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H-SETS AND APPLICATIONS ON H_v-GROUPS

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Abstract. In this paper, the notion of H-sets on H_v-groups is introduced and some related properties are investigated and some examples are given. In this regards, the concept of regular, strongly regular relations and homomorphism of H-sets are adopted. Also, the classical isomorphism theorems of groups are generalized to H-sets on H_v-groups. Finally, by using these concepts tensor product on H_v-groups is introduced and proved that the tensor product exists and is unique up to isomorphism.

1. Introduction

The concept of hypergroup was introduced in 1934 by a French mathematician F. Marty [3], at the 8^{th} Congress of Scandinavian Mathematicians. He published some notes on hypergroups, using them in different contexts such as algebraic functions, rational fractions and non commutative groups. One of these hyperstructures is H_v-structure[1, 2, 9] which is the largest class of hyperstructures. This concept was introduced by Vougiouklis in 1990 [9] at the fourth AHA congress. The concept of an H_v-structure is a generalization of the well-known algebraic hyperstructures such as hypergroup, hyperring, hypermodule and so on. Also, some axioms concerning the hyperstructures are replaced by their corresponding weak axioms [5, 7, 9, 6, 10, 11, 8].

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The fundamental relations are main tools to study hyperstructures ($\beta^*, \gamma^*, \epsilon^*$ etc.), which are defined in H_v-structures, as the smallest equivalences so that the quotients would be ordinary structures. Motivation to introduce H_v-structures:

(1) The quotient of a group with respect to an invariant subgroup is a group.
(2) Marty construct, the quotient of a group with respect to any subgroup is a hypergroup.
(3) The quotient of a group with respect to any partition (or equivalently to any equivalence relation) is an H_v-group.

The notion of a group acting on a set is one which links abstract algebra to nearly branch of mathematics such as linear algebra, and differential equation. Another application of group actions, the Sylow Theorems, which are essential to the classification of groups. The motivation for such an investigation is to generalize the concept of group acting on a set. We will introduce the notion of H-sets on H_v-groups that is a new hyperstructure and we investigate some property of this hyperstructure. Also, by the concept H-set, we define tensor product on H_v-groups that is a non-additive classical construction such as ring and module theory. Finally, we prove that tensor product exists and is unique up to isomorphism.

2. H-SETS

In this section, we present some notions about the H_v-groups and new concept left(right)-set on H_v-groups. Also, we construct quotient left(right)-sets by regular(strongly) equivalence relation and isomorphism theorems.

Definition 2.1. Let H be a nonempty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. Then, H is called a *canonical H_v-group*, when the following conditions hold:

(1) for every $x, y, z \in H$, $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset$,
(2) there exists $e \in H$, called identity, such that $x \circ e = e \circ x = x$,
(3) for every $x \in H$ there exists a unique element $x^{'} \in H$ and is called inverse such that $e \in (x \circ x^{'}) \cap (x^{'} \circ x)$,
(4) $z \in x \circ y$, implies that $y \in x^{'} \circ z$ and $x \in z \circ y^{'}$.

A nonempty subset N of a canonical H_v-hypergroup $(H, \circ)$ is called *subcanonical H_v-group* if $(N, \circ)$ is a canonical H_v-group.

Let H be a canonical H_v-group and N be a subcanonical H_v-group of H. Then, we define the equivalence relation $\equiv$ on H as follows:

$$h_1 \equiv h_2 \Longleftrightarrow \ h_1 \in h_2 + N.$$

This relation is denoted by N^* and $H^*(h)$ the equivalence class of the element h.

Definition 2.2 ([2], p.187)**.** Let H be a nonempty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation on H. Then, the hyperstructure $(H, \circ)$ is called *H_v-group* if

(1) $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset$,
(2) $x \circ H = H \circ x = H$,

where $x, y, z \in H$. An H_v-group $(H, \circ)$ is called *commutative*, when $x \circ y = y \circ x$, for every x and y of H.

Example 2.3. Let $\mathbb{R}$ be the set of real numbers and $M_n(\mathbb{R})$ be the set of all $n \times n$ matrices. Then, $M_n(\mathbb{R})$ is an H_v- group by following hyperoperation:

$$(a_{ij}) \oplus (b_{ij}) = \{(ra_{ij} + rb_{ij}) : r \in [0, 1]\}.$$

We have,

$$(a_{ij}) \oplus ((b_{ij}) \oplus (c_{ij})) = \{(ra_{ij} + rmb_{ij} + rmc_{ij})_{1 \leq i,j \leq n} : r, m \in [0, 1]\},$$

$$((a_{ij}) \oplus ((b_{ij})) \oplus (c_{ij}) = \{(tna_{ij} + tnb_{ij} + tc_{ij})_{1 \leq i,j \leq n} : t, n \in [0, 1]\}.$$

If $r = t = 0$, then

$$(0_{ij}) \subseteq (a_{ij}) \oplus ((b_{ij}) \oplus (c_{ij})) \cap ((a_{ij}) \oplus ((b_{ij})) \oplus (c_{ij}).$$

Also, we have the reproduction axiom.

