,\mathcal{A},\mu))$. Therefore

$$fh=hg=g(-h)=(-h)g=0$$

a.e. on $(X,\mathcal{A},\mu)$ and so $c(f,g)\leq 4$. If $c(f,g)=3$, then $\mu(\mathrm{co}Z_f\cap \mathrm{co}Z_g)=0$, by part (a), which is a contradiction.

If $\mu(\mathrm{co}Z_f\cap \mathrm{co}Z_g)=0$ and $\mu(Z_f\cap Z_g)=0$, then f is adjacent to g and $\mathrm{co}Z_f\cup \mathrm{co}Z_g=X$ a.e. on $(X,\mathcal{A},\mu)$. We set $h:=\frac{1}{2}f$ and $k:=\frac{1}{2}g$. Thus $fg=gh=hk=kf=0$ a.e. on $(X,\mathcal{A},\mu)$ and hence $c(f,g)\leq 4$. If $c(f,g)=3$, then $\mu(Z_f\cap Z_g)\neq 0$, by part (a), which is a contradiction.

Conversely, suppose that $c(f,g)=4$. We have two cases:

Case 1: $\mu(\mathrm{co}Z_f\cap \mathrm{co}Z_g)\neq 0$. Then f is not adjacent to g. Since $c(f,g)=4$, there exist two vertices h and k of $\Gamma(M(X,\mathcal{A},\mu))$ such that $fh=hg=gk=kf=0$ a.e. on $(X,\mathcal{A},\mu)$. Therefore $\mathrm{co}Z_h\subseteq Z_f$ and $\mathrm{co}Z_h\subseteq Z_g$ and so $\mathrm{co}Z_h\subseteq Z_f\cap Z_g$. Since $\mu(\mathrm{co}Z_h)\neq 0$, $\mu(Z_f\cap Z_g)\neq 0$.

Case 2: $\mu(\mathrm{co}Z_f\cap \mathrm{co}Z_g)=0$. Then f is adjacent to g. If $\mu(Z_f\cap Z_g)\neq 0$, then $\chi_{Z_f\cap Z_g}$ is a vertex of $\Gamma(M(X,\mathcal{A},\mu))$ and

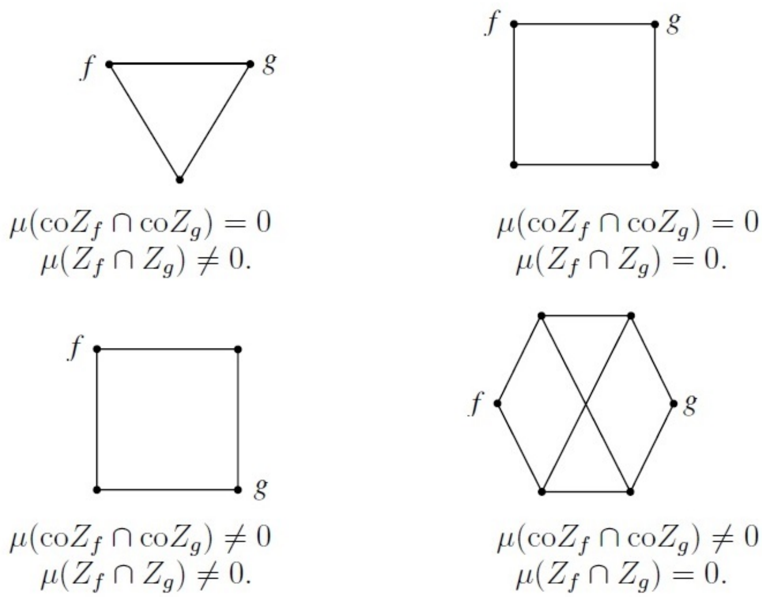
$$fg=g\chi_{Z_f\cap Z_g}=\chi_{Z_f\cap Z_g}f=0$$

a.e. on $(X,\mathcal{A},\mu)$. This means that $c(f,g)=3$, which is a contradiction.

