Journal of Algebraic Systems
Vol. 9, No. 2, (2022), pp 281-298

A METRIC-LIKE TOPOLOGY ON BL-ALGEBRAS

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Abstract. This paper is devoted to introduce a topology on BL-algebras, makes them semitopological algebras. For any BL-algebra $\mathcal{L} = (L, \wedge, \vee, *, \rightarrowtail, 0, 1)$, the introduced topology is defined by a distance-like function between elements of L which is defined by $a \leftrightarrow b = (a \rightarrowtail b) * (b \rightarrowtail a)$. We will show that when the continuous scale $[0, 1]$ is endowed to be a BL-algebra, then this topology admits some of the most important properties of the metric topology. Finally, we will show that this topology can be examined by a similar topology on dual of BL-algebras as well.

1. Introduction

Triangular norms and triangular conorms, shortly t-norms and s-norms, have been used in several areas of mathematics. Their origin have been goes back to [10, 11]. One of the areas that t-norms and s-norms have been appeared, is many-valued logics. Indeed, a t-norm (s-norm) is a generalization for the interpretation of the conjunction connective (disjunction connective) [7, 1].

Basic logic introduced by Hájek in the early of 1998 [6], is known as the logic of continuous t-norms. The algebraic counterpart of a propositional basic logic is a BL-algebra. MV-algebras, introduced by Chang [5] to prove the completeness theorem for Łukasiewicz logic, are special types of BL-algebras. A more general algebraic structure originated in logics without contractions is residuated lattice. The

DOI: 10.22044/jas.2021.10296.1509.
MSC(2010): Primary: 03G25; Secondary: 06F30, 06B30.
Keywords: BL-algebra, dual of BL-algebra, topological BL-algebra.
Received: 22 November 2020, Accepted: 14 May 2021.

oldest version of such structure appeared in classical logic is Boolean algebra.

Algebraic structures are studied in algebraic and topological point of view. Algebra studies the property of operations and algorithmic computations of a space, while topology provide a framework for understanding its geometric properties. Besides introducing the concept of BL-algebras [6], their algebraic and topological properties are of the most interesting research areas.

Bozooei et.al in [12, 4] introduced the notion of topological BL-algebras. In [3] they studied the metrizability of BL-algebras as well. The aim of this article is to introduce a metric-like topology on BL-algebras which makes them semitopological algebras in the sense of [4].

One of the biggest obstacles of extending the results of classical logic to basic logic is non-continuity of the interpretations of logical connectives. Therefore, the mentioned topology on BL-algebras could be seen as an applicable tool to extend the results of classical logic to Hájek Basic logic.

Here, for any BL-algebra $\mathcal{L} = (L, \wedge, \vee, *, \rightarrowtail, 0, 1)$, we define two topologies T_* and $\mathbf{T}_*$ on L and L^2 that all the operators of $\mathcal{L}$ becomes continuous function with respect to these topologies. The construction of T_* is based on the $*$-balls $B_r(a) = \{b : a \leftrightarrow b \geq r\}$ in which $a \leftrightarrow b = (a \rightarrowtail b) * (b \rightarrowtail a)$ and so it seems like a metric topology. We show that when the continuous scale $[0, 1]$ is endowed to be a BL-algebra, T_* admits some of the most important properties of the metric topology. This fact results in a simpler way for analysing T_*. Indeed, we show that when $[0, 1]$ is considered to be a BL-algebra, then T_* could be examined by a topology $T_\star$ on $[0, 1]$ as a dual of a BL-algebra. The studying of $T_\star$ on $[0, 1]$ as a dual of a BL-algebra in the cases that $T_\star$ forms a metric topology, has been the subject of the author conference paper [8].

The rest of the paper organized as follows: Section 2 presents a summary of t-norms, s-norms and BL-algebras. Section 3 introduces a topology T_* on any BL-algebra $\mathcal{L} = (L, \wedge, \vee, *, \rightarrowtail, 0, 1)$ which makes it a semitopological algebra. Section 4 shows when $[0, 1]$ is considered to be a BL-algebra, T_* admits some the important properties of the metric topology. Finally, Section 5 defines a dual concept for BL-algebras and examining T_* on $[0, 1]$ by a topology on its dual.

2. Preliminaries

In many-valued logics, t-norms and s-norms sometimes play the role of the interpretation of the conjunction and disjunction connective. Recall that a triangular norm, in shortly a t-norm, is a binary function T from $[0,1]^2$ into $[0,1]$ which is associative, commutative, non-decreasing on both arguments and $T(1,x)=x$ for all $x\in[0,1]$ [9, Definition 1.1]. The concept of t-conorm or s-norm reversed the boundary condition of the concept of t-norm. Thus an s-norm is an associative, commutative, and non-decreasing function S from the unite square into the unite interval satisfying for all $x\in[0,1]$ the boundary condition $S(0,x)=x$ [9, Definition 1.13].

Bellow, the most important t-norms and s-norms which are employed in the most significant many-valued logics as conjunction and disjunction are listed in Table 1.

t-norm	s-norm
$T_L(x,y)=\max\{0,x+y-1\}$	$S_L(x,y)=\min\{1,x+y\}$
$T_G(x,y)=\min\{x,y\}$	$S_G(x,y)=\max\{x,y\}$
$T_\pi(x,y)=x.y$	$S_\pi(x,y)=x+y-x.y$

Table 1. Łukasiewicz , Gödel , and Product t-norm and s-norm

In 1998, Hájek introduced a many-valued logic called *Basic logic* based on arbitrary continuous t-norms [6]. Indeed, Basic logic could be seen as an extension of the Łukasiewicz , Gödel , and Product logic.

Assume that T is a continuous t-norm and R_T is its residua which is defined by

$$z\le R_T(x,y) \text{ iff } T(z,x)\le y \tag{2.1}$$

for all $x,y,z\in[0,1]$. If *Prop* is generated from a set of atomic propositions P by formal operations $\{\&,\to,\perp\}$ and $e_0:P\to[0,1]$ is a function, there is a unique extension e of e_0, called an evaluation, satisfying the following rules [6, Section 2.2]:

- $e(\perp)=0$,
- $e(\varphi\&\psi)=T\left(e(\varphi),e(\psi)\right)$,
- $e(\varphi\to\psi)=R_T(e(\varphi),e(\psi))$.

The algebraic counter part of a theory in Basic logic, forms an algebra, called BL-algebra. Indeed, if for a theory $\Sigma\subseteq Prop$, we define

- $[\varphi]=\{\psi : T\vdash\varphi\leftrightarrow\psi\}$,
- $Lind(\Sigma)=\{[\varphi]:\varphi\in Prop\}$,

- $[\varphi] \le [\psi]$ iff $\Sigma \vdash (\varphi \to \psi)$,
- $[\top] = [\bot \to \bot]$,
- $[\varphi] * [\psi] = [\varphi \& \psi]$,
- $[\varphi] \rightarrowtail [\psi] = [\varphi \to \psi]$,

then, $(Lind(\Sigma), \le, *, \rightarrowtail, [\bot], [\top])$ forms a BL-algebra [6, Lemma 2.3.12]. Actually, we have the following definition for a BL-algebra.