Definition 2.4. Let $(H, \circ)$ be an H_v-group with identity and X be a nonempty set. Then, we say that X is a *left H-set*, if there is a hyperoperation $\mu : H \times X \longrightarrow P^*(X)$ from $H \times X$ into $P^*(X)$ with the properties:

(1) $\mu(h_1 \circ h_2, x) \cap \mu(h_1, \mu(h_2, x)) \neq \emptyset$,
(2) $x \in \mu(e, x)$,

where $x \in X$ and $h_1, h_2 \in H$ and

$$\mu(h_1 \circ h_2, x) = \bigcup_{t \in h_1 \circ h_2} \mu(t, x).$$

Let e be a scalar identity of H and $x = \mu(e, x)$, for every $x \in X$. Then, we say that X is a left H-set with unit. An element $h \in H$ is called *scalar*, when for every $x \in X$, the set $\mu(h, x)$ has only one element.

Example 2.5. Let S be a nonempty set and $\{A_x\}_{x\in H}$ be a partition of S, where $(H,\circ)$ is an H_v-group. Then, S is an H_v-group by following hyperoperation:

$$a\otimes b=\bigcup_{t\in x\circ y}A_t,$$

where $a\in A_x$ and $b\in A_y$. Also, H is a left S-set by following hyperoperation:

$$\begin{aligned}\mu: S\times H &\longrightarrow P^*(H)\\ (a,h) &\longrightarrow x\otimes h.\end{aligned}$$

Example 2.6. Let $(H,\circ)$ be a canonical H_v-group. Then, H is a H-set as follows:

$$\begin{aligned}\mu: H\times H &\longrightarrow P^*(H)\\ (x,h) &\longrightarrow x\circ h\circ x',\end{aligned}$$

where x' is an inverse of x.

Example 2.7. Let $(H,\circ)$ be an H_v-group and ρ be a regular relation on H. Then, H/ρ is a left H-set as follows:

$$\begin{aligned}\mu: H\times H/\rho &\longrightarrow P^*(H/\rho)\\ (h,\rho(x)) &\longrightarrow \{\rho(t): t\in h\circ x\}.\end{aligned}$$

Example 2.8. Let $(H,+)$ be a canonical H_v-group and N be a sub H_v-group of H. Then, we define the relation $\equiv$ on H as follows:

$$x\equiv y\Longleftrightarrow\ (x-y)\cap N\neq\emptyset.$$

This relation is equivalence on H. We define the equivalence class $x\in H$ by $N^*(x)$. Hence $H/N^*=\{N^*(x): x\in H\}$ is a left H-set by following hypeoperation:

$$\begin{aligned}\mu: H\times H/N^* &\longrightarrow P^*(H/N^*)\\ (h,N^*(x)) &\longrightarrow \{N^*(t): t\in h+x\}.\end{aligned}$$

In the same way, we can construct a right H-set. Also, we say that X is an (H_1,H_2)-set, when it is a left H_1-set and a right H_2-set and

$$\mu_2(\mu_1(h_1,x),h_2)=\mu_1(h_1,\mu(x,h_2)),$$

where $h_1\in H_1, h_2\in H_2$ and $x\in X$.

If H be a commutative H_v-group, then there is no distinction between a left and right H-sets.

It is clear that the cartesian product $X\times Y$ of a left H_1-set X and a right H_2-set Y is an (H_1,H_2)-set by the following hyperoperations:

$$\overline{\mu}_1(h_1,(x,y))=\{(t,y): t\in\mu_1(h_1,x)\},$$

$$\overline{\mu}_2((x,y),h_2)=\{(x,t): t\in\mu_2(y,h_2)\}.$$

Definition 2.9. Let X and Y be left H-sets and $\varphi : X \longrightarrow Y$ be a map. Then, we say that φ is a *morphism*, when

$$\varphi(\mu_1(h, x)) = \mu_2(h, \varphi(x)),$$

where $x \in X$ and $h \in H$.

Let ρ be an equivalence relation on X and A and B be nonempty subset of X. Then, we define

$$(A, B) \in \overline{\rho} \Longleftrightarrow \ \forall a \in A\ \exists b \in B : (a, b) \in \rho$$

and

$$\forall b \in B\ \exists a \in A : (a, b) \in \rho.$$

Also,

$$(A, B) \in \overline{\overline{\rho}} \Longleftrightarrow \ (a, b) \in \rho, \quad \text{for all } a \in A, b \in B.$$

Definition 2.10. Let X be a left H-set and ρ be an equivalence relation on X. Then, we say that ρ is *regular* on X, when

$$(x, y) \in \rho \Longrightarrow (\mu(h_1, x), \mu(h_2, x)) \in \overline{\rho},$$

and is called *strongly regular*, when

$$(x, y) \in \rho \Longrightarrow (\mu(h_1, x), \mu(h_2, x)) \in \overline{\overline{\rho}}.$$

By using a certain type of equivalence relations, we can construct quotient left H-sets as follows:

