Journal of Algebraic Systems
Vol. 1, No. 2, (2013), pp 91-100

A NEW PROOF OF THE PERSISTENCE PROPERTY FOR IDEALS IN DEDEKIND RINGS AND PRÜFER DOMAINS

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Abstract. In this paper, using elementary tools of commutative algebra, helps us prove the persistence property for two especial classes of rings. In fact, this paper has two main sections. In the first section, we let R be a Dedekind ring and I be a proper ideal of R. We prove that if $I_1, \ldots, I_n$ are non-zero proper ideals of R, then $\mathrm{Ass}^{\infty}(I_1^{k_1} \ldots I_n^{k_n}) = \mathrm{Ass}^{\infty}(I_1^{k_1}) \cup \cdots \cup \mathrm{Ass}^{\infty}(I_n^{k_n})$ for all $k_1, \ldots, k_n \geq 1$, where for an ideal J of R, $\mathrm{Ass}^{\infty}(J)$ is the stable set of associated primes of J. Moreover, we prove that every non-zero ideal in a Dedekind ring is Ratliff-Rush closed, normally torsion-free and also has a strongly superficial element. Especially, we show that if $\mathcal{R} = \mathcal{R}(R, I)$ is the Rees ring of R with respect to I, as a subring of $R[t, u]$ with $u = t^{-1}$, then $u\mathcal{R}$ has no irrelevant prime divisor. In the second section, we prove that every non-zero finitely generated ideal in a Prüfer domain has the persistence property with respect to weakly associated prime ideals. Finally, we extend the notion of persistence property of ideals to the persistence property for rings.

1. Introduction

Assume that R is a commutative Noetherian ring and I is an ideal of R. It is known by Brodmann [3] that the sets of associated prime ideals of I^k, which denote by $\mathrm{Ass}_R(R/I^k)$, stabilize, that is, there exists a positive integer k_0 such that $\mathrm{Ass}_R(R/I^k) = \mathrm{Ass}_R(R/I^{k_0})$ for all $k \geq k_0$.

MSC(2010): Primary: 13F05; Secondary: 13C13, 13P99
Keywords: Dedekind rings, Prüfer domains, Weakly associated prime ideals, Strongly superficial element, Ratliff-Rush closed ideal.
Received: 3 May 2013, Revised: 24 October 2013.

The smallest number k_0 for which this equality occurs is called the *index of stability of* I and $\mathrm{Ass}_R(R/I^{k_0})$ is called the *stable set of associated prime ideals of* I, which is denoted by $\mathrm{Ass}^{\infty}(I)$.

An ideal I is said to satisfy the *persistence property* if

$$\mathrm{Ass}(R/I) \subseteq \mathrm{Ass}(R/I^2) \subseteq \cdots \subseteq \mathrm{Ass}(R/I^k) \subseteq \cdots .$$

It is shown that not all ideals satisfy the persistence property (see [7]), but for some classes of monomial ideals such as edge ideals of graphs, vertex cover ideals of perfect graphs and polymatroidal ideals persistence have been shown(see [5], [8], [11], [14]).

Up to now, by using combinatorial tools, several papers have been published in order to describe the stable set of associated prime ideals for a monomial ideal and a square-free monomial ideal (see [1], [4], [6], [9]).

The whole of those papers are related to the polynomial ring while in this paper we examine the persistence property for ideals in two rings such that one of them is a Noetherian ring other than the polynomial ring and the other one is a non-Noetherian ring.

This paper has two main sections. In the first section, we let R be a Dedekind ring. Clearly, the class of Dedekind rings lies properly between the class of principal ideal domains and the class of Noetherian integral domains. If $I_1, \ldots, I_n$ are non-zero proper ideals of R, then we prove that $\mathrm{Ass}^{\infty}(I_1^{k_1} \ldots I_n^{k_n}) = \mathrm{Ass}^{\infty}(I_1^{k_1}) \cup \cdots \cup \mathrm{Ass}^{\infty}(I_n^{k_n})$ for all $k_1, \ldots, k_n \geq 1$ (see Corollary 2.7). In the sequel, we show that every non-zero ideal in a Dedekind ring is Ratliff-Rush closed, normally torsion-free (see Corollaries 2.5 and 3.7), and also has a strongly superficial element (see Corollary 4.4).