(c) If $c(f,g)=6$, then parts (a) and (b) imply that $\mu(\mathrm{co}Z_f\cap \mathrm{co}Z_g)\neq 0$ and $\mu(Z_f\cap Z_g)=0$. Conversely, since $\mu(\mathrm{co}Z_f\cap \mathrm{co}Z_g)\neq 0$ and $\mu(Z_f\cap Z_g)=0$, then by part (c) of Theorem 2.5, $d(f,g)=3$. Hence there exist vertices h and k such that $fh=hk=kg=0$ a.e. on $(X,\mathcal{A},\mu)$. Now if some vertex t is adjacent to g, then $\mathrm{co}Z_t\subseteq Z_g$ and $\mathrm{co}Z_h\subseteq Z_f$ a.e. on $(X,\mathcal{A},\mu)$ imply that

$$\mu(\mathrm{co}Z_t\cap \mathrm{co}Z_h)\leq \mu(Z_f\cap Z_g)=0$$

and so t is adjacent to h. This shows that $c(f,g)\geq 5$. But $d(f,g)=3$ implies that f is not adjacent to t and hence $c(f,g)\geq 6$. If we consider the vertices $p := 2h$ and $q := 2k$, then we have a cycle with vertices f, g, h, k, p and q, and so $c(f, g) = 6$.



□

Theorem 3.9. *Let* $[f]$ *and* $[g]$ *be two vertices of* $\Gamma_E(M(X, \mathcal{A}, \mu))$. *Then the following properties hold:*

(a) $c([f], [g]) = 3$ *if and only if* $\mu(coZ_f \cap coZ_g) = 0$ *and* $\mu(Z_f \cap Z_g) \neq 0$.

(b) $c([f], [g]) = 4$ *if and only if one of the following statements hold:*

(1)$\mu(coZ_f \cap coZ_g) = 0$, $\mu(Z_f \cap Z_g) = 0$ *and both* coZ_f *and* coZ_g *are not near-zero sets.*

(2) $\mu(coZ_f \cap coZ_g) \neq 0$, $\mu(Z_f \cap Z_g) \neq 0$ *and* $Z_f \cap Z_g$ *is not near-zero.*

(c) $c([f], [g]) = 5$ *if and only if* $\mu(coZ_f \cap coZ_g) \neq 0$, $\mu(Z_f \cap Z_g) \neq 0$ *and* $Z_f \cap Z_g$ *is a near-zero set.*

(d) $c([f], [g]) = 6$ *if and only if* $\mu(coZ_f \cap coZ_g) \neq 0$, $\mu(Z_f \cap Z_g) = 0$ *and both* $coZ_f \backslash coZ_g$ *and* $coZ_g \backslash coZ_f$ *are not near-zero sets.*

(e) $c([f], [g]) = \infty$ *if and only if one of the following statements hold:*

(1) $\mu(coZ_f \cap coZ_g) = 0$, $\mu(Z_f \cap Z_g) = 0$ *and* coZ_f *or* coZ_g *is near-zero.*

(2) $\mu(coZ_f \cap coZ_g) \neq 0$, $\mu(Z_f \cap Z_g) = 0$ *and* $coZ_f \backslash coZ_g$ *or* $coZ_g \backslash coZ_f$ *is near-zero.*

Proof. (a) Assume that $c([f], [g]) = 3$. Then $c(f, g) = 3$ and so $\mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) = 0$ and $\mu(Z_f \cap Z_g) \neq 0$, by Theorem 3.8(a). Conversely, suppose that $\mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) = 0$ and $\mu(Z_f \cap Z_g) \neq 0$. Then $\chi_{Z_f \cap Z_g}$ is a vertex of $\Gamma(M(X, \mathcal{A}, \mu))$ and $fg = g\chi_{Z_f \cap Z_g} = f\chi_{Z_f \cap Z_g} = 0$ a.e. on $(X, \mathcal{A}, \mu)$. It is easy to check that $[f] \cap [g] \cap [\chi_{Z_f \cap Z_g}] = \emptyset$ and $c([f], [g]) = 3$.