Theorem 2.11. *Let X be a left H-set and ρ be a regular relation on X. Then, X/ρ is a left H-set by following hyperoperation:*

$$\widehat{\mu}(h, \rho(x)) = \{\rho(t) : t \in \mu(h, x)\}.$$

and $\pi : X \longrightarrow X/\rho$ is a morphism

Proof. Suppose that $\rho(x_1) = \rho(x_2)$. Let $(x_1, x_2) \in \rho$ and since ρ is regular, we have $(\mu(h, x_1), \mu(h, x_2)) \in \overline{\rho}$. This implies that for every $t_1 \in \mu(h, x_1)$, there exists $t_2 \in \mu(h, x_2)$ such that $(t_1, t_2) \in \rho$. Hence $\widehat{\mu}(h, \rho(x_1)) \subseteq \widehat{\mu}(h, \rho(x_2))$. In the same way, $\widehat{\mu}(h, \rho(x_2)) \subseteq \widehat{\mu}(h, \rho(x_1))$. Thus, the hyperoperation defined on X/ρ is well-defined. Also,

$$\begin{aligned}
\widehat{\mu}(h_1, \widehat{\mu}(h_2, \rho(x)) = \widehat{\mu}(h_1, \{\rho(t) : t \in \mu(h_2, x)\}) &= \bigcup_{t \in \mu(h_2, x)} \widehat{\mu}(h_1, \rho(t)) \\
&= \bigcup_{t \in \mu(h_2, x), t_1 \in \mu(h_1, t)} \rho(t_1) \\
&= \bigcup_{t_1 \in \mu(h_1, \mu(h_2, x))} \rho(t_1).
\end{aligned}$$

By a similar argument, we have

$$\widehat{\mu}(h_1 \circ h_2, \rho(x)) = \bigcup_{t_1 \in \mu(h_1 \circ h_2, x)} \rho(t_1).$$

Also, $\mu(h_1, \mu(h_2, x)) \cap \mu(h_1 \circ h_2, x) \neq \emptyset$, implies that

$$\widehat{\mu}(h_1, \widehat{\mu}(h_2, \rho(x)) \cap \widehat{\mu}(h_1 \circ h_2, \rho(x)) \neq \emptyset.$$

Also,

$$\rho(x) \in \widehat{\mu}(e, \rho(x)) = \{\rho(t) : t \in \mu(e, x)\}.$$

Therefore, X/ρ is a left H-set. Also,

$$\begin{aligned} \pi(\mu(h, x)) = \{\rho(t) : t \in \mu(h, x)\} \quad &= \widehat{\mu}(t, \rho(x)) \\ &= \widehat{\mu}(t, \pi(x)). \end{aligned}$$

Hence, π is a morphism and this completes the proof. □

Theorem 2.12. *Let X and Y be left H-sets and $\varphi : X \longrightarrow Y$ be a morphism. Then, the relation*

$$ker\varphi = \{(x_1, x_2) \in X \times X : \varphi(x_1) = \varphi(x_2)\},$$

is a regular relation on X and there is a monomorphism $\widehat{\varphi} : X/ker\varphi \longrightarrow Y$ such that $im\widehat{\varphi} = im\varphi$ and $\widehat{\varphi} \circ \pi = \varphi$, where $\pi : X \longrightarrow X/ker\varphi$ is a natural map.

Proof. Suppose that $(x_1, x_2) \in ker\varphi$. Hence $\varphi(x_1) = \varphi(x_2)$. This implies that for every $h \in H$,

$$\varphi(\mu(h, x_1)) = \mu(h, \varphi(x_1)) = \mu(h, \varphi(x_2)) = \varphi(\mu(h, x_2)),$$

and for every $t_1 \in \mu(h, x_1)$, there exists $t_2 \in \mu(h, x_2)$ such that $\varphi(t_1) = \varphi(t_2)$. Also, for every $t_2 \in \mu(h, x_2)$ there exists $t_1 \in \mu(h, x_1)$ such that $(t_1, t_2) \in ker\varphi$. Thus, the relation $ker\varphi$ is regular. Let us denote $ker\varphi$ by k. We define $\widehat{\varphi} : X/ker\varphi \longrightarrow Y$ defined by $\widehat{\varphi}(k(x)) = \varphi(x)$, where $x \in X$. Then, $\widehat{\varphi}$ is both well-defined and one to one, since

$$k(x_1) = k(x_2) \Longleftrightarrow (x_1, x_2) \in k \Longleftrightarrow \varphi(x_1) = \varphi(x_2).$$

Also, for every $x \in X$ and $h \in H$,

$$\begin{aligned} \widehat{\varphi}(\widehat{\mu}(h, k(x)) = \widehat{\varphi}(\{k(t) : t \in \mu(h, x)\}) \quad &= \{\varphi(t) : t \in \mu(h, x)\} \\ &= \varphi(\mu(h, x)) \\ &= \mu(h, \varphi(x)) \\ &= \mu(h, \widehat{\varphi}(k(x)). \end{aligned}$$

Hence $\widehat{\varphi}$ is a morphism. Clearly, $im(\widehat{\varphi}) = im\varphi$ and $\widehat{\varphi} \circ \pi = \varphi$. □

Let X be a left H-set, σ_1 and σ_2 be regular relations on X such that $\sigma_1 \subseteq \sigma_2$. Then, there is a morphism α from X/σ_1 onto X/σ_2 such that $\alpha \circ \pi_1 = \pi_2$, where $\pi_1 : X \longrightarrow X/\sigma_1$ and $\pi : X \longrightarrow X/\sigma_2$ are natural morphisms. The morphism α given by

$$\alpha(\sigma_1(x)) = \sigma_2(x), \ x \in X,$$

and the regular relation $ker\alpha$ on X/σ_1 given by

$$ker\alpha = \{(\sigma_1(a), \sigma_1(b)) : (a, b) \in \sigma_2\}.$$

We write $ker\alpha$ by σ_1/σ_2.