In the second section, we assume that R is a Prüfer domain. In fact, a Prüfer domain is a type of commutative ring that generalizes Dedekind rings in a non-Noetherian context. In other words, a commutative ring is a Dedekind ring if and only if it is a Prüfer domain and a Noetherian ring. Since, in general, Prüfer domains are not Noetherian rings, we consider weakly associated prime ideals and prove that every non-zero finitely generated ideal in a Prüfer domain has the persistence property (see Theorem 5.8).

It is necessary to note that some results of this paper can be found in [16]. Throughout this paper the symbol $\mathbb{N}$ (respectively $\mathbb{N}_0$) will always denote the set of positive integers (respectively nonnegative integers).

2. Persistence property and the stable sets in Dedekind rings

We first give some necessary definitions and theorem which used throughout this paper.

Remark 2.1. Assume that $I_1, \dots, I_n$ are ideals in a commutative ring R which are pairwise coprime. Then, for all $1 \le i, j \le n$ with $i \neq j$, $\sqrt{I_i^{k_i} + I_j^{k_j}} = \sqrt{\sqrt{I_i} + \sqrt{I_j}} = \sqrt{I_i + I_j} = R$ for all $k_i, k_j \in \mathbb{N}$. Hence $I_1^{k_1}, \dots, I_n^{k_n}$ are also pairwise coprime for all $k_1, \dots, k_n \in \mathbb{N}$.

The following theorem is very useful and will be used frequently throughout the paper.

Theorem 2.2. [10, Theorem 6.10] *The following conditions on an integral domain R are equivalent.*

(i) *R is a Dedekind domain;*
(ii) *every proper ideal in R is uniquely a product of a finite number of prime ideals;*
(iii) *every non-zero ideal in R is invertible;*
(iv) *every fractional ideal of R is invertible;*
(v) *the set of all fractional ideals of R is a group under multiplication;*
(vi) *every ideal in R is projective;*
(vii) *every fractional ideal of R is projective;*
(viii) *R is Noetherian, integrally closed and every non-zero prime ideal is maximal;*
(ix) *R is Noetherian and, for every non-zero prime ideal $\mathfrak{p}$ of R, the localization $R_{\mathfrak{p}}$ of R at $\mathfrak{p}$ is a discrete valuation ring.*

Now, we investigate the persistence property in Dedekind rings.

Theorem 2.3. *Let R be a Dedekind ring and I be a proper ideal of R. Then I has the persistence property. Furthermore,* $\mathrm{Ass}^{\infty}(I) = \mathrm{Ass}_R(R/I)$.

Proof. If I is a non-zero ideal of R, then, by Theorem 2.2 (ii), $I = \mathfrak{p}_1^{\alpha_1} \dots \mathfrak{p}_n^{\alpha_n}$, where $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are distinct prime ideals of R and $\alpha_1, \dots, \alpha_n$ are positive integers. Also, by Theorem 2.2 (viii), $\mathfrak{p}_i$ is a maximal ideal of R for all $i = 1, \dots, n$. Fix $k \in \mathbb{N}$. Then, by Remark 2.1, $\mathfrak{p}_1^{k\alpha_1}, \dots, \mathfrak{p}_n^{k\alpha_n}$ are pairwise coprime, and so, by [12, Theorem 1.4], we have the following isomorphism

$$R/\mathfrak{p}_1^{k\alpha_1} \dots \mathfrak{p}_n^{k\alpha_n} \cong R/\mathfrak{p}_1^{k\alpha_1} \oplus \cdots \oplus R/\mathfrak{p}_n^{k\alpha_n}.$$

Hence

$$\begin{aligned}\mathrm{Ass}_R(R/I^k) &= \mathrm{Ass}_R(R/\mathfrak{p}_1^{k\alpha_1}\dots\mathfrak{p}_n^{k\alpha_n})\\ &= \mathrm{Ass}_R(R/\mathfrak{p}_1^{k\alpha_1}\oplus\cdots\oplus R/\mathfrak{p}_n^{k\alpha_n})\\ &= \{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}.\end{aligned}$$

Thus $\mathrm{Ass}_R(R/I^k) = \mathrm{Ass}_R(R/I)$ for all $k \in \mathbb{N}$, and so $\mathrm{Ass}^\infty(I) = \mathrm{Ass}_R(R/I)$. □

Now, we recall the following definition.