Definition 2.1. [6, Deinition 2.3.3] A *BL-algebra* is an algebra $\mathcal{L} = (L, \wedge, \vee, *, \rightarrowtail, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ satisfying the following properties:

(BL1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice with the greatest element 1 and the smallest element 0,
(BL2) $(L, *, 1)$ is an Abelian monoid,
(BL3) $\rightarrowtail$ is the residua of $*$, i.e., $c \le a \rightarrowtail b$ iff $c * a \le b$ for all $a, b, c \in L$,
(BL4) $a \wedge b = a * (a \rightarrowtail b)$ for all $a, b \in L$,
(BL5) $(a \rightarrowtail b) \vee (b \rightarrowtail a) = 1$ for all $a, b \in L$.

For any continuous t-norm T and its residua R_T,

$$[0,1]_T = ([0,1], \min, \max, T, R_T, 0, 1)$$

forms a BL-algebra [6, Chapter 2]. Conversely, when the continuous scale $[0, 1]$ endowed to be a BL-algebra, the binary operator $*$ becomes a continuous t-norm on $[0, 1]$ [2]. The standard BL-algebra on the real segment $[0, 1]$ which is defined by the continuous t-norm $*$, is denoted by $[0, 1]_*$.

The following fact, used several times in the outcome results of the paper. Its proof can be found in [6, Chapter 2].

Lemma 2.2. *Let $\mathcal{L} = (L, \wedge, \vee, *, \rightarrowtail, 0, 1)$ be a BL-algebra. The following properties holds in $\mathcal{L}$.*

(B1) $a * b = b * a$ *and* $(a * b) * c = a * (b * c)$,
(B2) $a * (a \rightarrowtail b) \le b$ *and* $a \le b \rightarrowtail (a * b)$,
(B3) $a \le b$ *iff* $a \rightarrowtail b = 1$,
(B4) *if* $a \le b$ *then* $a * c \le b * c$, $c \rightarrowtail a \le c \rightarrowtail b$, *and* $a \rightarrowtail c \ge b \rightarrowtail c$,
(B5) $a * 0 = 0$,
(B6) $(a \vee b) * c = (a * c) \vee (b * c)$,
(B7) $a * b \le a$ *and* $a \le b \rightarrowtail a$,
(B8) $a \vee b = \big((a \rightarrowtail b) \rightarrowtail b\big) \wedge \big((b \rightarrowtail a) \rightarrowtail a\big)$,
(B9) $(a \rightarrowtail b) \le (b \rightarrowtail c) \rightarrowtail (a \rightarrowtail c)$,
(B10) $(a \rightarrowtail b) * (b \rightarrowtail c) \le (a \rightarrowtail c)$,
(B11) $a \rightarrowtail (b \rightarrowtail c) = (a * b) \rightarrowtail c$,
(B12) $a \rightarrowtail (b \rightarrowtail c) = b \rightarrowtail (a \rightarrowtail c)$,

(B13) $a \rightarrowtail a = 1$,
(B14) $a \rightarrowtail b \leq (a * c) \rightarrowtail (b * c)$,
(B15) $(a \rightarrowtail b) * (c \rightarrowtail d) \leq (a * c) \rightarrowtail (b * d)$.

3. A topology on BL-algebras makes them semitopological algebras

In this section, a topology on arbitrary BL-algebras introduced which makes them semitopological algebras.

From now on, we denote (a_1, a_2) shortly by $\mathbf{a}$. The following crucial definition is needed for Definition 3.3.

Definition 3.1. Let $\mathcal{L} = (L, \wedge, \vee, *, \rightarrowtail, 0, 1)$ be a BL-algebra. An element $a \in L$ is called *strongly less than* 1, denoted by $a \ll 1$, whenever for any $b \in L$, $a \vee b = 1$ implies that $b = 1$. Furthermore, $\leftrightarrow$ and $\Leftrightarrow$ are operators on L and L^2 which are defined respectively as follows:

$$a \leftrightarrow b = (a \rightarrowtail b) * (b \rightarrowtail a)\ ,\ \mathbf{a} \Leftrightarrow \mathbf{b} = (a_1 \leftrightarrow b_1) * (a_2 \leftrightarrow b_2). \quad (3.1)$$

In the following lemma, some of the properties of the notions $\ll$, $\leftrightarrow$, and $\Leftrightarrow$ are established.

Lemma 3.2. *Let $\mathcal{L} = (L, \wedge, \vee, *, \rightarrowtail, 0, 1)$ be a BL-algebra.*

(L1) $0 \ll 1$.
(L2) *For any $a \in L$, if $a \ll 1$, then $a < 1$.*
(L3) *For any $a, b \in L$, if $b < a \ll 1$, then $b \ll 1$,*
(L4) *For any $a, b \in L$, if $a \ll 1$ and $b \ll 1$, then $a \vee b \ll 1$.*
(L5) *For any $a, b \in L$, $a \rightarrowtail b \geq a \leftrightarrow b$,*
(L6) *Both of $\leftrightarrow$ and $\Leftrightarrow$ are symmetric.*
(L7) *For any $a \in L$ and $\mathbf{a} \in L^2$, $a \leftrightarrow a = 1$ and $\mathbf{a} \Leftrightarrow \mathbf{a} = 1$.*
(L8) *For any $a, b, c \in L$, $(a \leftrightarrow c) * (c \leftrightarrow b) \leq a \leftrightarrow b$.*
(L9) *For any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in L^2$, $(\mathbf{a} \Leftrightarrow \mathbf{c}) * (\mathbf{c} \Leftrightarrow \mathbf{b}) \leq \mathbf{a} \Leftrightarrow \mathbf{b}$.*
(L10) *For any $a, b \in L$, $a \leftrightarrow b = 1$ iff $a = b$.*

Proof.

L1) By (BL1), we know that $0 \vee x = x$ for any $x \in L$. So, for any $x \in L$, $0 \vee x = 1$ implies that $x = 1$. Therefore $0 \ll 1$.
L2) On the contrary, if $a = 1$ then for $x \neq 1$, $a \vee x = 1$, which is in contradiction with $a \ll 1$.
L3) For an arbitrary $x \in L$, assume that $b \vee x = 1$. Since $b < a$, $b \vee x \leq a \vee x$. So, $a \vee x = 1$ which together with $a \ll 1$ implies that $x = 1$. Thus $b \ll 1$.
L4) For an arbitrary $x \in L$, assume that $(a \vee b) \vee x = 1$. Thus, $a \vee (b \vee x) = 1$. Therefore, $b \vee x = 1$. Hence, $x = 1$ that is $a \vee b \ll 1$.

L5, L6, and L7) Follows respectively from (B7), (B1), and (B13).