Theorem 2.13. *Let X be a left H-set, σ_1, σ_2 be regular relations such that $\sigma_1 \subseteq \sigma_2$. Then, σ_1/σ_2 is a regular relation on X/σ_2 and*

$$(X/\sigma_2)/(\sigma_1/\sigma_2) \cong (X/\sigma_1).$$

Proof. The proof is straightforward. □

Definition 2.14. Let X be a left H-set on a canonical H_v-group H. Then, X is called *invertible*, when

$$x \in \mu(h, y) \Longrightarrow \; y \in \mu(h^{'}, x),$$

where $x, y \in X$.

Let X be an invertible left H-set. Then, we define an equivalence relation $\sim$ on X as follows:

$$x \sim y \Longleftrightarrow \; \exists h \in H : x \in \mu(h, y).$$

The equivalence class $x \in X$ is called *orbital* and denoted by $orb(x)$. Hence $X = \bigcup_{x\in X} orb(x)$ and when X is a finite set $|X| = \sum_{x\in X} |orb(x)|$. Also, the stabilizer $x \in X$ is defined as follows:

$$stab(x) = \{g \in H : g \in \mu(g, x)\}.$$

Example 2.15. Let X be a left H-set and ρ be a strongly regular relation on X. Then, X/ρ is a H-set as follows:

$$\begin{aligned} \widehat{\mu} : H \times X/\rho \; &\longrightarrow P^*(X/\rho) \\ (h, \rho(x)) \; &\longrightarrow \{\rho(t) : t \in \mu(h, x)\}, \end{aligned}$$

where $\rho(x) \in X/\rho$ and $h \in H$. Since ρ is a strongly regular relation, $| \; \widehat{\mu}(h, \rho(x)) \; |= 1$. Hence,

$$orb(\rho(x)) = \{\rho(t) : t \in \mu(h, x), \; h \in H\},$$

$$stab(\rho(x)) = \{h \in H : \rho(x) = \widehat{\mu}(h, \rho(x))\}.$$

Proposition 2.16. *Let X be a left H-set on canonical H_v-group H and ρ be a strongly regular relation on X. Then, $stab(\rho(x))$ is a H_v-subgroup of H.*

Proof. Suppose that $h_1, h_2 \in stab(\rho(x))$. Hence, $\rho(x) = \widehat{\mu}(h_1, \rho(x))$ and $\rho(x) = \widehat{\mu}(h_2, \rho(x))$. This implies that

$$\widehat{\mu}(h_1 \circ h_2, \rho(x)) \cap \widehat{\mu}(h_1, \widehat{\mu}(h_2, \rho(x)) \neq \emptyset.$$

Thus,

$$\widehat{\mu}(h_1 \circ h_2, \rho(x)) = \widehat{\mu}(h_1, \widehat{\mu}(h_2, \rho(x)) = \widehat{\mu}(h_1, \rho(x)) = \rho(x).$$

Then, for every $h \in h_1 \circ h_2$, $\widehat{\mu}(h, \rho(x)) = \rho(x)$ and $h_1 \circ h_2 \subseteq stab(\rho(x))$. Also, we can see $stab(\rho(x))$ is closed with respect to the inverse. $\square$

Theorem 2.17. *Let X be a left H-set on commutative canonical H_v-group $(H, +)$ and ρ be a strongly regular relation on X. Then,*

$$\mid orb(\rho(x)) \mid = \mid H/S^* \mid,$$

where $S = stab(\rho(x))$.

Proof. Suppose that $\varphi : H/S^* \longrightarrow orb(\rho(x))$ defined by

$$\varphi(S^*(h)) = \widehat{\mu}(h, \rho(x)).$$

Let $S^*(h_1) = S^*(h_2)$. Then, $h_1 \in h_2 + s$, for some $s \in stab(\rho(x))$ and

$$\widehat{\mu}(h_1, \rho(x))) \subseteq \widehat{\mu}(h_2 + s, \rho(x)) = \widehat{\mu}(h_2, \widehat{\mu}(s, \rho(x)) = \widehat{\mu}(h_2, \rho(x)).$$

Hence $\widehat{\mu}(h_1, \rho(x))) = \widehat{\mu}(h_2, \rho(x))$ and φ is well-defined. Also, $\varphi(S^*(h_1)) = \varphi(S^*(h_2))$, implies that

$$\begin{aligned} \rho(x) = \widehat{\mu}(0, \rho(x)) \subseteq \widehat{\mu}(h_1 - h_1, \rho(x)) &= \widehat{\mu}(-h_1, \widehat{\mu}(h_1, \rho(x)) \\ &= \widehat{\mu}(-h_1, \widehat{\mu}(h_2, \rho(x)) \\ &= \widehat{\mu}(h_2 - h_1, \rho(x)). \end{aligned}$$