(b) Suppose that $c([f], [g]) = 4$. Then $c(f, g) \leq 4$. If $c(f, g) = 3$, then $c([f], [g]) = 3$, by part (a) and Theorem 3.8(a), which is a contradiction. Therefore $c(f, g) = 4$ and we have two cases, by Theorem 3.8(b):

Case 1: $\mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) = 0$ and $\mu(Z_f \cap Z_g) = 0$. Then f is adjacent to g and $\mathrm{co}Z_f \cap \mathrm{co}Z_g = X$ a.e. on $(X, \mathcal{A}, \mu)$. If $\mathrm{co}Z_f$ is near-zero, then for every $h, k \in \Gamma(M(X, \mathcal{A}, \mu))$ such that $fg = gh = hk = kf = 0$ a.e. on $(X, \mathcal{A}, \mu)$, $h \in [f]$, which is a contradiction.

Case 2: $\mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) \neq 0$ and $\mu(Z_f \cap Z_g) \neq 0$. If $Z_f \cap Z_g$ is near-zero, then for every $h, k \in \Gamma(M(X, \mathcal{A}, \mu))$ such that $fh = hg = gk = kf = 0$, $\mathrm{co}Z_h = Z_f \cap Z_g$ and $\mathrm{co}Z_k = Z_f \cap Z_g$ a.e. on $(X, \mathcal{A}, \mu)$. This means that $h, k \in [\chi_{Z_f \cap Z_g}]$, which is a contradiction.

Conversely, if $\mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) = 0$, $\mu(Z_f \cap Z_g) = 0$ and both $\mathrm{co}Z_f$ and $\mathrm{co}Z_g$ are not near-zero sets, then f is adjacent to g and there exist $A \subseteq \mathrm{co}Z_f$ and $B \subseteq \mathrm{co}Z_g$ such that $\mu(A) < \mu(\mathrm{co}Z_f)$, $\mu(B) < \mu(\mathrm{co}Z_g)$, $\mu(A) \neq 0$ and $\mu(B) \neq 0$. Thus $[f][g] = [f][\chi_B] = [\chi_B][\chi_A] = [\chi_A][g] = 0$ and hence $c([f], [g]) \leq 4$. If $c([f], [g]) = 3$, then $\mu(Z_f \cap Z_g) \neq 0$, by part (a), which is a contradiction.

If $\mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) \neq 0$, $\mu(Z_f \cap Z_g) \neq 0$ and $Z_f \cap Z_g$ is not near-zero, then f is not adjacent to g and there exist two measurable sets $A, B \subseteq Z_f \cap Z_g$ such that $A \cap B = \emptyset$ and $\chi_A, \chi_B \in \Gamma(M(X, \mathcal{A}, \mu))$. Therefore $[f][\chi_A] = [\chi_A][g] = [g][\chi_B] = [\chi_B][f] = 0$ and so $c([f], [g]) \leq 4$. If $c([f], [g]) = 3$, then $\mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) = 0$, by part (a), which is a contradiction.

(c) Suppose that $c([f], [g]) = 5$. Then $c(f, g) \leq 5$. By Theorem 3.8, $c(f, g) = 3$ or 4. If $c(f, g) = 3$, then $d([f], [g]) = 3$, by part (a) and Theorem 3.8(a), which is a contradiction. Therefore $c(f, g) = 4$ and we have two cases, by Theorem 3.8(b):

Case 1: $\mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) = 0$ and $\mu(Z_f \cap Z_g) = 0$. If $\mathrm{co}Z_f$ and $\mathrm{co}Z_g$ are not near-zero, then $c([f], [g]) = 4$, by part (b), which is a contradiction. If $\mathrm{co}Z_f$ is near-zero, then for every vertex h such that $[gh] = 0$, $h \in [f]$. Therefore $c([f].[g]) = \infty$.

Case 2: $\mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) \neq 0$ and $\mu(Z_f \cap Z_g) \neq 0$. If $Z_f \cap Z_g$ is not near-zero, then $c([f], [g]) = 4$, by part (b), which is a contradiction.

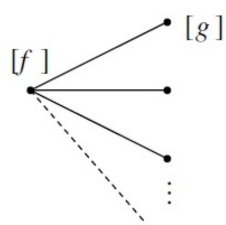
Therefore $c([f], [g]) = 5$ implies that $\mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) \neq 0$, $\mu(Z_f \cap Z_g) \neq 0$ and $Z_f \cap Z_g$ is a near-zero set.