Definition 2.4. Let I be an ideal in a commutative Noetherian ring R. Then I is called *normally torsion-free* if $\mathrm{Ass}_R(R/I^k) \subseteq \mathrm{Ass}_R(R/I)$ for all $k \in \mathbb{N}$.

Theorem 2.3 has the following immediate consequence.

Corollary 2.5. *Let R be a Dedekind ring and I be a proper ideal in R. Then I is normally torsion-free.*

Theorem 2.6. *Let R be a Dedekind ring and $I_1,\dots,I_n$ be non-zero proper ideals of R. Then*

$$\mathrm{Ass}^\infty(I_1^{k_1}\dots I_n^{k_n}) = \bigcup_{i=1}^{n}\mathrm{Ass}_R(R/I_i)$$

for all $k_1,\dots,k_n \in \mathbb{N}$.

Proof. Since R is a Dedekind ring, by Theorem 2.2 (ii), we have $I_i = \mathfrak{p}_{i,1}^{\alpha_{i,1}}\dots\mathfrak{p}_{i,t_i}^{\alpha_{i,t_i}}$, where $\mathfrak{p}_{i,1},\dots,\mathfrak{p}_{i,t_i}$ are distinct prime ideals in R and $\alpha_{i,1},\dots,\alpha_{i,t_i} \in \mathbb{N}$ for all $i = 1,\dots,n$. Also, by Theorem 2.2 (viii), it follows that $\mathfrak{p}_{i,1},\dots,\mathfrak{p}_{i,t_i}$ are maximal ideals in R for all $i = 1,\dots,n$. So we can rewrite $I_1^{k_1}\dots I_n^{k_n} = \mathfrak{q}_1^{m_1}\dots\mathfrak{q}_s^{m_s}$ for some maximal ideals $\mathfrak{q}_1,\dots,\mathfrak{q}_s$ of R and $m_1,\dots,m_s \in \mathbb{N}$, where

$$\{\mathfrak{q}_1,\dots,\mathfrak{q}_s\} = \{\mathfrak{p}_{i,j} | i = 1,\dots,n,\ j = 1,\dots,t_i\}.$$

By Remark 2.1, the ideals $\mathfrak{q}_1^{m_1 d},\dots,\mathfrak{q}_s^{m_s d}$ are pairwise coprime for all $d \in \mathbb{N}$. Now, by [12, Theorem 1.4], we have that

$$\begin{aligned}R/(I_1^{k_1}\dots I_n^{k_n})^d &= R/\mathfrak{q}_1^{m_1 d}\dots\mathfrak{q}_s^{m_s d}\\ &\cong R/\mathfrak{q}_1^{m_1 d}\times\cdots\times R/\mathfrak{q}_s^{m_s d}\\ &\cong R/\mathfrak{q}_1^{m_1 d}\oplus\cdots\oplus R/\mathfrak{q}_s^{m_s d}\end{aligned}$$

for all $d \in \mathbb{N}$. Hence, by Theorem 2.3, we obtain

$$\begin{aligned}\mathrm{Ass}_R(R/(I_1^{k_1}\dots I_n^{k_n})^d) &= \mathrm{Ass}_R(R/\mathfrak{q}_1^{m_1 d}\oplus\cdots\oplus R/\mathfrak{q}_s^{m_s d})\\ &= \{\mathfrak{p}_{i,j} | i=1,\dots,n,\ j=1,\dots,t_i\}\\ &= \bigcup_{i=1}^{n}\mathrm{Ass}_R(R/I_i)\end{aligned}$$

for all $d \in \mathbb{N}$. Thus

$$\mathrm{Ass}_R(R/I_1^{k_1}\dots I_n^{k_n}) = \mathrm{Ass}^\infty(I_1^{k_1}\dots I_n^{k_n}) = \bigcup_{i=1}^{n}\mathrm{Ass}_R(R/I_i)$$

for all $k_1,\dots,k_n \in \mathbb{N}$. □

Corollary 2.7. *Let R be a Dedekind ring and $I_1,\dots,I_n$ be non-zero proper ideals in R. Then $\mathrm{Ass}^\infty(I_1^{k_1}\dots I_n^{k_n}) = \mathrm{Ass}^\infty(I_1^{k_1})\cup\cdots\cup\mathrm{Ass}^\infty(I_n^{k_n})$ for all $k_1,\dots,k_n \in \mathbb{N}$.*

