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SOME INEQUALITIES FOR POLYNILPOTENT MULTIPLIER OF POWERFULL p-GROUPS

M. ALIZADEH SANATI

Abstract. In this paper, we present some inequalities for the order, the exponent, and the number of generators of the polynilpotent multiplier, the Baer invariant with respect to the variety of polynilpotent groups of class row $(c_1, \ldots, c_t)$ of a powerful p-group. Our results extend some of Mashakekhy and Maohammadzadeh's in 2007 to polynilpotent multipliers.

1. Introduction

Let $R \rightarrowtail G \twoheadrightarrow F$ be a free presentation of an arbitrary group G. If $\mathcal{V}$ is the variety of polynilpotent groups of class row $(c_1, \ldots, c_t), \mathcal{N}_{c_1,\ldots,c_t}$, then the Baer invariant of group G with respect to this variety which we call it *the polynilpotent multiplier* of G is as follows:

$$\mathcal{N}_{c_1,\ldots,c_t}M(G) \simeq \frac{R \cap \gamma_{c_1,c_2,\ldots,c_t}(F)}{\gamma_{c_1,c_2,\ldots,c_t}(R,F)},$$

where $\gamma_{c_1,c_2,\ldots,c_t}(F) = \gamma_{c_t+1}\left(\ldots\left(\gamma_{c_2+1}\left(\gamma_{c_1+1}(F)\right)\right)\ldots\right)$ is the term of iterated lower central series of F and by corollary 6.14 of [5], we have

$$\gamma_{c_1,c_2,\ldots,c_t}(R,F) = [R,_{c_1} F,_{c_2} \gamma_{c_1+1}(F), \ldots,_{c_t} \gamma_{c_{t-1}+1}\left(\ldots \gamma_{c_1+1}(F)\ldots\right)].$$

In particular, if $t = 1$ and $c_1 = c$ then the above notation will be the c-nilpotent multiplier of G, $\mathcal{N}_c\left(M(G)\right) \simeq \left(R \cap \gamma_{c+1}(F)\right)/[R,_c F]$. Also,

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the case $c = 1$ is the much studied Schur multiplier of G and denoted by $\mathcal{M}(G)$.

Historically, there have been several papers from the beginning of the twentieth century trying to find some structures for the well-known notion the Schur multiplier and its varietal generalization the Baer invariant of some famous products of groups, such as the direct product, the free product and the nilpotent product. Determining these Baer invariants of a given group is known to be very useful for classification of groups into isologism classes. Also structures of Baer invariants are very essential for studying varietal capability and covering groups.

In the Mid-20th Century the relationship between the exponent of the Schur multiplier of a p-group and the exponent of the group itself was examined, whether $e\left(\mathcal{M}(G)\right) \leq e(G)$. In 1973 a group of exponent 4 was constructed [1] whereas its Schur multiplier has exponent 8, hence the conjecture is not true in general. In 1973 Jones [6] proved that the exponent of the Schur multiplier of a finite p-group of class $c \geq 2$ and exponent p^e is at most $p^{e(c-1)}$. A result of Ellis [3] shows that if G is a p-group of class $k \geq 2$ and exponent p^e, then $exp(\mathcal{M}^{(c)}(G)) \leq p^{e[k/2]}$, where $[k/2]$ denotes the smallest integer n such that $n \geq k/2$. For $c = 1$, Moravec [12] showed that $[k/2]$ can be replaced by $2[log_2 k]$ which is an improvement if $k \geq 11$. Also he proved that if G is a metabelian group of exponent p, then $exp(\mathcal{M}(G))$ divides p. Kayvanfar and Sanati [7] proved that $exp(\mathcal{M}(G))|exp(G)$ when G is a finite p-group of class 3, 4 or 5 under some arithmetical conditions on p and the exponent of G. In 1987 Lubotzky and Mann [8] presented some inequalities for the Schur multiplier of a powerful p-group. They gave a bound for the order, the exponent and the number of generators of the Schur multiplier of a powerful p-group. Then Mashayekhy and Mohammadzadeh generalized Lubotzky and Mann's results to the nilpotent multipliers. Their consequences improve the previous inequalities for powerful p-groups. In this paper we will extend some results of Mashayekhy and Mohammadzadeh [10] to the polynilpotent multipliers and give some upper bounds for the order, the exponent and the number of generators of the polynilpotent multiplier of a d-generator powerful p-group G. Finally, by giving some examples of groups we will show tightness of our results.