L8) By (B10) for any $a, b, c \in L$, $(a \rightarrowtail c) * (c \rightarrowtail b) \leq (a \rightarrowtail b)$. Similarly, $(b \rightarrowtail c) * (c \rightarrowtail a) \leq (b \rightarrowtail a)$. Now using (B4) twice together with (B1) implies (L8).

$$\begin{aligned} a \leftrightarrow b &= (a \rightarrowtail b) * (b \rightarrowtail a) \\ &\geq \big((a \rightarrowtail c) * (c \rightarrowtail b)\big) * (b \rightarrowtail a) \\ &\geq \big((a \rightarrowtail c) * (c \rightarrowtail b)\big) * \big((b \rightarrowtail c) * (c \rightarrowtail a)\big) \\ &= \big((a \rightarrowtail c) * (c \rightarrowtail a)\big) * \big((b \rightarrowtail c) * (c \rightarrowtail b)\big) \\ &= (a \leftrightarrow c) * (b \leftrightarrow c) \end{aligned}$$

L9) Follows immediately from (L8) together with (B1).

$$\begin{aligned} \mathbf{a} \Leftrightarrow \mathbf{b} &= (a_1 \leftrightarrow b_1) * (a_2 \leftrightarrow b_2) \\ &\geq \big((a_1 \leftrightarrow c_1) * (c_1 \leftrightarrow b_1)\big) * \big((a_2 \leftrightarrow c_2) * (c_2 \leftrightarrow b_2)\big) \\ &= \big((a_1 \leftrightarrow c_1) * (a_2 \leftrightarrow c_2)\big) * \big((c_1 \leftrightarrow b_1) * (c_2 \leftrightarrow b_2)\big) \\ &= (\mathbf{a} \Leftrightarrow \mathbf{c}) * (\mathbf{c} \Leftrightarrow \mathbf{b}). \end{aligned}$$

L10) One direction is obvious from (L7). For the other direction, if $a \leftrightarrow b = 1$, then $(a \rightarrowtail b) * (b \rightarrowtail a) = 1$. So, by (L5) $a \rightarrowtail b = 1$ and $b \rightarrowtail a = 1$. Now, by (B3) $b \leq a$ and $a \leq b$ which implies that $a = b$.

□

Now, the expected topology on BL-algebras which makes them topological algebras is as follows.

Definition 3.3. Let $\mathcal{L} = (L, \wedge, \vee, *, \rightarrowtail, 0, 1)$ be a BL-algebra. For any elements $a, r \in L$ that $r \ll 1$, the *$*$-ball around a of radius r* is the set

$$B_r(a) = \{b \in L : a \leftrightarrow b > r\}.$$

Similarly the *$*$-ball around $\mathbf{a} \in L^2$ of radius* $r \ll 1$ is the set

$$\mathbf{B}_r(\mathbf{a}) = \{\mathbf{b} \in L^2 : \mathbf{a} \Leftrightarrow \mathbf{b} > r\}.$$

A subset G of L is called an $*$-open set if for every $a \in G$ there exists a radius $r \ll 1$ such that $B_r(a) \subseteq G$. $*$-open subsets of L^2 defined similarly.

Remark 3.4. By (L7), $a \in B_r(a)$ and similarly $\mathbf{a} \in \mathbf{B}_r(\mathbf{a})$. Moreover, if $r \geq s$ then $B_r(a) \subseteq B_s(a)$ and $\mathbf{B}_r(\mathbf{a}) \subseteq \mathbf{B}_s(\mathbf{a})$.

Theorem 3.5. *With the notations in Definition 3.3, the family of all $*$-open subsets of L form a topology on L denoted by T_*, called the "open ball topology". Similarly $\mathbf{T}_* = \{A : A$ is an $*$-open subset of $L^2\}$ is a topology on L^2.*

Proof. Obviously $\emptyset, L \in T_*$. Assume that $A, B \in T_*$. If $a \in A \cap B$, then since A and B are $*$-open sets, there exist $r_A \ll 1$ and $r_B \ll 1$ such that $B_{r_A}(a) \subseteq A$ and $B_{r_B}(a) \subseteq B$. By (L4), $r = r_A \vee r_B \ll 1$. Since $r \geq r_A$, Remark 3.4 implies that $B_r(a) \subseteq B_{r_A}(a)$. Similarly, $B_r(a) \subseteq B_{r_B}(a)$. Thus $B_r(a) \subseteq B_{r_A}(a) \cap B_{r_B}(a) \subseteq A \cap B$. Hence, $A \cap B$ is an $*$-open set.

Now, let $\{G_i\}_{i \in I}$ be a family of $*$-open sets and $G = \cup_{i \in I} G_i$. If G is empty there is noting to prove. Assume that $G \neq \emptyset$ and $a \in G$. So, there is $i \in I$ such that $a \in G_i$. Since G_i is an $*$-open set, there exists $r \ll 1$ such that $B_r(a) \subseteq G_i \subseteq G$. Thus G is an $*$-open set.

The second part will be proved by a similar argument. □

The following examples describe the introduced topology on BL-algebras more precisely.

Example 3.6. Let $L = [0, 1]$, $a * b = \max\{0, a + b - 1\}$ which is the Łukasiewicz t-norm. Then, by residuation relation 2.1 one can verify that $a \rightarrowtail b = \min\{1, 1 + b - a\}$ [6, Theorem 2.1.8]. To calculate $a \leftrightarrow b$, if $a \leq b$ then $b - a \geq 0$ and therefore $a \rightarrowtail b = 1$ and $b \rightarrowtail a = 1 + a - b$ and therefore $a \leftrightarrow b = 1 * (1 + a - b) = 1 + a - b$. Similarly, if $b \leq a$, then $a \leftrightarrow b = 1 + b - a$. Consequently, we have $a \leftrightarrow b = 1 - |a - b|$. In addition, $[0, 1]$ is a linearly ordered BL-algebra and therefore $a \ll b$ and $a < b$ have the same meaning. So, for any $a \in [0, 1]$ and $r < 1$,

$$\begin{aligned} B_r(a) &= \{b : a \leftrightarrow b > r\} \\ &= \{b \in [0,1] : 1 - |a - b| > r\} \\ &= \{b \in [0,1] : |a - b| < 1 - r\} \\ &= (a - (1 - r), a + (1 - r)) \cap [0, 1] \\ &= (a - (1 - r), a + (1 - r)). \end{aligned}$$

For example $B_{0.5}(a) = (a - 0.5, a + 0.5)$ and $B_{0.2}(a) = (a - 0.8, a + 0.8)$ and $B_{0.7}(a) = (a-0.3, a+0.3)$. Verily, the $*$-ball $B_r(a)$ in T_* is the open ball around a of radius $1 - r$ in the Euclidean topology and therefore T_* is equivalent to the Euclidean topology on $[0, 1]$.

Example 3.7. Let $L = \{0, a, b, c, 1\}$. Define $*$ and $\rightarrowtail$ on L as follows.