Proof. For $k_1,\dots,k_n \in \mathbb{N}$, consider the ideals $I_1^{k_1},\dots,I_n^{k_n}$ of Dedekind ring R. Then, by Theorems 2.3 and 2.6, we have the following equalities

$$\begin{aligned}\mathrm{Ass}^\infty(I_1^{k_1}\dots I_n^{k_n}) =&\mathrm{Ass}_R(R/I_1^{k_1}\dots I_n^{k_n})\\ =&\bigcup_{i=1}^{n}\mathrm{Ass}_R(R/I_i)\\ =&\bigcup_{i=1}^{n}\mathrm{Ass}_R(R/I_i^{k_i})\\ =&\mathrm{Ass}^\infty(I_1^{k_1})\cup\cdots\cup\mathrm{Ass}^\infty(I_n^{k_n}).\end{aligned}$$

□

Proposition 2.8. *Let $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ be distinct non-zero prime ideals in a Dedekind ring R. Then there exists an ideal I such that $\mathrm{Ass}_R(R/I) = \mathrm{Ass}^\infty(I) = \{\mathfrak{p}_1,\dots,\mathfrak{p}_n\}$.*

Proof. In view of Theorem 2.3, it is enough to put $I := \mathfrak{p}_1\dots\mathfrak{p}_n$. □

3. A class of Ratliff-Rush closed ideals

We begin with the following Theorem.

Theorem 3.1. *Let I be a non-zero ideal in a Dedekind ring R. Then $(I^{k+i} :_R I^i) = I^k$ for all $k,i \in \mathbb{N}$.*

Proof. Fix $i \in \mathbb{N}$. It is easy to see that $I^i(I^{k+i} :_R I^i) = I^{k+i}$ for all $k \in \mathbb{N}$. By Theorem 2.2 (iii), I^i is an invertible ideal, and so, by [10, p. 401], there exists a fractional ideal J of R such that $JI^i = R$.

Now, by multiplying the equation $I^i(I^{k+i} :_R I^i) = I^{k+i}$ in J, we obtain $(I^{k+i} :_R I^i) = I^k$ for all $k \in \mathbb{N}$, as desired. $\square$

The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let I be a non-zero ideal in a Dedekind ring R. Then, for all $k \in \mathbb{N}$, we have the following equalities*

$$(I^{k+1} :_R I) = I^k \text{ and } (I^{k+1} :_R I^k) = I.$$

Notation 3.3. *For an ideal I of a commutative ring R, we set*

$$I^* := \bigcup_{n\in\mathbb{N}} (I^{n+1} :_R I^n).$$

Proposition 3.4. *Let I be a non-zero ideal in a Dedekind ring R. Then $I^{k^*} = I^k$ for all $k \in \mathbb{N}$.*

Proof. Fix $k \in \mathbb{N}$. By Theorem 3.1, we have that

$$(I^k)^* = \bigcup_{n\in\mathbb{N}} ((I^k)^{n+1} :_R (I^k)^n) = \bigcup_{n\in\mathbb{N}} (I^{nk+k} :_R I^{nk}) = I^k.$$

$\square$

Corollary 3.5. *Let I be a non-zero ideal in a Dedekind ring R. Then $I^* = I$.*

Let I be an arbitrary ideal in a commutative Noetherian ring R. From the maximal condition on ideals of R, it follows that there exist ideals I^* in R which are maximal with respect to the condition

$$I^{*n} = I^n \text{ for all large } n.$$

Ratliff and Rush, in [16, Theorem 2.1], proved that if I is a regular ideal (that is, I contains a non-zerodivisor), then there exists a unique such I^*, which can be presented in terms of I as follows:

$$I^* := \bigcup_{n\in\mathbb{N}} (I^{n+1} :_R I^n).$$

In fact, the eventual stable value of the ascending chain $(I^2 :_R I) \subseteq (I^3 :_R I^2) \subseteq \cdots \subseteq (I^{i+1} :_R I^i) \subseteq \cdots$ is I^*.