Conversely, suppose that $\mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) \neq 0$, $\mu(Z_f \cap Z_g) \neq 0$ and $Z_f \cap Z_g$ is a near-zero set. By Theorem 2.5(b), $d(f, g) = 2$ and there exists a vertex h such that $fh = gh = 0$ a.e. on $(X, \mathcal{A}, \mu)$. Since

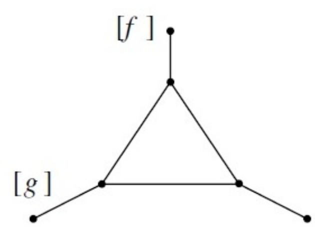
$Z_f \cap Z_g$ is near-zero, then $h \in [\chi_{Z_f \cap Z_g}]$. On the other hand $d(f,g) = 3$ in the zero-divisor graph of $\mathrm{co}Z_f \cup \mathrm{co}Z_g$, by Theorem 2.5(c). Therefore there exists two vertices k and t such that $fk = kt = tg = 0$ a.e. on $(X, \mathcal{A}, \mu)$. Since $d(f,g) = 3$ in the zero-divisor graph of $\mathrm{co}Z_f \cup \mathrm{co}Z_g$, $[f] \cap [k] \cap [t] \cap [g] \cap [h] = \emptyset$ and therefore $c([f],[g]) = 5$.

(d) Suppose that $c([f],[g]) = 6$. Then $\mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) \neq 0$ and $\mu(Z_f \cap Z_g) = 0$, by parts (a), (b) and (c). If $\mathrm{co}Z_f \backslash \mathrm{co}Z_g$ is near-zero, then $[g]$ is only adjacent to $[\chi_{\mathrm{co}Z_f \backslash \mathrm{co}Z_g}]$, which is a contradiction. Conversely, suppose that $\mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) \neq 0$, $\mu(Z_f \cap Z_g) = 0$ and both $\mathrm{co}Z_f \backslash \mathrm{co}Z_g$ and $\mathrm{co}Z_g \backslash \mathrm{co}Z_f$ are not near-zero sets. Then $c([f],[g]) \geq 6$, by parts (a), (b) and (c). Since $\mathrm{co}Z_f \backslash \mathrm{co}Z_g$ and $\mathrm{co}Z_g \backslash \mathrm{co}Z_f$ are not near-zero sets, there exists $A, B \in M_\mu$ such that $A \subsetneq \mathrm{co}Z_f \backslash \mathrm{co}Z_g$, $B \subsetneq \mathrm{co}Z_g \backslash \mathrm{co}Z_f$, $\mu(A) \neq \mu(\mathrm{co}Z_f \backslash \mathrm{co}Z_g)$ and $\mu(B) \neq \mu(\mathrm{co}Z_g \backslash \mathrm{co}Z_f)$. Therefore $[f], [g], [\chi_A], [\chi_B], [1-\chi_A]$ and $[1-\chi_B]$ are different classes in $\Gamma_E(M(X, \mathcal{A}, \mu))$ and $[f][\chi_B] = [\chi_B][\chi_A] = [\chi_A][g] = [g][1-\chi_A] = [1-\chi_A][1-\chi_B] = [1-\chi_B][f] = 0$.

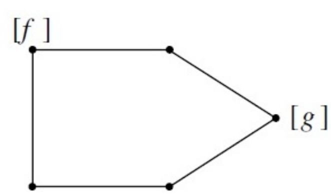
(e) The proof of this part is a consequence of the proofs of parts (c) and (d).



$\mu(Z_f \cap Z_g) = 0$
$\mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) = 0$
$\mathrm{co}Z_f$ is near-zero.



$\mu(Z_f \cap Z_g) = 0$
$\mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) \neq 0$
$|M_\mu| = 6.$



$\mu(\mathrm{co}Z_f \cap \mathrm{co}Z_g) \neq 0$
$\mu(Z_f \cap Z_g) \neq 0$
$Z_f \cap Z_g$ is near-zero.