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	c	c	a
b	0	c	b	c	b
c	0	c	c	c	c
1	0	a	b	c	1

$\rightarrowtail$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	0	1	1	1	1
1	0	a	b	c	1

Obviously, L is a BL-algebra. The Hasse diagram of L will be illustrated in Figure 1.

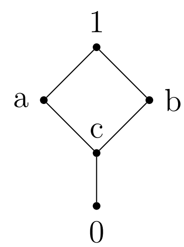

Figure 1. Hasse diagram of L

So, an easy argument leads to the following table for $\leftrightarrow$.

$\leftrightarrow$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	c	b	a
b	0	c	1	a	b
c	0	b	a	1	c
1	0	a	b	c	1

By the Hasse diagram of L, $R = \{r : r \ll 1\} = \{0, c\}$. All $*$-balls of T_*, i.e. $B_r(x) = \{y : x \leftrightarrow y > r\}_{r \in R, x \in L}$, are as follows:

$$\begin{array}{ll} B_r(0) = \{0\}\ \forall r \in R & B_0(x) = \{a, b, c, 1\}\ \forall x > 0 \\ B_c(a) = \{a, c, 1\} & B_c(b) = \{b, c, 1\} \\ B_c(c) = \{a, b, c\} & B_c(1) = \{a, b, 1\} \end{array}$$

So, $T_* = \{\emptyset, \{0\}, \{a, b, c, 1\}, \{0, a, b, c, 1\}\}$. Note that by definition of $*$-open sets, an $*$-ball is not necessarily an $*$-open set.

Above examples showed that the introduced topology on BL-algebras is not trivial. Indeed, this topology is obtained from the distance between elements of a BL-algebra with respect to the $\leftrightarrow$.

In Section 5, we will show the open ball topology on BL-algebras could be described more explicitly by a kind of duality between algebras. The topological equivalence of T_* and the Euclidean topology in Example 3.9 could be explained by this duality as well.

Besides (BL4) and (B8), the following theorem indicates that for any BL-algebra $\mathcal{L} = (L, \wedge, \vee, *, \rightarrowtail, 0, 1)$, all the operators of $\mathcal{L}$ are continuous functions with respect to the introduced topologies T_* and $\mathbf{T}_*$ on L and L^2.

Theorem 3.8. *If $\mathcal{L} = (L, \wedge, \vee, *, \rightarrowtail, 0, 1)$ is a BL-algebra and T_* and $\mathbf{T}_*$ are the same as in Theorem 3.5, then the mappings $* : (L^2, \mathbf{T}_*) \to (L, T_*)$ and $\rightarrowtail : (L^2, \mathbf{T}_*) \to (L, T_*)$ would be continuous functions.*

Proof. Consider an $*$-open set $A \in T_*$. We must verify that the inverse images of A, $*^{-1}(A)$ and $\rightarrowtail^{-1}(A)$ are $*$-open subsets of L^2.

Firstly, consider a point $\mathbf{a} \in *^{-1}(A)$, that is $a_1 * a_2 \in A$. Since A is an $*$-open set, there exists $r \ll 1$ such that $B_r(a_1 * a_2) \subseteq A$. To finalize the first part of proof, we will show that $\mathbf{B}_r(\mathbf{a}) \subseteq *^{-1}(A)$. Consider an element $\mathbf{b} \in \mathbf{B}_r(\mathbf{a})$. So, $\mathbf{a} \Leftrightarrow \mathbf{b} > r$, that is

$$(a_1 \leftrightarrow b_1) * (a_2 \leftrightarrow b_2) > r. \tag{3.2}$$

In addition since by (B15) we have

$$(a_1 * a_2) \rightarrowtail (b_1 * b_2) \geq (a_1 \rightarrowtail b_1) * (a_2 \rightarrowtail b_2)$$

and

$$(b_1 * b_2) \rightarrowtail (a_1 * a_2) \geq (b_1 \rightarrowtail a_1) * (b_2 \rightarrowtail a_2),$$

so, applying (B4) twice and then using (B1) and 3.2 we get

$$\begin{aligned}
&(a_1 * a_2) \leftrightarrow (b_1 * b_2)\\
&= \big((a_1 * a_2) \rightarrowtail (b_1 * b_2)\big) * \big((b_1 * b_2) \rightarrowtail (a_1 * a_2)\big)\\
&\geq \big((a_1 \rightarrowtail b_1) * (a_2 \rightarrowtail b_2)\big) * \big((b_1 * b_2) \rightarrowtail (a_1 * a_2)\big)\\
&\geq \big((a_1 \rightarrowtail b_1) * (a_2 \rightarrowtail b_2)\big) * \big((b_1 \rightarrowtail a_1) * (b_2 \rightarrowtail a_2)\big)\\
&= \big((a_1 \rightarrowtail b_1) * (b_1 \rightarrowtail a_1)\big) * \big((a_2 \rightarrowtail b_2) * (b_2 \rightarrowtail a_2)\big)\\
&= (a_1 \leftrightarrow b_1) * (a_2 \leftrightarrow b_2)\\
&> r.
\end{aligned}$$

Thus $b_1 * b_2 \in B_r(a_1 * a_2) \subseteq A$. Hence $\mathbf{b} \in *^{-1}(A)$, completes the first part of the proof.

Secondly, to prove that $\rightarrowtail^{-1}(A)$ is an $*$-open subset of L^2, consider a point $\mathbf{a} \in \rightarrowtail^{-1}(A)$. So $a_1 \rightarrowtail a_2 \in A$. Since A is an $*$-open set, there exists $r \ll 1$ that $B_r(a_1 \rightarrowtail a_2) \subseteq A$. To prove that $\rightarrowtail^{-1}(A)$ is $*$-open, we investigate that $\mathbf{B}_r(\mathbf{a}) \subseteq \rightarrowtail^{-1}(A)$. To this end, if $\mathbf{b} \in \mathbf{B}_r(\mathbf{a})$, then

$$\mathbf{a} \Leftrightarrow \mathbf{b} > r. \tag{3.3}$$

By (B9) $a_1 \rightarrowtail b_1 \leq (b_1 \rightarrowtail b_2) \rightarrowtail (a_1 \rightarrowtail b_2)$ and by (BL3)

$$(a_1 \rightarrowtail b_1) * (b_1 \rightarrowtail b_2) \leq (a_1 \rightarrowtail b_2). \tag{3.4}$$

Again by (B9) $a_1 \rightarrowtail b_2 \leq (b_2 \rightarrowtail a_2) \rightarrowtail (a_1 \rightarrowtail a_2)$ and therefore by 3.4

$$(a_1 \rightarrowtail b_1) * (b_1 \rightarrowtail b_2) \leq (b_2 \rightarrowtail a_2) \rightarrowtail (a_1 \rightarrowtail a_2).$$

Now applying (BL3) we have

$$\big((a_1 \rightarrowtail b_1) * (b_1 \rightarrowtail b_2)\big) * (b_2 \rightarrowtail a_2) \leq (a_1 \rightarrowtail a_2)$$

which besides (B1) leads to

$$\big((a_1 \rightarrowtail b_1) * (b_2 \rightarrowtail a_2)\big) * (b_1 \rightarrowtail b_2) \leq (a_1 \rightarrowtail a_2).$$