Definition 3.6. The ideal I^* is called the *Ratliff-Rush ideal associated with* I or the *Ratliff-Rush closure of* I. A regular ideal I for which $I^* = I$ is called *Ratliff-Rush closed*.

As an application of Corollary 3.5, we have the following result.

Corollary 3.7. *Every non-zero ideal in a Dedekind ring is Ratliff-Rush closed.*

4. Existence of a strongly superficial element

Suppose that I is an arbitrary ideal in a commutative ring R and $k \in \mathbb{N}$. An element $x \in R$ is called a *superficial element of degree k for I* if $x \in I^k$ and there exists $c \in \mathbb{N}$ such that $(I^{n+k} :_R x) \cap I^c = I^n$ for all $n \geq c$. Also, if $(I^{n+k} :_R x) = I^n$ for all $n \in \mathbb{N}$, we say that x is a *strongly superficial element of degree k for I* (see [13, Definition 4.1.2]). In this section we show that every non-zero ideal in a Dedekind ring has a superficial element. To do this, we recall the following definitions.

Definition 4.1. Let $I = (b_1, \ldots, b_k)$ be an ideal in a Noetherian ring R, let t be an indeterminate and let $u = t^{-1}$. Then the *Rees ring* $\mathcal{R} = \mathcal{R}(R, I)$ *of R with respect to I* is the subring $\mathcal{R} = R[tb_1, \ldots, tb_k, u]$ of $R[t, u]$ (see [15, Definition 2.3]).

Definition 4.2. Let I be an ideal in a Noetherian ring R and let $\mathcal{R} = \mathcal{R}(R, I)$. Then a homogeneous ideal H in $\mathcal{R}$ is said to be *irrelevant* if H contains all homogeneous elements of sufficiently large degree; otherwise, H is said to be *relevant* (see [15, Definition 3.1]).

Theorem 4.3. *The following statements hold for a non-zero ideal I in a Dedekind ring R:*

(i) *$u\mathcal{R}$ has no irrelevant prime divisor.*
(ii) *There exist $k \in \mathbb{N}$ and b in I^k such that $(I^{n+k} :_R b) = I^n$ for all $n \in \mathbb{N}$.*
(iii) *$I^{n+1} = I^n \cap (I^{n+1} :_R \mathfrak{p}) \cap (I^{n+2} :_R I)$ for all $n \in \mathbb{N}_0$ and all prime ideals $\mathfrak{p}$ in R such that $I \subseteq \mathfrak{p}$.*

Proof. By Corollary 3.2, $(I^{n+1} :_R I) = I^n$ for all $n \in \mathbb{N}$. Now, by [15, Remark 3.6.2], the statements hold. □

Corollary 4.4. *Every non-zero ideal in a Dedekind ring has a strongly superficial element.*

Proof. Let I be a non-zero ideal in a Dedekind ring R. By Theorem 4.3 (ii), there exist $k \in \mathbb{N}$ and b in I^k such that $(I^{n+k} :_R bR) = I^n$ for all $n \in \mathbb{N}$. Thus I has a strongly superficial element of degree k. □

5. Persistence property in Prüfer domains

There are many equivalent definitions of a Prüfer domain. We begin with the following one.

Definition 5.1. A *Prüfer domain R* is an integral domain in which every non-zero finitely generated ideal is invertible.

Theorem 5.2. *Let I be a non-zero finitely generated ideal in a Prüfer domain R. Then $(I^{k+i} :_R I^i) = I^k$ for all $k, i \in \mathbb{N}$.*

Proof. Fix $i \in \mathbb{N}$. It is easy to see that $I^i(I^{k+i} :_R I^i) = I^{k+i}$ for all $k \in \mathbb{N}$. Since I^i is a non-zero finitely generated ideal of R, by [2, Theorem 1.1(6)], I^i is a cancellation ideal. Thus $(I^{k+i} :_R I^i) = I^k$ for all $k \in \mathbb{N}$, as desired. □

The following result is an immediate consequence of Theorem 5.2.