□

4. Continuity properties of $\Gamma(M(X, \mathcal{A}, \mu))$

In this section, we assume that μ is a measure on a locally compact Hausdorff space X which has the properties stated in Riesz Representation Theorem [21, Theorem 2.14]. Since the continuous functions played such a prominent role in the construction of Borel measures, it seems reasonable to expect that there are some interesting relations between continuous functions and the zero-divisor graph of the ring of measurable functions. In the following, we shall give two main theorems of this kind. In the first theorem, we approximate the vertices of $\Gamma(M(X, \mathcal{A}, \mu))$ by the vertices of the zero-divisor graph of $C_C(X)$, denoted by $\Gamma(C_C(X))$. In the second theorem, we give a relation between continuity and the edges of $\Gamma(M(X, \mathcal{A}, \mu))$.

We recall that a *Hausdorff space* is a topological space in which distinct points have disjoint neighbourhoods. A topological space X is *called locally compact*, if every point $x \in X$ has a compact neighbourhood. A topological space X is a *completely regular space* if given any closed set $F \subseteq X$ and any point $x \in X$ that does not belong to F, then there is a continuous function f from X to the real line $\mathbb{R}$ such that $f(x) = 0$ and, for every $y \in F$, $f(y) = 1$. The *support* of a function f on a topological space X is the closure of the set $\{x \in X : f(x) \neq 0\}$, denoting by $\mathrm{supp}(f)$. The collection of all continuous functions on a completely regular Hausdorff space X whose support is compact is denoted by $Cc(X)$. For every function $f : X \longrightarrow [-\infty, +\infty]$, $|f| = \sup\{|f(x)| : x \in X\}$. The reader is referred to [11, 14] for undefined terms and concepts.

To enter the discussion, we recall that a corollary of the Lusin theorem [21, Theorem 2.24]: Suppose that f is a complex measurable function on X, $\mu(A) < \infty$, $f(x) = 0$ if $x \notin A$ and $|f| \leq 1$. Then there exists a sequence $g_n \in C_C(X)$ such that $|g_n| \leq 1$ and $f(x) = \lim g_n(x)$ a.e. on $(X, \mathcal{A}, \mu)$.

Theorem 4.1. *For every vertex f of $\Gamma(M(X, \mathcal{A}, \mu))$ which $\mu(\mathrm{co}\, Z_f) < \infty$ and $|f| \leq 1$, there exists a sequence of vertices $\{f_n\}$ of $\Gamma(C_C(X))$ such that*

$$f(x) = \lim f_n(x) \;\; a.e. \;\; on \;\; (X, \mathcal{A}, \mu).$$

Proof. Let f be a vertex of $\Gamma(M(X, \mathcal{A}, \mu))$, $|f| \leq 1$ and $\mu(\mathrm{co}\, Z_f) < \infty$. Using Lusin Theorem [21, Theorem 2.24], for every $n \in \mathbb{N}$, there exists $f_n \in C_C(X)$ such that

$$\mu(E_n = \{x : f(x) \neq f_n(x)\}) < 2^{-n}.$$

We claim that every $x \in X$ lies in at most finitely many of the sets E_n. Let $g := \Sigma_{n=1}^{\infty} \chi_{E_n}$ and

$$K := \{x \in X : x \text{ lies in infinitely many } E_n\}.$$

It is easy to check that $x \in K$ if and only if $g(x) = \infty$. Now we have

$$\int_X g d\mu = \int_X \Sigma_{n=1}^{\infty} \chi_{E_n} d\mu = \Sigma_{n=1}^{\infty} \int_X \chi_{E_n} d\mu = \sum_{n=1}^{\infty} \mu(E_n) \leq \sum_{n=1}^{\infty} 2^{-n} = 1.$$