Again (BL3) implies that

$$(b_1 \rightarrowtail b_2) \rightarrowtail (a_1 \rightarrowtail a_2) \geq (a_1 \rightarrowtail b_1) * (b_2 \rightarrowtail a_2). \tag{3.5}$$

Analogously

$$(a_1 \rightarrowtail a_2) \rightarrowtail (b_1 \rightarrowtail b_2) \geq (b_1 \rightarrowtail a_1) * (a_2 \rightarrowtail b_2). \tag{3.6}$$

Now, applying 3.5, 3.6, and (B4) we see that

$$\begin{aligned}
&(a_1 \rightarrowtail a_2) \leftrightarrow (b_1 \rightarrowtail b_2) \\
&= \big((a_1 \rightarrowtail a_2) \rightarrowtail (b_1 \rightarrowtail b_2)\big) * \big((b_1 \rightarrowtail b_2) \rightarrowtail (a_1 \rightarrowtail a_2)\big) \\
&\geq \big((b_1 \rightarrowtail a_1) * (a_2 \rightarrowtail b_2)\big) * \big((b_1 \rightarrowtail b_2) \rightarrowtail (a_1 \rightarrowtail a_2)\big) \\
&\geq \big((b_1 \rightarrowtail a_1) * (a_2 \rightarrowtail b_2)\big) * \big((a_1 \rightarrowtail b_1) * (b_2 \rightarrowtail a_2)\big) \\
&= \big((b_1 \rightarrowtail a_1) * (a_1 \rightarrowtail b_1)\big) * \big((a_2 \rightarrowtail b_2) * (b_2 \rightarrowtail a_2)\big) \\
&= (a_1 \leftrightarrow b_1) * (a_2 \leftrightarrow b_2) \\
&= \mathbf{a} \Leftrightarrow \mathbf{b}.
\end{aligned}$$

Therefore, 3.3 implies that $(a_1 \rightarrowtail a_2) \leftrightarrow (b_1 \rightarrowtail b_2) > r$ which means that is $(b_1 \rightarrowtail b_2) \in B_r(a_1 \rightarrowtail a_2) \subseteq A$. Hence $\mathbf{b} \in \rightarrowtail^{-1} (A)$. ☐

Although the continuous scale $[0,1]_* = ([0,1], \wedge, \vee, *, \rightarrowtail, 0, 1)$ endowed to be a BL-algebra and consequently $*$ becomes a continuous t-norm, all the operators of $[0,1]_*$ are not necessarily continuous with respect to the usual topology on $[0,1]$. However, Theorem 3.8 shows that the introduced topologies T_* and $\mathbf{T}_*$ makes all the operators of any BL-algebra continuous.

Example 3.9. Let $L = [0,1]$. Consider the Gödel t-norm on $[0,1]$, that is $a * b = \min\{a, b\}$. The residuation relation 2.1 implies that $a \rightarrowtail b = \begin{cases} 1 & a \leq b \\ b & a > b \end{cases}$ [6, Theorem 2.1.8]. An easy argument shows that in spite of the continuity of $*$ with respect to the Euclidean topology on $[0,1]$ and $[0,1]^2$, the function $\rightarrowtail$ is not a continuous function. However, Theorem 3.8 shows that both $*$ and $\rightarrowtail$ are continuous functions with respect to the topologies T_* and $\mathbf{T}_*$ on $[0,1]$ and $[0,1]^2$.

Now, for any BL-algebra $\mathcal{L} = (L, \wedge, \vee, *, \rightarrowtail, 0, 1)$, we are going to show that the introduced topology T_* makes $\mathcal{L}$ a semitopological BL-algebra.

Recall from [12] and [4] that a semitopological algebra is an algebra $\mathcal{L} = (L, *)$ of type (2) together with a topology τ on L such that for all $\delta \in L$ the maps $*_l^\delta : (L, \tau) \to (L, \tau)$ and $*_r^\delta : (L, \tau) \to (L, \tau)$ defined respectively by $*_l^\delta(x) = \delta * x$ and $*_r^\delta(x) = x * \delta$ are continuous functions.

Definition 3.10. Let $\mathcal{L} = (L, \wedge, \vee, *, \rightarrowtail, 0, 1)$ be a BL-algebra. If there exists a topology τ on L such that for any $\Box \in \{\wedge, \vee, *, \rightarrowtail\}$ and any $\delta \in L$ the maps $\Box_l^\delta : (L, \tau) \to (L, \tau)$ and $\Box_r^\delta : (L, \tau) \to (L, \tau)$ defined respectively by $\Box_l^\delta(x) = \delta \Box x$ and $\Box_r^\delta(x) = x \Box \delta$ are continuous functions, then $(\mathcal{L}, \tau)$ is called a *semitopological BL-algebra.*

Theorem 3.11. *If $\mathcal{L} = (L, \wedge, \vee, *, \rightarrowtail, 0, 1)$ is a BL-algebra and T_* is the same as in Theorem 3.5, then $(\mathcal{L}, T_*)$ forms a semitopological BL-algebra.*

Proof. Besides (BL4) and (B8) it's enough to clarify that for any $\delta \in L$ the mappings $*_l^\delta$,$*_r^\delta$, $\rightarrowtail_l^\delta$, and $\rightarrowtail_r^\delta$ are continuous functions. We only do the proof for $*_l^\delta$ and others will be proved in a similar way.

Let $A \in T_*$. We must prove that $(*_l^\delta)^{-1}(A) \in T_*$. So, assume that $a \in (*_l^\delta)^{-1}(A)$ that is $*_l^\delta(a) \in A$. Hence $\delta * a \in A$. Since A is an $*$-open set, there exists $r \ll 1$ such that $B_r(\delta * a) \subseteq A$. We claim that $B_r(a) \subseteq (*_l^\delta)^{-1}(A)$ which implies that $(*_l^\delta)^{-1}(A)$ is an $*$-open set and fulfills the proof. To this end, consider an arbitrary element $b \in B_r(a)$ that is $a \leftrightarrow b > r$. Now, (B15) implies that

$$
\begin{aligned}
(\delta * b) \leftrightarrow (\delta * a) &= (\delta * b) \rightarrowtail (\delta * a)\big) * \big((\delta * a) \rightarrowtail (\delta * b)\big) \\
&\geq \big((\delta \rightarrowtail \delta) * (b \rightarrowtail a)\big) * \big((\delta \rightarrowtail \delta) * (a \rightarrowtail b)\big) \\
&= \big(1 * (b \rightarrowtail a)\big) * \big(1 * (a \rightarrowtail b)\big) \\
&= (b \rightarrowtail a) * (a \rightarrowtail b) \\
&= a \leftrightarrow b \\
&> r.
\end{aligned}
$$

So $\delta * b \in B_r(\delta * a) \subseteq A$ means that $*_l^\delta(b) \in A$. Therefore $b \in (*_l^\delta)^{-1}(A)$ which completes the proof. $\square$

4. Some properties of the open ball topology on $[0, 1]$

Example 3.7 shows that the $*$-balls are not necessarily $*$-open set. Furthermore, it shows that T_* does not admit the weakest separation axiom T_0. However, when the continuous scale $[0, 1]$ endowed to be a BL-algebra, we will show that T_* admits some nice properties.