Corollary 5.3. *Let I be a non-zero finitely generated ideal in a Prüfer domain R. Then $(I^{k+1} :_R I^k) = I$ for all $k \in \mathbb{N}$.*

Proposition 5.4. *Let I be a non-zero finitely generated ideal in a Prüfer domain R and $I^* := \bigcup_{n\in\mathbb{N}}(I^{n+1} :_R I^n)$. Then ${I^k}^* = I^k$ for all $k \in \mathbb{N}$.*

Proof. Fix $k \in \mathbb{N}$. By Theorem 5.2, we obtain

$$(I^k)^* = \bigcup_{n\in\mathbb{N}}((I^k)^{n+1} :_R (I^k)^n) = \bigcup_{n\in\mathbb{N}}(I^{nk+k} :_R I^{nk}) = I^k.$$

□

Corollary 5.5. *Let I be a non-zero finitely generated ideal in a Prüfer domain R. Then $I^* = I$.*

There are several variant definitions of associated primes in the literature. The following definition is more standard for associated prime ideals of non-Noetherian rings.

Definition 5.6. Let $N \subseteq M$ be modules over a ring R. The set

$$\{\mathfrak{p} \in \mathrm{Spec}(R) |\ \mathfrak{p} \text{ is minimal over } (N :_R m) \text{ for some } m \in M\}$$

is called *the set of weakly associated primes of* M/N, and is denoted $\widetilde{\mathrm{Ass}}_R(M/N)$.

If I is an ideal, then *the weakly associated primes of the ideal* I are the weakly associated primes of the module R/I.

Proposition 5.7. *Let R be a (not necessary Noetherian) commutative ring, I an ideal, J a finitely generated ideal of R, and $\mathfrak{p}$ a prime ideal of R. If $\mathfrak{p} \in \widetilde{\mathrm{Ass}}_R(R/(I :_R J))$, then $\mathfrak{p} \in \widetilde{\mathrm{Ass}}_R(R/I)$.*

Proof. Suppose that $J = (x_1, \dots, x_n)$ and choose $\mathfrak{p} \in \widetilde{\mathrm{Ass}}_R(R/(I :_R J))$. Then there exists $y \in R$ such that $\mathfrak{p}$ is minimal over $((I :_R J) :_R y)$. On the other hand,

$$((I :_R J) :_R y) = (I :_R yJ) = \bigcap_{i=1}^{n}(I :_R x_iy).$$

Since $\bigcap_{i=1}^{n}(I :_R x_i y) \subseteq \mathfrak{p}$, it follows that there exists $i \in \mathbb{N}$ with $1 \leq i \leq n$ such that $(I :_R x_i y) \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is minimal over $(I :_R x_i y)$, we have that $\mathfrak{p} \in \widetilde{\mathrm{Ass}}_R(R/I)$, as claimed. $\square$

Now, we present the main result in this section.

Theorem 5.8. *Every non-zero finitely generated ideal in a Prüfer domain has the persistence property.*

Proof. Let I be a non-zero finitely generated ideal in a Prüfer domain R. By Theorem 5.2, we have $(I^{n+1} :_R I) = I^n$ for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and consider $\mathfrak{p} \in \widetilde{\mathrm{Ass}}_R(R/I^n)$. Then $\mathfrak{p} \in \widetilde{\mathrm{Ass}}_R(R/(I^{n+1} :_R I))$. Now, by Proposition 5.7, it follows that $\mathfrak{p} \in \widetilde{\mathrm{Ass}}_R(R/I^{n+1})$ and the proof is complete. $\square$

Now, we extend the notion of persistence property of ideals to the persistence property for rings.

Definition 5.9. Let R be a commutative ring. Then we say that R has the persistence property when any finitely generated ideal of R has the persistence property with respect to weakly associated prime ideals.

By Theorem 2.3 and Theorem 5.8, it follows that Dedekind rings and Prüfer domains have the persistence property.

We conclude this section with a question focusing on further research.