2. Main results

This section deals with prerequisite concepts and results which will be used in the next section. We use techniques involving the concept of basic commutators. Here is the definition. Let X be an arbitrary

subset of a free group, and select an arbitrary total order for X. The basic commutators on X, their weight w_t, and the ordering among them are defined as follows:
(i) The elements of X are basic commutators of weight one, ordered according to the total order previously chosen.
(ii) Having defined the basic commutators of weight less than n, a basic commutator of weight n is $c = [b, a]$, where:
(a) b and a are basic commutators and $w_t(b) + w_t(a) = n$, and
(b) $b > a$, and if $b = [b_1, b_2]$, then $a \geq b_2$.
(iii) The basic commutators of weight n follow those of weight less than n. The basic commutators of weight n are ordered among themselves in any total order, but the most common used total order is lexicographic order; that is, if $[b_1, a_1]$ and $[b_2, a_2]$ are basic commutators of weight n, then $[b_1, a_1] < [b_2, a_2]$ if and only if $b_1 < b_2$ or $b_1 = b_2$ and $a_1 < a_2$.

Theorem 2.1. *(M . Hall [4]) Let F be a free group on $\{x_1, x_2, \ldots, x_d\}$. Then for all $1 \leq i \leq n$, $\dfrac{\gamma_n(F)}{\gamma_{n+i}(F)}$ is a free abelian group freely generated by the basic commutators of weights $n, n+1, \ldots, n+i-1$ on the letters $\{x_1, x_2, \ldots, x_d\}$.*

Theorem 2.2. *(Witt Formula [4]). The number of basic commutators of weight n on d generators is given by the following formula:*

$$\chi_n(d) = \frac{1}{n} \sum_{m|n} \mu(m) d^{\frac{n}{m}}$$

where $\mu(m)$ is the Mobious function which defined to be

$$\mu(m) = \begin{cases} 1 & ;\ m = 1 \\ 0 & ;\ m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}\ \exists\ \alpha_i > 1 \\ (-1)^s & ;\ m = p_1 p_2 \cdots p_s, \end{cases}$$

where the p_i are distinct prime numbers.

Powerful p-groups were introduced in 1987 by Lubotzky and Mann. A number of applications are given, including results on Schur multipliers. Powerful p-groups are used in the study of automorphisms of p-groups, the solution of the restricted Burnside problem, the classification of finite p-groups via the coclass conjectures, and provided an excellent method of understanding analytic pro-p-groups. We will discuss some of them in this section. A finite p-group G is called *powerful*, if

$$\begin{cases} G' \subseteq G^p;\ p \text{ is odd} \\ G' \subseteq G^4;\ p = 2. \end{cases}$$

It can be easily proved that N is powerfully embedded in G if and only if $N/[N,G,G]$ is powerfully embedded in $G/[N,G,G]$ (Theorem 1.1, [8]). Hence to prove that a normal subgroup N is powerfully embeddeding G we can assume that (i) $[N,G,G]=1$, (ii) $N^p=1(N^4=1$ for $p=2)$ and try to show that $[N,G]=1$, and $(iii)[N,G]^2=1$ whenever $p=2$.

Any powerfully embedded subgroup is itself a powerful p-group and must be normal in the whole group. Also a p-group is powerful exactly when it is powerfully embedded in itself. This property is not subgroup-inherited [8], but it is closed with respect to homomorphic image and direct sum. We will require some standard properties of powerful p-groups in the following lemma.

Lemma 2.3. *The following statements hold for a powerful p-group G.*
(i) $\gamma_i(G), G^i, G^p = \phi(G)$ *(the Frattini subgroup of G) are powerfully embedded in G.*
(ii) If $G=<a_1,a_2,...,a_d>$*, then* $G^{p^i}=<a_1^{p^i},a_2^{p^i},\cdots,a_d^{p^i}>$.
(iii) If $H\leq G$*, then* $d(H)\leqslant d(G)$.

In order to prove the main results we need the following lemma, which is proved by Mashayekhy and Mohammadzadeh [10].

Lemma 2.4. *Let* F/R *be a free presentation of a powerful d-generator p-group G. Let* $Z=R/[R,_cF]$ *and* $H=F/[R,_cF]$*, so that* $G\simeq H/Z$*. Then* $\gamma_{c+1}(H)$ *is powerfully embedded in H and* $d(\gamma_{c+1}(H))\leqslant\chi_{c+1}(d)$.