Hence, for some $s \in h_2 - h_1$ such that $\rho(x) = \widehat{\mu}(s, \rho(x))$. Thus, $s \in stab(\rho(x))$ and $S^*(h_1) = S^*(h_2)$ and the map φ is one to one. $\square$

Corollary 2.18. *Let X be a left H-set and ρ be a strongly regular relation on X such that $\mid H/S^* \mid$ is finite. Then, the order H/S^* divide $\mid X/\rho \mid$.*

Definition 2.19 ([2], p. 188)**.** Let $(H_1, \circ)$ and $(H_2, *)$ be H_v-groups. Then, a map $\varphi : H_1 \longrightarrow H_2$ is called a *strong homomorphism*, when

$$\varphi(x_1 \circ x_2) = \varphi(x_1) * \varphi(x_2),$$

for every $x_1, x_2 \in H_1$. An injective and onto strong homomorphism is called an *isomorphism*.

Definition 2.20. Let X_1 and X_2 be left H_1 and H_2-sets, respectively, $\varphi : H_1 \longrightarrow H_2$ be an isomorphism and $\lambda : X_1 \longrightarrow X_2$ be a bijective morphism. Then, we say that X_1 and X_2 are equivalent when,

$$\lambda(\mu_1(h_1, x_1)) = \mu_2(\varphi(h_1), \lambda(x_1)),\ \lambda^{-1}(\mu_2(h_2, x_2)) = \mu_1(\varphi^{-1}(h_2), \lambda^{-1}(x_2)),$$

where $h_1 \in H_1, h_2 \in H_2$ and $x_1 \in X_1, x_2 \in X_2$. We write $X_1 \sim X_2$, when X_1 and X_2 are equivalent. When X_1 and X_2 are (H_1, H_2)-sets, we say that X_1 and X_2 are equivalent, if X_1 and X_2 are equivalent as left H_1-set and right H_2-set.

Proposition 2.21. *Let X_1 and X_2 be equivalent left H_1- and H_2-sets and $x_1 \in X_1$. Then,*

$$stab(x_1) \simeq stab(\lambda(x_1)).$$

Proof. The proof is straightforward. □

3. Tensor product

In this section by left(right) H-sets we introduce a new type of superstructures that will be called tensor product.

Definition 3.1. Let X, Y and Z be (H_1, H_2)-, (H_2, H_3)- and (H_1, H_3)-sets, respectively. Then, a map $\varphi : X \times Y \longrightarrow Z$ is called *bimap*, if for every $x \in X$, $y \in Y$ and h_2 of H_2,

$$\varphi(\mu_1(x, h_2), y) = \varphi(x, \mu_2(h_2, y)).$$

Definition 3.2. Let X, Y and T be (H_1, H_2)-, (H_2, H_3)- and (H_1, H_3)-sets, respectively, and $\psi : X \times Y \longrightarrow T$ be a bimap. Then, a pair (T, ψ) is called *tensor product* of X and Y over H_2, if for every (H_1, H_3)-set C and every bimap $\beta : X \times Y \longrightarrow C$ there exists a unique bimap $\overline{\beta} : T \longrightarrow C$ such that $\overline{\beta} \circ \psi = \beta$.

Let X and Y be (H_1, H_2) and (H_2, H_3)-sets, respectively. Then, we define the relation ρ on $X \times Y$ as follows:

$$\rho = \{((t_1, t_2), (t_3, t_4)) : t_1 = \mu_1(t_3, h_2) \ t_4 = \mu_2(h_2, t_2)\},$$

where h_2 is a scalar element of H_2. The relation ρ is reflexive and symmetric. Let ρ^* be transitive closure of ρ and we denote a typical element $\rho^*(x, y)$ of $X \otimes Y$ by $x \otimes y$. For any two nonempty subsets A of X and B of Y, we define

$$A \otimes B = \bigcup_{a \in A, b \in B} a \otimes b.$$

We note that by definition of ρ for every scalar element $h_2 \in H_2$,

$$\mu_1(x, h_2) \otimes y = x \otimes \mu_2(h_2, y).$$

Proposition 3.3. *Let X and Y be (H_1, H_2)- and (H_2, H_3)-sets, respectively. Then, $x_1 \otimes y_1 = x_2 \otimes y_2$ if and only if there exist $a_1, a_2, ..., a_{n-1}$ in X, $b_1, b_2, ..., b_{n-1}$ in Y and scalar elements $s_1, s_2, ..., s_n$, $t_1, t_2, ..., t_{n-1}$ in H_2 such that*

$$\begin{aligned} x_1 = \mu_1(a_1, s_1),\ \mu_1(a_1, t_1) = \mu_1(a_2, s_2),\ , ...\mu_1(a_i, t_i) &= \mu_1(a_{i+1}, s_{i+1}) \\ &\vdots \\ \mu_1(a_{n-1}, t_{n-1}) &= \mu_1(x_2, s_n) \ (*). \end{aligned}$$

$$\begin{aligned} \mu_2(s_1, y_1) = \mu_2(t_1, b_1),\ \mu_2(s_2, b_1) &= \mu_2(t_2, b_2), ..., \mu_2(s_{i+1}, b_i) \\ &= \mu_2(t_{i+1}, b_{i+1}) \\ &\vdots \\ \mu_2(s_n, b_{n-1}) &= y_2 \qquad (**). \end{aligned}$$