Question 5.10. *Let S be a commutative ring and R be a subring of S such that R has not the persistence property with respect to weakly associated prime ideals. Then is it true that S has not the persistence property with respect to weakly associated prime ideals?*

For example, suppose that K is a field and $S = K[x_1, x_2, x_3, \ldots]$ is the polynomial ring in infinitely many variables $x_1, x_2, x_3, \ldots$. It is common that the ring S is a non-Noetherian ring and that $S = \cup_{n\in\mathbb{N}} K[x_1, \ldots, x_n]$ such that the subring $R = K[x_1, \ldots, x_n]$, for all $n \in \mathbb{N}$, is a Noetherian ring. Now, suppose that the ideal I is generated by monomials $x_1x_2^2x_3, x_2x_3^2x_4, x_3x_4^2x_5, x_4x_5^2x_1$, and $x_5x_1^2x_2$ in the Noetherian ring $R = K[x_1, x_2, x_3, x_4, x_5]$. Note that $\widetilde{\mathrm{Ass}}_R(R/I) = \mathrm{Ass}_R(R/I)$ and $\widetilde{\mathrm{Ass}}_R(R/I^2) = \mathrm{Ass}_R(R/I^2)$. Also, we have that $(x_1, x_2, x_3, x_4, x_5) \in \mathrm{Ass}_R(R/I)$, but $(x_1, x_2, x_3, x_4, x_5) \notin \mathrm{Ass}_R(R/I^2)$. Therefore the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5]$ has not the persistence property. Now, can we conclude that S has not the persistence property with respect to weakly associated prime ideals?

Acknowledgments

The author wish to thank two anonymous reviewers for their helpful feedback and valuable suggestions. Also the author would like to thank Professor Kazem Khashyarmanesh, Ferdowsi University of Mashhad for his helpful comments on this paper.

References

1. S. Bayati, J. Herzog and G. Rinaldo, On the stable set of associated prime ideals of a monomial ideal, *Arch. Math.* **98**(3) (2012), 213-217.
2. S. Bazzoni and S. Glaz, Prüfer rings, *Multiplicative Ideal Theory in Commutative Algebra*, Springer (2006), 55-72.
3. M. Brodmann, Asymptotic stability of $\mathrm{Ass}(M/I^nM)$, *Proc. Amer. Math. Soc.* **74** (1979), 16-18.
4. J. Chen, S. Morey and A. Sung, The stable set of associated primes of the ideal of a graph, *Rocky Mountain J. Math.* **32**(1) (2002), 71-89.
5. C. Francisco, H. T. Hà and A. Van Tuyl, Colorings of hypergraphs, perfect graphs and associated primes of powers of monomial ideals, *J. Algebra* **331** (2011), 224-242.
6. H. T. Hà and S. Morey, Embedded associated primes of powers of square-free monomial ideals, *J. Pure Appl. Algebra* **214**(4) (2010), 301-308.
7. J. Herzog and T. Hibi, The depth of powers of an ideal, *J. Algebra* **291** (2005), 534-550.
8. J. Herzog and A. A. Qureshi, Persistence and stability properties of powers of ideals, Preprint, 2012, arXiv: 1208.4684v1.
9. J. Herzog, A. Rauf and M. Vladoiu, The stable set of associated prime ideals of a polymatroidal ideal, *J. Algebraic Combin.* (to appear).
10. T. W. Hungerford, *Algebra,* Graduate Texts in Mathematics 73 Springer-Verlag, New York, 1974.
11. J. MartÍnez-Bernal, S. Morey and R. H. Villarreal, Associated primes of powers of edge ideals, *J. Collect. Math.* (to appear).
12. H. Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.
13. S. McAdam and L. J. Ratliff, Sporadic and irrelevant prime divisors, *Trans. Amer. Math. Soc.* **303** (1987), 311-324.
14. S. Morey and R. H. Villarreal, Edge ideals: algebraic and combinatorial properties, *J. Progress in commutative Algebra, Combinatorics and Homology* **1** (2012), 85-126.
15. L. J. Ratliff, On prime divisors of I^n, n large, *Michigan Math. J.* **23** (1976), 337-352.
16. L. J. Ratliff and D. E. Rush, Two notes on reductions of ideals, *Indiana Univ. Math. J.* **27** (1978), 929-934.

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