First of all, the following theorem shows that when $[0, 1]$ is endowed to be a BL-algebra, then like metric spaces, $*$-balls are $*$-open set.

Theorem 4.1. *If $[0, 1]_* = ([0, 1], \min, \max, *, \rightarrowtail, 0, 1)$ is a BL-algebra, then, $*$-balls are $*$-open set.*

Proof. Note that since $[0, 1]$ is linearly ordered, $\ll$ and $<$ have the same meaning. For any $*$-ball $B_r(a)$ and any $b \in B_r(a)$, we must find $\epsilon < 1$

such that $B_\epsilon(b) \subseteq B_r(a)$. To this end, let $\epsilon = (a \leftrightarrow b) \rightarrowtail r$. Since $b \in B_r(a)$, $a \leftrightarrow b > r$ and therefore (B3) implies that

$$\epsilon = (a \leftrightarrow b) \rightarrowtail r < 1.$$

Now, if $c \in B_\epsilon(b)$ then $b \leftrightarrow c > \epsilon$ that is $b \leftrightarrow c > (a \leftrightarrow b) \rightarrowtail r$. Thus, (BL3) implies that $(b \leftrightarrow c) * (a \leftrightarrow b) > r$. Consequently by (L8) and (B1) we get

$$a \leftrightarrow c \geq (a \leftrightarrow b) * (b \leftrightarrow c) = (b \leftrightarrow c) * (a \leftrightarrow b) > r$$

which means that $c \in B_r(a)$. Hence $B_\epsilon(b) \subseteq B_r(a)$ which entails that $B_r(a)$ is an $*$-open set. □

Now, we want to examine the most famous separation axiom for T_* on $[0, 1]$ such as a BL-algebra.

Theorem 4.2. *If* $[0, 1]_* = ([0, 1], \min, \max, *, \rightarrowtail, 0, 1)$ *is a BL-algebra, then,* T_* *would be a Hausdorff topology on* $[0, 1]$.

Proof. Since $[0, 1]_*$ is a BL-algebra, so $*$ is a continuous t-norm on $[0, 1]$ (with respect to the usual topology on $[0, 1]$). Now, if a, b are two distinct element of $[0, 1]$, then there exists $r < 1$ that

$$a \leftrightarrow b < r * r. \tag{4.1}$$

Indeed, otherwise $a \leftrightarrow b \geq r * r$ for any $r < 1$, which together with the fact that $*$ is a continuous t-norm, implies that $a \leftrightarrow b = 1$ and therefore by (L10) $a = b$, a contradiction. To complete the proof, we show that $B_r(a) \cap B_r(b) = \emptyset$. Indeed, if $c \in B_r(a) \cap B_r(b)$, then (L8) and 4.1 leads to the following contradiction.

$$a \leftrightarrow b \geq (a \leftrightarrow c) * (c \leftrightarrow b) \geq r * r > a \leftrightarrow b.$$

□

5. Describing the open ball topology by means of duality

In this section, following our conference article [8], we show that if one consider a dual notion of BL-algebras, then the open ball topology could be described by a metric-like topology.

Firstly, consider the following dual notion for BL-algebras.

Definition 5.1. An *SL-algebra*, is an algebra $\mathcal{L} = (L, \curlywedge, \curlyvee, \star, \dashrightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ that satisfies the following conditions.

(SL1) $(L, \curlywedge, \curlyvee, 0, 1)$ is a bounded lattice with the greatest element 1 and the smallest element 0. Note that here, $a \leq b$ iff $a \curlyvee b = a$. So, $a \curlyvee b = \inf\{a, b\}$ and $a \curlywedge b = \sup\{a, b\}$.

(SL2) $(L, \star, 0)$ is an Abelian monoid,

(SL3) $\dashrightarrow$ is the residua of $\star$, i.e., $a \geq b \dashrightarrow c$ iff $a \star b \geq c$ for all $a, b, c \in L$,
(SL4) $a \curlywedge b = a \star (a \dashrightarrow b)$ for all $a, b \in L$,
(SL5) $(a \dashrightarrow b) \curlyvee (b \dashrightarrow a) = 0$ for all $a, b \in L$.

Note that (SL3) implies that $b \dashrightarrow c = \inf\{a : a \star b \geq c\}$.

Example 5.2. If S is a continuous s-norm, and the residua of S is defined by $R_S(a, b) = \inf\{c : S(c, a) \geq b\}$, then

$$[0,1]_S = ([0,1], \max, \min, S, R_S, 0, 1)$$

forms an SL-algebra.

When $[0,1]_\star = ([0,1], \curlywedge, \curlyvee, \star, \dashrightarrow, 0, 1)$ is endowed to be an SL-algebra, then $\star$ becomes a continuous s-norm, $\dashrightarrow$ would be the residua of $\star$ and therefore $\curlywedge$ and $\curlyvee$ becomes the maximum and minimum functions, respectively [2, dual form of Proposition 3].

The following theorem is an obvious consequence of duality between BL-algebras and SL-algebras which follows from Proposition 2.2.

Theorem 5.3. *Let $\mathcal{L} = (L, \curlywedge, \curlyvee, \star, \dashrightarrow, 0, 1)$ be an SL-algebra. The following properties hold in $\mathcal{L}$.*

(S1) $a \star b = b \star a$ *and* $(a \star b) \star c = a \star (b \star c)$,
(S2) $a \star (a \dashrightarrow b) \geq b$ *and* $a \geq b \dashrightarrow (a \star b)$,
(S3) $a \geq b$ *iff* $a \dashrightarrow b = 0$,
(S4) *if* $a \geq b$ *then* $a \star c \geq b \star c$, $c \dashrightarrow a \geq c \dashrightarrow b$, *and* $a \dashrightarrow c \leq b \dashrightarrow c$,
(S5) $a \star 1 = 1$,
(S6) $(a \curlyvee b) \star c = (a \star c) \curlyvee (b \star c)$,
(S7) $a \star b \geq a$ *and* $a \geq b \dashrightarrow a$,
(S8) $a \curlyvee b = \big((a \dashrightarrow b) \dashrightarrow b\big) \curlywedge \big((b \dashrightarrow a) \dashrightarrow a\big)$,
(S9) $(a \dashrightarrow b) \geq (b \dashrightarrow c) \dashrightarrow (a \dashrightarrow c)$,
(S10) $(a \dashrightarrow b) \star (b \dashrightarrow c) \geq (a \dashrightarrow c)$,
(S11) $a \dashrightarrow (b \dashrightarrow c) = (a \star b) \dashrightarrow c$,
(S12) $a \dashrightarrow (b \dashrightarrow c) = b \dashrightarrow (a \dashrightarrow c)$,
(S13) $a \dashrightarrow a = 0$,
(S14) $a \dashrightarrow b \geq (a \star c) \dashrightarrow (b \star c)$,
(S15) $(a \dashrightarrow b) \star (c \dashrightarrow d) \geq (a \star c) \dashrightarrow (b \star d)$,

The key point that we interested in dual of BL-algebras, is that when $[0,1]_\star = ([0,1], \max, \min, \star, \dashrightarrow, 0, 1)$ is endowed to be an SL-algebra, then in most cases the dual of the notion $\leftrightarrow$ forms a metric.