This implies that $g \in L^1(X, \mathcal{A}, \mu)$ and so $\mu(K) = 0$. Thus for every $x \in X$ and all large enough n, $f(x) = f_n(x)$ and hence

$$f(x) = \lim f_n(x) \text{ a.e. on } (X, \mathcal{A}, \mu).$$

Now, we claim that $\{f_n\}$ has a subsequence of the vertices of $\Gamma(C_C(X))$. If for infinitely many $n \in \mathbb{N}$, $\mu(\mathrm{co}Z_{f_n}) = 0$, then there exists a sequence $\{n_k\} \subseteq \mathbb{N}$ such that for every $k \in \mathbb{N}$, $\mu(\mathrm{co}Z_{f_{n_k}}) = 0$ and $f(x) = \lim f_{n_k}(x)$ a.e. on $(X, \mathcal{A}, \mu)$. According to the assumptions and measure properties, for every $n \in \mathbb{N}$,

$$\mu(\mathrm{co}Z_f) \leq \mu(\mathrm{co}Z_{f_{n_k}}) + \mu(E_n) \leq 2^{-n}.$$

This means that $\mu(\mathrm{co}Z_f) = 0$, which is a contradiction. Now suppose that for infinitely many $n \in \mathbb{N}$, $\mu(Z_{f_n}) = 0$. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that for every $k \in \mathbb{N}$, $\mu(Z_{f_{n_k}}) = 0$. Therefore for every $k \in \mathbb{N}$, f_{n_k} is unit a.e. on $(X, \mathcal{A}, \mu)$ and $\mu(Z_f \backslash E_{n_k}) = 0$. As a consequence of the assumptions, for every $k \in \mathbb{N}$,

$$\mu(Z_f) \leq \mu(E_{n_k}) \leq 2^{-n_k}.$$

This implies that $\mu(Z_f) = 0$, which is a contradiction. Therefore without considering the elements of $\{f_n\}$ which their cozero sets are not in M_μ, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that all members are in $\Gamma(C_C(X))$. □

In order to establish a relation between continuity and the edges of $\Gamma(M(X, \mathcal{A}, \mu))$, we need the following definition.

Definition 4.2. A measurable function $f \in M(X, \mathcal{A}, \mu)$ is called *ϵ-continuous* if

$$\mu(\{x \in X : f \text{ is not continuous at } x\}) < \epsilon.$$

Now, we find a relationship between the edges of the graph $\Gamma(M(X, \mathcal{A}, \mu))$ and the edges of $\Gamma(C_C(X))$.

Theorem 4.3. *Let $f, g \in \Gamma(M(X, \mathcal{A}, \mu))$, $|f| \leq 1$, $|g| \leq 1$ and $\sum_{n=1}^{\infty} \epsilon_n$ be a convergence series in real line $\mathbb{R}$. Then f is adjacent to g if and* *only if there exist two sequences* $\{f_n\}$ *and* $\{g_n\}$ *in* $\Gamma(C_C(X))$ *such that the following statements hold:*

(1) *For every* $n \in \mathbb{N}$, f_n *and* g_n *are* ϵ_n*-continuous.*
(2) $\{f_n\}$ *and* $\{g_n\}$ *pointwise convergence to* f *and* g, *respectively.*
(3) $\{f_n\}$ *and* $\{g_n\}$ *are the parts of a complete bipartite graph.*

Proof. Using Lusin Theorem [21, Theorem 2.24], for every $n \in \mathbb{N}$, there exists $h_n \in C_C(X)$ such that

$$\mu(E_n = \{x : f(x) \neq h_n(x)\}) < \epsilon_n.$$

As the proof of Theorem 4.1, since $\sum_{n=1}^{\infty} \epsilon_n < \infty$, $f(x) = \lim h_n(x)$ a.e. on $(X, \mathcal{A}, \mu)$. For each $n \in \mathbb{N}$, we define

$$f_n(x) := \begin{cases} h_n(x) & x \in E_n^c, \\ 0 & x \in E_n. \end{cases}$$

It is easy to check that f_n is ϵ_n-continuous function and $\mu(Z_{f_n}) \geq \mu(Z_{h_n}) \neq 0$, for every $n \in \mathbb{N}$. If for infinitely many $n \in \mathbb{N}$, $\mu(\mathrm{co}Z_{f_n}) = 0$, then there exists $\{n_k\} \subseteq \mathbb{N}$ such that $\mu(\mathrm{co}Z_{f_{n_k}}) = 0$ and $\{f_{n_k}\}$ pointwise converges to f, for every $k \in \mathbb{N}$. This means that $\mu(\mathrm{co}Z_f) = 0$, which is a contradiction. Therefore we can assume that $f_n \in \Gamma(C_C(X))$, for every $n \in \mathbb{N}$. On the other hand, for ever $n \in \mathbb{N}$,