The next theorem is an interesting result of this lemma.

Theorem 2.5. *Let G be a powerful p-group d-generators and* $R\rightarrowtail F\twoheadrightarrow G$ *be a free presentation for G. Put* $Z=\dfrac{R}{\gamma_{c_1,c_2,...,c_t}(R,F)}$ *and* $H=\dfrac{F}{\gamma_{c_1,c_2,...,c_t}(R,F)}$*. Then* $G\cong H/Z$ *and* $\gamma_{c_1,c_2,...,c_t}(H)$ *is powerfully embeded in H and* $d(\gamma_{c_1,c_2,...,c_t}(H))\leq\chi_{c_t+1}(\ldots(\chi_{c_1+1}(d))\ldots)$.

Proof. Consider $H_1=F/[R,_{c_1}F]$ and $Z_1=R/[R,_{c_1}F]$ and apply Lemma 2.4 for $G=H_1/Z_1$. One conclude $\gamma_{c_1+1}(F/[R,_{c_1}F])$ is powerfully embedded in H_1 and $d(\gamma_{c_1+1}(H_1))\leqslant\chi_{c_1+1}(d)$. Now, consider $F_1:=\gamma_{c_1+1}(F)$, $R_1:=[R,_{c_1}F]$ and $G_1:=\gamma_{c_1+1}(H_1)$. By Nielsen-Schreier' theorem F_1 is a free group and $R_1\rightarrowtail F_1\twoheadrightarrow G_1$ is a free presentation. Put $H_2=F_1/[R_1,_{c_2}F_1]=\gamma_{c_1+1}(F)/[R,_{c_1}F,_{c_2}\gamma_{c_1+1}(F)]$ and $Z_2=R_1/[R_1,_{c_2}F_1]$. Then $G_1=H_2/Z_2$ and by noting that $d(G_1)\leqslant\chi_{c_1+1}(d)$, Lemma 2.4 for G_1 states

$$\gamma_{c_2+1}(H_2)=\gamma_{c_2+1}\left(\gamma_{c_1+1}(F)\right)/[R,_{c_1}F,_{c_2}\gamma_{c_1+1}(F)]=\gamma_{c_1,c_2}(F)/\gamma_{c_1,c_2}(R,F)$$

is powerfully embedded in H_2 and $d(\gamma_{c_2+1}(H_2))\leqslant\chi_{c_2+1}(\chi_{c_1+1}(d))$. By continuing this method the result holds. □

An interesting corollary of this theorem is as follows.

Theorem 2.6. *Let G be powerful p-group with $d(G) = d$. Then*
$$d(\mathcal{N}_{c_1,\ldots,c_t}M(G)) \leqslant \chi_{c_t+1}\left(\ldots\left(\chi_{c_1+1}(d)\right)\ldots\right).$$

Proof. Let F/R be a free presentation of G with $Z = R/\gamma_{c_1,c_2,\ldots,c_t}(R,F)$ and $H = F/\gamma_{c_1,c_2,\ldots,c_t}(R,F)$. Then $G \simeq H/Z$ and the above result and Lemma 2.3(iii) implies that

$$d\left(\frac{R\cap\gamma_{c_1,c_2,\ldots,c_t}(R,F)}{\gamma_{c_1,c_2,\ldots,c_t}(R,F)}\right) \leqslant d\left(\frac{\gamma_{c_1,c_2,\ldots,c_t}(F)}{\gamma_{c_1,c_2,\ldots,c_t}(R,F)}\right) \leqslant \chi_{c_t+1}\left(\ldots\left(\chi_{c_1+1}(d)\right)\ldots\right).$$

Hence the result follows. □

Theorem 2.7. *For each powerful p-group G, $exp(\mathcal{N}_{c_1,\ldots,c_t}M(G))|exp(G)$.*

By applying Theorems 2.6 and 2.7 we have

Theorem 2.8. *Let G be a powerful d-generator p-group and $exp(G) = p^e$. Then*
$$|\mathcal{N}_{c_1,\ldots,c_t}M(G)| \leqslant p^{\chi_{c_t+1}\left(\ldots\left(\chi_{c_1+1}(d)\right)\ldots\right)e}.$$

If in the above theorems $t = 1$, we have the following results of Mashayekhy and Mohammadzadeh [10].