Example 5.4. Let $L = [0,1]$, $a \star b = \min\{0, a + b\}$ which is the Łukasiewicz s-norm, and $\dashrightarrow$ be the residua of $\star$. For $a, b \in [0,1]$

- if $a \geq b$ then by (S4) and (SL2) for any $c \in [0,1]$,

$$c \star a \geq c \star b \geq 0 \star b = b$$

and therefore

$$a \dashrightarrow b = \inf\{c : c \star a \geq b\} = \inf\{c : c \in [0,1]\} = 0,$$

- if $a < b$ then $c \star a \geq b$ iff $c + a \geq b$ iff $c \geq b - a$ and therefore

$$a \dashrightarrow b = \inf\{c : c \star a \geq b\} = \inf\{c : c \geq b - a\} = b - a.$$

Thus

$$a \dashrightarrow b = \begin{cases} 0 & a \geq b \\ b - a & a < b \end{cases}.$$

Now , if $a \geq b$ then $a \dashrightarrow b = 0$, $b \dashrightarrow a = a - b$, and so we have

$$(a \dashrightarrow b) \star (b \dashrightarrow a) = \min\{1, 0 + a - b\} = a - b.$$

Similarly if $a \leq b$ then $(a \dashrightarrow b) \star (b \dashrightarrow a) = b - a$. Thus,

$$(a \dashrightarrow b) \star (b \dashrightarrow a) = |a - b|$$

which is the Euclidean metric on $[0,1]$.

Example 5.5. Let $L = [0,1]$, $a \star b = \max\{a, b\}$ which is the Gödel s-norm, and $\dashrightarrow$ be the residua of $\star$. An argument such as the one in Example 5.4 shows that

$$a \dashrightarrow b = \begin{cases} 0 & a \geq b \\ b & a < b \end{cases}$$

and

$$(a \dashrightarrow b) \star (b \dashrightarrow a) = \begin{cases} 0 & a = b \\ \max\{a, b\} & a \neq b \end{cases}.$$

Again, note that $(a \dashrightarrow b) \star (b \dashrightarrow a)$ is a metric on $[0,1]$.

Certainly we have the following fact.

Theorem 5.6. *Assume that $\mathcal{L} = (L, \curlywedge, \curlyvee, \star, \dashrightarrow, 0, 1)$ is an SL-algebra. Suppose that the mappings $d_\star : L \times L \to L$ is defined by*

$$d_\star(a, b) = (a \dashrightarrow b) \star (b \dashrightarrow a).$$

Then,

(1) $\forall a, b,\ d_\star(a, b) = 0$ iff $a = b$,
(2) $\forall a, b,\ d_\star(a, b) = d_\star(b, a)$,
(3) $\forall a, b, c,\ d_\star(a, b) \leq d_\star(a, c) \star d_\star(c, b)$,
(4) if $L = [0,1]$ and for any $a, b \in [0,1]$, $a \star b \leq S_L(a, b)$, then $d_\star$ is a metric on $[0,1]$.

Proof. 1) Follows as like as (L10) from (S13), (S7), and (S3).
2) Follows as like as (L6) from (S1).
3) Follows as like as (L8) from (S10) and (S4).

4) If $a \star b \leq S_L(a,b)$ holds for any $a, b \in [0,1]$, then (3) implies that $d_\star(a,b) \leq d_\star(a,c) \star d_\star(c,b) \leq d_\star(a,c) + d_\star(c,b)$, for any $a, b, c \in [0,1]$, means that $d_\star$ is a metric on $[0,1]$. □

A similar argument such as Theorem 5.6 holds for the dual notion of $\Leftrightarrow$ which is denoted by $\mathbf{d}_\star$.

Theorem 5.7. *Assume that $\mathcal{L} = (L, \curlywedge, \curlyvee, \star, \dashrightarrow, 0, 1)$ is an SL-algebra and define the mappings $\mathbf{d}_\star : L^2 \times L^2 \to L$ by*

$$\mathbf{d}_\star(\mathbf{a}, \mathbf{b}) = d_\star(a_1, b_1) \star d_\star(a_2, b_2).$$

Then,

(1) $\forall \mathbf{a}, \mathbf{b}$, $\mathbf{d}_\star(\mathbf{a}, \mathbf{b}) = 0$ *iff* $\mathbf{a} = \mathbf{b}$,
(2) $\forall \mathbf{a}, \mathbf{b}$, $\mathbf{d}_\star(\mathbf{a}, \mathbf{b}) = \mathbf{d}_\star(\mathbf{b}, \mathbf{a})$,
(3) $\forall \mathbf{a}, \mathbf{b}, \mathbf{c}$, $\mathbf{d}_\star(\mathbf{a}, \mathbf{b}) \leq \mathbf{d}_\star(\mathbf{a}, \mathbf{c}) \star \mathbf{d}_\star(\mathbf{b}, \mathbf{c})$,
(4) If $L = [0,1]$ and for any $a, b \in [0,1]$, $a \star b \leq S_L(a,b)$, then $\mathbf{d}_\star$ is a metric on $[0,1]^2$.

Proof. Similar to the proof of Theorem 5.6. □

Now, for any SL-algebra $\mathcal{L} = (L, \curlywedge, \curlyvee, \star, \dashrightarrow, 0, 1)$, the metric-like topologies on L and L^2 could be constructed as the one introduced for BL-algebras in Theorem 3.5 which made all the operators of $\mathcal{L}$ continuous.

Theorem 5.8. *Let $\mathcal{L} = (L, \curlywedge, \curlyvee, \star, \dashrightarrow, 0, 1)$ be an SL-algebra. For an element $a \in L$, write $a \gg 0$ whenever for any $b \in L$, $a \curlyvee b = 0$ implies that $b = 0$. For any $a \in L$ and $\mathbf{a} \in L^2$ and $r \gg 0$, suppose that $N_r(a) = \{b \in L : d_\star(a,b) < r\}$ and $\mathbf{N}_r(\mathbf{a}) = \{\mathbf{b} \in L^2 : \mathbf{d}_\star(\mathbf{a}, \mathbf{b}) < r\}$. Then*

$$T_\star = \big\{G : G \subseteq L \text{ and } \forall a \in G, \exists r \gg 0 \text{ such that} \big(N_r(a) \subseteq G\big)\big\}$$

and

$$\mathbf{T}_\star = \big\{G : G \subseteq L^2 \text{ and } \forall \mathbf{a} \in G, \exists r \gg 0 \text{ such that} \big(N_r(\mathbf{a}) \subseteq G\big)\big\}$$

form topologies on L and L^2, respectively. Furthermore, the mappings $\star : (L^2, \mathbf{T}_\star) \to (L, T_\star)$ and $\dashrightarrow : (L^2, \mathbf{T}_\star) \to (L, T_\star)$ are continuous functions.