Proof. Suppose that we have $(*)$ and $(**)$. Then,

$$\begin{aligned} x_1 \otimes y_1 = \mu_1(a_1, s_1) \otimes y_1 = a_1 \otimes \mu_2(s_1, y_1) &= a_1 \otimes \mu_2(t_1, b_1) \\ &\vdots \\ &= \mu_1(x_2, s_n) \otimes b_1 \\ &= x_2 \otimes \mu_2(s_n, b_{n-1}) \\ &= x_2 \otimes y_2. \end{aligned}$$

Conversely, assume that $x_1 \otimes y_1 = x_2 \otimes y_2$. By definition there are $(t_i, s_i) \in X \times Y$, $1 \leq i \leq n$ such that

$$(t_1, s_1) = (x_1, y_1),\ (t_n, s_n) = (x_2, y_2)$$

and $((t_i, s_i), (t_{i+1}, s_{i+1})) \in \rho$. By definition of ρ, we have $(*)$ and $(**)$. □

Proposition 3.4. *Let X and Y be (H_1, H_2)- and (H_2, H_3)-sets, respectively. Then, $X \otimes Y$ is an (H_1, H_3)-set by following hyperoperations:*

$$\begin{aligned} \overline{\mu}_1 : H_1 \times X \otimes Y &\longrightarrow P^*(X \otimes Y) \\ h_1 \cdot (x \otimes y) &= \{t \otimes y : t \in \mu_1(h_1, x)\}, \end{aligned}$$

$$\begin{aligned} \overline{\mu}_2 : X \otimes Y \times H_3 &\longrightarrow P^*(X \otimes Y) \\ (x \otimes y) \cdot h_3 &= \{x \otimes t : t \in \mu_2(y, h_3)\}, \end{aligned}$$

where $x \otimes y \in X \otimes Y$, $h_1 \in H_1$ and $h_3 \in H_3$.

Proof. Let $x_1 \otimes y_1 = x_2 \otimes y_2$. Then, by Proposition 3.3,

$$\begin{aligned} x_1 = \mu_1(a_1, s_1),\ \mu_1(a_1, t_1) = \mu_1(a_2, s_2),\ ..., \mu_1(a_i, t_i) &= \mu_1(a_{i+1}, s_{i+1}) \\ &\vdots \\ \mu_1(a_{n-1}, t_{n-1}) &= \mu_1(x_2, s_n). \end{aligned}$$

$$\begin{aligned} \mu_2(s_1, y_1) = \mu_2(t_1, b_1),\ \mu_2(s_2, b_1) &= \mu_2(t_2, b_2), ..., \mu_2(s_{i+1}, b_i) \\ &= \mu_2(t_{i+1}, b_{i+1}) \\ &\vdots \\ \mu_2(s_n, b_{n-1}) &= y_2. \end{aligned}$$

We have,

$$\begin{array}{ll} \mu_1(h_1, x_1) & = \mu_1(\mu_1((h_1, a_1), s_1),\ \mu_1(\mu_1((h_1, a_1), t_1) \\ & = \mu_1((\mu_1(h_1, a_2), s_2) \\ & \vdots \\ \mu_1(\mu_1(h_1, a_i), t_i) & = \mu_1(\mu_1(h_1, a_{i+1}), s_{i+1})) \\ & \vdots \\ \mu_1(h_1 a_{n-1}), t_{n-1}) & = \mu_1(\mu_1(h_1, x_2), s_n). \end{array}$$

Assume that $w \in \mu_1(h_1, x_1)$. Since s_1 is a scalar element, then there exists $w_1 \in \mu_1(h_1, a_1)$ such that $w = \mu_1(w_1, s_1)$. Also, since s_2 is a scalar element, then there exists $w_2 \in \mu_1(h_1, a_2)$ such that

$$\mu_1(w_1, t_1) = \mu_1(w_2, s_2).$$

After a finite process, we have $w_1, w_2, ..., w_n \in X$ such that

$$\begin{array}{rl} w = \mu_1(w_1, s_1),\ \mu_1(w_1, t_1) = \mu_1(w_2, s_2), ..., \mu_1(w_i, t_i) & = \mu_1(w_{i+1}, s_{i+1}) \\ & \vdots \\ & = \mu_1(w_{n-1}, t_{n-1}) \\ & = w_n s_n. \end{array}$$

Hence, $w \otimes y_1 = w_n \otimes y_1 \in h_1 x_2 \otimes y_2$. Thus,

$$\widehat{\mu}(h_1, x_1 \otimes y_1) \subseteq \widehat{\mu}(h_2, x_2 \otimes y_2).$$

In the same way, we can see that $\widehat{\mu}(h_2, x_2 \otimes y_2) \subseteq \widehat{\mu}(h_1, (x_1 \otimes y_1)$. Therefore, the hyperoperation $\widehat{\mu}$ defined on $X \otimes Y$ is well-defined. Also,

$$\widehat{\mu}(h_1, \widehat{\mu}(h_2, (x \otimes y)) \cap \widehat{\mu}(h_1 h_2, (x \otimes y) \neq \emptyset.$$