Corollary 2.9. *Let G be powerful p-group with $d(G) = d$ with $exp(G) = p^e$. Then*
$$d(\mathcal{N}_cM(G)) \leqslant \chi_{c+1}(d)\ ,\ exp(\mathcal{N}_cM(G))|exp(G)\ ,\ |\mathcal{N}_cM(G)| \leqslant p^{\chi_{c+1}(d)e}.$$

We give an explicit example showing the tightness of our results. By [9] the polynilpotent multiplier of finitely generated abelian groups can be calculated explicitly as follows:

Lemma 2.10. *Suppose that $G \cong \mathbb{Z}^m \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_d}$ is finitely generated abelian group where $n_{i+1}|n_i$ for all $1 \leq i \leq d-1$. Then*
$$\mathcal{N}_{c_1,\ldots,c_t}M(G) = \mathbb{Z}^{(f_m)} \oplus \mathbb{Z}_{n_1}^{(f_{m+1}-f_m)} \oplus \mathbb{Z}_{n_2}^{(f_{m+2}-f_{m+1})} \oplus \cdots \oplus \mathbb{Z}_{n_d}^{(f_{m+d}-f_{m+d-1})},$$
where $f_i = \chi_{c_t+1}(\chi_{c_{t-1}+1}(\cdots(\chi_{c_1+1}(i))\ldots))$ and $\mathbb{Z}_n^{(s)}$ denotes the direct sum of s copies of the cyclic group $\mathbb{Z}_n$.

We apply the above result for finite abelian p-groups and by noting that $m = 0$ and $f_0 = f_1 = 0$ conclude the following facts.

Example 2.11. Consider finite abelian p-group $G \cong \mathbb{Z}_{p^{\alpha_1}} \oplus \mathbb{Z}_{p^{\alpha_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{\alpha_d}}$ where $\alpha_1, \alpha_2, \ldots, \alpha_d$ are positive integers and $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_d$. Then
$$\mathcal{N}_{c_1,\ldots,c_t}M(G) = \mathbb{Z}_{p^{\alpha_2}}^{(f_2)} \oplus \mathbb{Z}_{p^{\alpha_3}}^{(f_3-f_2)} \oplus \cdots \oplus \mathbb{Z}_{p^{\alpha_d}}^{(f_d-f_{d-1})} \quad (*),$$

(i) $d(\mathcal{N}_{c_1,\ldots,c_t}M(G)) = f_d = \chi_{c_t+1}(\chi_{c_{t-1}+1}(\cdots(\chi_{c_1+1}(d))\cdots))$, where $d = d(G)$. Hence the bound of Theorem 2.6 is attained and the best one in the abelian case.
(ii) $exp(\mathcal{N}_{c_1,\ldots,c_t}M(G)) = p^{\alpha_2}$, whereas $exp(G) = p^{\alpha_1}$. Hence the bound of Theorem 2.7 is attained when $\alpha_1 = \alpha_2$ and it is the best one in the abelian case.
(iii) $|\mathcal{N}_{c_1,\ldots,c_t}M(G)| = p^{\alpha_2 f_2 + \sum_{i=3}^{d}\alpha_i(f_i - f_{i-1})} \leqslant p^{\alpha_1 f_d}$. Hence the bound of Theorem 2.8 is attained if and only if $\alpha_1 = \alpha_2 = \cdots = \alpha_k$.
We are going to give two nonabelian powerful p-groups in order to compute explicitly the number of generators, the order and the exponent of their $(2,2)$-nilpotent multipliers and then compare these numbers with bounds obtained.

Example 2.12. (i) The finite group $G = \langle a, b : a^2 = 1, aba = b^{-3}\rangle$ is a powerful 2-generated 2-group with the order 16 and the exponent 8. By [[2], Fig.2, ♯ 13] $\mathcal{N}_{2,2}M(G) = \mathcal{N}_2M(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \mathbb{Z}_2$, and hence $|\mathcal{N}_{2,2}M(G)| = 2, d(\mathcal{N}_{2,2}M(G)) = 1$, and $exp(\mathcal{N}_{2,2}M(G)) = 2$. It is seen that the bound of Theorem 2.6 is attained.