Proof. Similar to the proof of Theorems 3.5 and 3.8 with dual notions. □

Now, for any continuous t-norm $*$, if $\star$ is defined by

$$a \star b = 1 - \big((1-a) * (1-b)\big),$$

then $*$ and $\star$ are called dual and one can examine the open ball topology on the BL-algebra $[0,1]_* = ([0,1], \min, \max, *, \rightarrowtail, 0, 1)$ by the metric-like topology on $[0,1]_\star = ([0,1], \max, \min, \star, \dashrightarrow, 0, 1)$.

Example 5.9. We know that dual of $a *_L b = \max\{1, a+b-1\}$ is $a \star_L b = \min\{1, a+b\}$. So, the open ball topology on the BL-algebra

$$[0,1]^t_L = ([0,1], \min, \max, *_L, \rightarrowtail_L, 0, 1)$$

can be examined with the metric topology on the SL-algebra

$$[0,1]^s_L = ([0,1], \max, \min, \star_L, \dashrightarrow_L, 0, 1).$$

In this special case, the open ball topology on $[0,1]^t_L$ (Example 3.9) and the metric topology on $[0,1]^s_L$ (Example 5.4) are equivalent. Indeed, replacing any element b with $1-b$ in the $\star$-balls of $([0,1]^s_L, T_\star)$ gives an $*$-ball in $([0,1]^t_L, T_*)$. For example $N_{0.1}(0.7) = (0.6, 0.8)$ corresponds to $B_{0.1}(0.3) = (0.2, 0.4)$, $N_{0.3}(0.1) = [0, 0.4)$ corresponds to $B_{0.3}(0.9) = (0.6, 1]$, and so forth.

Example 5.10. Let $a *_G b = \max\{a, b\}$ that is the Gödel t-norm. We know that $\rightarrowtail_G = \begin{cases} 1 & a \leq b \\ b & a > b \end{cases}$ [6, Chapter2]. An argument such as the one in Example 5.5, shows that

$$(a \rightarrowtail_G b) *_G (b \rightarrowtail_G a) = \begin{cases} 1 & a = b \\ \min\{a, b\} & a \neq b \end{cases}.$$

Therefore, there are two kinds of $*$-balls:

$r \leq a) \ : \ B_r(a) = \{b : (a \rightarrowtail_G b) \star_G (b \rightarrowtail_G a) \geq r\} = [r, 1]$,
$r > a) \ : \ B_r(a) = \{b : (a \rightarrowtail_G b) \star_G (b \rightarrowtail_G a) \geq r\} = \{a\}$.

Note that the only singleton in $[0,1]^t_G$ which is not an $*$-ball is $\{1\}$. So, T_* is a little coarser than the discreet topology on $[0,1]$.

On the other hand, if $a \star_G b = \min\{a, b\}$ which is the Gödel s-norm, Example 5.5 shows that

$$(a \dashrightarrow_G b) \star_G (b \dashrightarrow_G a) = \begin{cases} 0 & a = b \\ \max\{a, b\} & a \neq b \end{cases}.$$

So, the $\star$-balls of the $T_\star$ topology on $[0,1]^s_G$ are as follows:

$r < a) \ : \ N_r(a) = \{b : (a \dashrightarrow_G b) \star_G (b \dashrightarrow_G a) \leq r\} = \{a\}$,
$r \geq a) \ : \ N_r(a) = \{b : (a \dashrightarrow_G b) \star_G (b \dashrightarrow_G a) \leq r\} = [0, r]$.

In this case, the only singleton in $[0,1]^s_G$ which is not an $\star$-ball is $\{0\}$.

So, in this case, T_* and $T_\star$ are not equivalent but we could examine each of them by another. For example,

- since for any $a > 0$ the singleton $\{a\}$ is open in $T_\star$, by replacing a with $1-a$ we get that for any $a < 1$ the singleton $\{a\}$ is open in T_*,
- since for any $r > 0$ the set $[0, r]$ is open in $T_\star$, by replacing any element b with $1-b$ we know that for any $r < 1$ the set $[r, 1]$ is open in T_*,

- since $\{\frac{1}{n}\}_{n\in\mathbb{N}} \cup \{0\}$ is a compact subset of $([0,1], T_\star)$, so replacing any element b with $1-b$ leads to the fact that $\{1-\frac{1}{n}\}_{n\in\mathbb{N}} \cup \{1\}$ is a compact subset of $([0,1], T_*)$.

Final remarks

In this paper we introduced a topology on BL-algebras that makes them semitopological algebras. One of the advantages of this topology, is the study of model theoretic properties of Basic logic. In Łukasiewicz logic the continuity of the interpretation of logical connectives make it possible to extend some of the results of model theory of classical logic to Łukasiewicz logic. However in Basic logic, this study did not developed as like as the Łukasiewicz logic and the introduced topology maybe smoothed the future way of this study. Finally, another possible research that maybe facilitated by the introduced topology, is the study of stone topology for BL-algebras.

6. Acknowledgments

We greatly appreciate the constructive suggestions of reviewers. Their comments would increase the overall quality of the work.

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Journal of Algebraic Systems

A METRIC-LIKE TOPOLOGY ON BL-ALGEBRAS

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یک توپولوژیِ مشابهِ متر روی BL-جبرها

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هدف اصلی این مقاله معرفیِ یک توپولوژی روی BL-جبرها است که آن‌ها را به جبرهای توپولوژیک تبدیل می‌کند. اگر $\mathcal{L} = (L, \wedge, \vee, *, \rightarrowtail, ۰, ۱)$ یک BL-جبر باشد، آن‌گاه تابعی که با ضابطه $a \leftrightarrow b = (a \rightarrowtail b) * (b \rightarrowtail a)$ روی L تعریف می‌شود مشابه یک متر است و ما به کمک آن یک توپولوژی روی L تعریف می‌کنیم. نشان می‌دهیم در حالتی که بازه بسته واحد $[۰, ۱]$ به عنوان یک BL-جبر در نظر گرفته شود، آن‌گاه بعضی از مهمترین خواص توپولوژی متریک در این توپولوژی برقرار است. در نهایت نشان می‌دهیم که این توپولوژی را می‌توان با یک توپولوژی روی دوگان BL-جبرها نیز مطالعه کرد.

کلمات کلیدی: BL-جبر، دوگان BL-جبر، BL-جبرِ توپولوژیک.