$\square$

Corollary 3.5. *Let X and Y be (H_1, H_2)- and (H_2, H_3)-sets, respectively. Then, a map $\pi : X \times Y \longrightarrow X \otimes Y$ is a bimap.*

Example 3.6. Let $(H, \circ)$ be a commutative H_v-group and ρ be a strongly regular relation on H. Then, $X = H/\rho$ is a left H-set as follows:

$$\begin{array}{rl} \mu : H \times H/\rho & \longrightarrow P^*(H) \\ (h, \rho(x)) & \longrightarrow \{\rho(t) : t \in h \circ x\}. \end{array}$$

Since ρ is a strongly regular relation on H, every element of H is a scalar element of X. Thus,

$$\rho(a) \otimes \rho(b) = \{(\rho(x), \rho(y)) : x \in h \circ a,\ b \in h \circ y\}.$$

Also,

$$orb(\rho(x)) = \{\rho(t) : t \in h \circ x,\ h \in H\}.$$

Theorem 3.7. *Let X and Y be (H_1, H_2)- and (H_2, H_3)- sets, respectively. Then, $(X \otimes Y, \pi)$ is a tensor product over H_2.*

Proof. Suppose that Z is an (H_1, H_2)-set and $\beta : X \times Y \longrightarrow Z$ be a bimap. We define $\overline{\beta} : X \otimes Y \longrightarrow Z$ by

$$\overline{\beta}(x \otimes y) = \beta(x, y),$$

where $x \in X$ and $y \in Y$. Let $x_1 \otimes y_1 = x_2 \otimes y_2$. Then, by Proposition 3.3, we have

$$\begin{aligned} \beta(x_1, y_1) = \beta(\mu_1(a_1, s_1), y_1) \ &= \beta(a_1, \mu_2(s_1, y_1)) \\ &\vdots \\ &= \beta(\mu_1(x_2, s_2), b_{n-1}) \\ &= \beta(x_2, \mu_2(s, b_{n-1})) \\ &= \beta(x_2, y_2). \end{aligned}$$

Hence, $\overline{\beta}(x_1 \otimes y_1) = \beta(x_2 \otimes y_2)$ and $\overline{\beta}$ is well-defined. Also,

$$\begin{aligned} \overline{\beta}(\widehat{\mu}_1(h_1, x \otimes y)) = \beta(\mu_1(h_1, x) \otimes y) \ &= \beta(\mu_1(h_1, x), y) \\ &= \mu_3(h_1, \beta(x, y)) \\ &= \mu_3(h_1, \beta(x \otimes y)), \end{aligned}$$

where $h_1 \in H_1$, $x \otimes y \in X \otimes Y$. In the same way, we can see that $\overline{\beta}(\widehat{\mu}_3(x \otimes y, h_3)) = \mu_3(\beta(x \otimes y), h_3)$. Thus, $\overline{\beta}$ is a morphism. Also,

$$\overline{\beta} \circ \pi(x, y) = \beta(x \otimes y) = \beta(x, y),$$

implies that $\overline{\beta} \circ \pi = \beta$. If $\beta_1 : X \otimes Y \longrightarrow Z$ be an another morphism such that $\beta_1 \circ \pi = \beta$, then we have

$$\beta_1(x \otimes y) = \beta_1(\pi(x, y)) = \beta_1 \circ \pi(x, y) = \overline{\beta} \circ \pi(x, y) = \overline{\beta}(x \otimes y).$$

Therefore, $\overline{\beta}$ is unique with respect to this properties. $\square$

Theorem 3.8. *Let X_1, X_2 and Y_1, Y_2 be (H_1, H_2)- and (H_2, H_3)-sets, respectively such that $X_1 \sim X_2$ and $Y_1 \sim Y_2$. Then, $X_1 \otimes Y_1 \sim X_2 \otimes Y_2$.*

Proof. Suppose that $X_1 \sim X_2$ and $Y_1 \sim Y_2$. By definition, there exist isomorphisms $\varphi_i : H_i \longrightarrow H_{i+1}$ for $1 \leq i \leq 3$ and $\lambda_1 : X_1 \longrightarrow X_2$ and $\lambda_2 : Y_1 \longrightarrow Y_2$ such that

$$\begin{aligned} \lambda_1(\mu_1(h_1, x_1)) \ &= \mu_2(\varphi_1(h_1), \lambda_1(x_1)), \ \lambda_1^{-1}(\mu_2(h_2, x_2)) \\ &= \mu_1(\varphi_1^{-1}(h_2), \lambda_1^{-1}(x_2)), \end{aligned}$$

where $x_1 \in X_1, x_2 \in X_2$ and $h_1 \in H_1, h_2 \in H_2$.

$$\begin{aligned} \lambda_2(\mu_2(h_2, y_1)) \ &= \mu_3(\varphi_2(h_2), \lambda_2(y_1)) \\ \lambda_2^{-1}(\mu_2(h_3, y_2)) \ &= \mu_2(\varphi^{-1}(h_3), \lambda_2^{-1}(y_2)), \end{aligned}$$

where $y_1 \in Y_1, y_2 \in Y_2$ and $h_2 \in H_2, h_3 \in H_3$.