(ii) Consider the finite 3-group $G = \langle a, b : a^3 = 1, a^{-1}ba = b^{-2}\rangle$. It is easy to see that G is a powerful 3-group and $|G| = 27, d(G) = 2, exp(G) = 9$. By [[2] , Fig.2 , ♯ 40] $\mathcal{N}_{2,2}M(G) = \mathcal{N}_2M(\mathbb{Z}_3 \oplus \mathbb{Z}_3) = \mathbb{Z}_3$, and hence $|\mathcal{N}_{2,2}M(G)| = 3, d(\mathcal{N}_{2,2}M(G)) = 1$, and $exp(\mathcal{N}_{2,2}M(G)) = 3$. It is also seen that the bound of Theorem 2.6 is attained.

The next example shows that one cannot omit the powerfulness condition in the theorems 2.6 and 2.8.

Example 2.13. Let p be any odd prime and let s, t be positive integers with $s \geq t$. Consider the following finite d-generator p-group with nilpotency class 2:

$$P_{s,t} = \left\langle y_1, \ldots, y_d : y_i^{p^s} = [y_j, y_k]^{p^t} = [[y_j, y_k], y_i] = 1, 1 \leq i, j, k \leq d, j \neq k \right\rangle$$

Clearly $P_{1,1}^p = 1$ and so in general $P_{s,t}$ is not a powerful p-group. By [11] the c_1-nilpotent multiplier of $P_{s,t}$ is as follows: $\mathcal{N}_{c_1}M(P_{s,t}) = \mathbb{Z}_{p^s}^{(\chi_{c_1+1}(d))} \oplus \mathbb{Z}_{p^t}^{(\chi_{c_1+2}(d))}$. Put $d_1 = \chi_{c_1+1}(d)$ and $d_2 = \chi_{c_1+2}(d)$. By applying $(*)$ when $t = 1$ we have

$$\begin{aligned}\mathcal{N}_{c_1,c_2}M(P_{s,t}) &= \mathcal{N}_{c_2}M\left(\mathbb{Z}_{p^s}^{(\chi_{c_1+1}(d))} \oplus \mathbb{Z}_{p^t}^{(\chi_{c_1+2}(d))}\right)\\ &= \mathcal{N}_{c_2}M\left(\mathbb{Z}_{p^s}^{(d_1)} \oplus \mathbb{Z}_{p^t}^{(d_2)}\right)\\ &= \mathbb{Z}_{p^s}^{(b_2)} \oplus \mathbb{Z}_{p^s}^{(b_3-b_2)} \oplus \cdots \oplus \mathbb{Z}_{p^s}^{(b_{d_1}-b_{d_1-1})} \oplus \mathbb{Z}_{p^t}^{(b_{d_1+1}-b_{d_1})} \oplus \cdots \oplus \mathbb{Z}_{p^t}^{(b_{d_1+d_2}-b_{d_1+d_2-1})},\end{aligned}$$

where $b_i = \chi_{c_2+1}(i)$. Now we have the following relations.
(i) $d\,(\mathcal{N}_{c_1,c_2}M(P_{s,t})) = \chi_{c_2+1}(\chi_{c_1+1}(d) + \chi_{c_1+2}(d)) > \chi_{c_1+1}(\chi_{c_1+1}(d))$. Hence the condition of being powerful cannot be omitted from Theorem 2.6.
(ii) $exp\,(\mathcal{N}_{c_1,c_2}M(P_{s,t})) = p^s = exp(P_{s,t})$.
(iii) $|\mathcal{N}_{c_1,c_2}M(P_{s,t})| = p^{(s-t)\chi_{c_2+1}(d_1)+t\chi_{c_2+1}(d_1+d_2)} > p^{s\chi_{c_2+1}(\chi_{c_1+1}(d))}$.
It means powerfulness is also a necessary condition for the bound of Theorem 2.8.

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نامساوی‌هایی برای ضربگر چندپوچ‌توان p-گروه‌های توانمند

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در این مقاله، نامساوی‌هایی برای مرتبه، نما و تعداد مولدهای ضربگر چندپوچ‌توان، پایای بئر نسبت به چندگونای گروه‌های چندپوچ‌توان از ردیف رده $(c_۱, c_۲, \ldots, c_t)$، برای p-گروه‌های توانمند ارائه می‌شود. این نتایج تعمیم قسمتی از مقاله‌ی مشترک مشایخی و محمدزاده در سال ۲۰۰۷ به ضربگر چندپوچ‌توان است.

کلمات کلیدی: p-گروه‌های توانمند، ضربگر چندپوچ‌توان، مرتبه، نما، حداقل تعداد مولدهای یک گروه.