$$\mu(\{x : f(x) \neq f_n(x)\}) \leq \mu(E_n) < \epsilon_n.$$

This means that $f(x) = \lim f_n(x)$ a.e. on $(X, \mathcal{A}, \mu)$. If for $m, n \in \mathbb{N}$, f_n is adjacent to f_m, then

$$\mu(\mathrm{co}Z_f) \leq \mu(E_n) + \mu(E_m) \leq \epsilon_n + \epsilon_m.$$

Now if for infinitely many $m, n \in \mathbb{N}$, f_m is adjacent to f_n, $\mu(\mathrm{co}Z_f) = 0$, which is a contradiction. Therefore without considering the elements of $\{f_n\}$ which they are adjacent, there exists a subsequence of $\{f_n\}$ such that f_n is not adjacent to f_m, for every $m, n \in \mathbb{N}$. Similarly, there exists a sequence of ϵ-continuous functions $\{g_n\}$ in $\Gamma(C_C(X))$ such that $\{g_n\}$ pointwise convergence to g and g_n is not adjacent to g_m, for every $m, n \in \mathbb{N}$. By the definition of $\{f_n\}$ and $\{g_n\}$, $\mathrm{co}Z_{f_n} \subseteq \mathrm{co}Z_f$ and $\mathrm{co}Z_{g_n} \subseteq \mathrm{co}Z_g$, for every $n \in \mathbb{N}$. Now since f is adjacent to g, for every $m, n \in \mathbb{N}$, f_n is adjacent to g_m. Therefore $\{f_n\}$ and $\{g_n\}$ are the parts of a bipartite graph.

Conversely, assume that $\{f_n\}$ and $\{g_n\}$ are two sequences in $\Gamma(C_C(X))$ such that the conditions (1), (2) and (3) are true. Now suppose that $\mu(\mathrm{co}Z_{f_k} \cap \mathrm{co}Z_g) \neq 0$, for $k \in \mathbb{N}$. Since for every $m, n \in \mathbb{N}$, f_n and g_m are adjacent, $\{g_n\}$ pointwise convergence to $g(1 - \chi_{\mathrm{co}Z_{f_k} \cap \mathrm{co}Z_g})$, which is a contradiction. This means that for every $n \in \mathbb{N}$, f_n is adjacent to g. Similarly, for every $n \in \mathbb{N}$, g_n is adjacent to f. Therefore by the assumptions, f is adjacent to g. $\square$

Remark 4.4. According to Theorems 4.1 and 4.3, in some cases, for the study of $\Gamma(M(X, \mathcal{A}, \mu))$, we can use the behavior of the members of $\Gamma(C_C(X))$ and ε-continuous functions. The question that arises is that how can we characterize the features of the graph $\Gamma(M(X, \mathcal{A}, \mu))$ by the continuous functions?

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ZERO-DIVISOR GRAPH OF THE RINGS OF REAL MEASURABLE FUNCTIONS WITH THE MEASURES

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گراف مقسوم علیه صفر حلقه‌های توابع اندازه پذیر با اندازه‌ها

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فرض کنیم $M(X,\mathcal{A},\mu)$ حلقه توابع انداره پذیر روی فضای اندازه پذیر $(X,\mathcal{A})$ با اندازه μ باشد. در این مقاله گراف مقسوم علیه صفر $M(X,\mathcal{A},\mu)$ که با $\Gamma(M(X,\mathcal{A},\mu))$ نمایش داده می‌شود را مطالعه می‌کنیم. ارتباط بین خواص گرافی $\Gamma(M(X,\mathcal{A},\mu))$، خواص حلقه‌ای $M(X,\mathcal{A},\mu)$ و خواص اندازه‌ای $(X,\mathcal{A},\mu)$ را ارایه می‌دهیم. در نهایت خواص پیوستگی $\Gamma(M(X,\mathcal{A},\mu))$ را بررسی می کنیم.

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