We define $\lambda : X_1 \otimes Y_1 \longrightarrow X_2 \otimes Y_2$, by $\lambda(x \otimes y) = \lambda_1(x) \otimes \lambda_2(y)$, where $x \in X_1$ and $y \in Y_1$. Let $x_1 \otimes y_1 = x_2 \otimes y_2$, where $x_1, x_2 \in X_1$ and $y_1, y_2 \in Y_1$. Then,

$$\begin{aligned} x_1 = \mu_1(a_1, s_1),\ \mu_1(a_1, t_1) = \mu_1(a_2, s_2),\ \ , ...\mu_1(a_i, t_i) &= \mu_1(a_{i+1}, s_{i+1}) \\ &\vdots \\ \mu_1(a_{n-1}, t_{n-1}) &= \mu_1(x_2, s_n). \end{aligned}$$

$$\begin{aligned} \mu_2(s_1, y_1) = \mu_2(t_1, b_1),\ \mu_2(s_2, b_1) &= \mu_2(t_2, b_2) \\ &\vdots \\ \mu_2(s_{i+1}, b_i) &= \mu_2(t_{i+1}, b_{i+1}) \\ &\vdots \\ \mu_2(s_n, b_{n-1}) &= y_2. \end{aligned}$$

where $s_1, s_2, ..., s_n,\ \ t_1, t_2, ..., t_{n-1}$ are scalar elements in H_2 and $a_i \in X_1,\ b_i \in Y_1,\ 1 \leq i \leq n-1$. This implies that

$$\begin{aligned} \lambda_1(x_1) = \lambda_1(\mu_1(a_1, s_1)),\ \lambda_1(\mu_1(a_1, t_1)) &= \lambda_1(\mu_1(a_2, s_2)) \\ &\vdots \\ \lambda_1(\mu_1(a_i, t_i)) &= \lambda_1(\mu_1(a_{i+1}, s_{i+1})) \\ &\vdots \\ \lambda_1(\mu_1(a_{n-1}, t_{n-1})) &= \lambda_1(\mu_1(x_2, s_n)). \end{aligned}$$

$$\begin{aligned} \lambda_2(\mu_2(s_1, y_1)) = \lambda_2(\mu_2(t_1, b_1)),\ \lambda_2(\mu_2(s_2, b_1)) &= \lambda_2(\mu_2(t_2, b_2)) \\ &\vdots \\ \lambda_2(\mu_2(s_{i+1}, b_i)) &= \lambda_2(\mu_2(t_{i+1}, b_{i+1})) \\ &\vdots \\ \lambda_2(\mu_2(s_n, b_{n-1})) &= \lambda_2(y_2). \end{aligned}$$

Hence,

$$\begin{aligned} \lambda_1(x_1) = \mu_1^{'}(\lambda_1(a_1), \varphi_2(s_1)),\ \mu_1^{'}(\lambda_1(a_1), \varphi_2(t_1)) &= \mu_1^{'}(\lambda_1(a_2), \varphi_2(s_2)) \\ &\vdots \\ \mu_1^{'}(\lambda_1(a_i), \varphi_2(t_i)) &= \mu_1^{'}(\lambda_1(a_{i+1}), \varphi_2(s_{i+1})) \\ &\vdots \\ \mu_1^{'}(\lambda_1(a_{n-1}), \varphi_2(t_{n-1})) &= \mu_1^{'}(\lambda_1(x_2), \varphi_2(s_n)). \end{aligned}$$

$$\begin{aligned}
\mu_2'(\varphi_2(s_1),\lambda_2(y_1)) &= \mu_2'(\varphi_2(t_1),\lambda_2(b_1))\\
\mu_2'(\varphi_2(s_2),\lambda_2(b_1)) &= \mu_2'(\varphi_2(t_2),\lambda_2(b_2))\\
&\vdots\\
\mu_2'(\varphi_2(s_{i+1}),\lambda_2(b_i)) &= \mu_2'(\varphi_2(t_{i+1}),\lambda_2(b_{i+1}))\\
&\vdots\\
\mu_2'(\varphi_2(s_n),\lambda_2(b_{n-1})) &= \lambda_2(y_2).
\end{aligned}$$

Thus, $\lambda_1(x_1)\otimes\lambda_2(y_1)=\lambda_1(x_2)\otimes\lambda_2(y_2)$ and the map λ is well-defined. Also,

$$\begin{aligned}
\lambda(\overline{\mu}_1(h_1,x_1\otimes y_1))=\lambda(\mu_1(h_1,x_1)\otimes y_1)) &= \lambda_1(\mu_1(h_1,x_1)\otimes\lambda_2(y_1)\\
&= \mu_1'(\varphi(h_1),\lambda_1(x_1))\otimes\lambda_2(y_1)\\
&= \overline{\mu}_1'(\varphi(h_1),\lambda_1(x_1)\otimes\lambda_2(y_1))\\
&= \overline{\mu}_1'(\varphi(h_1),\lambda(x_1\otimes y_1)).
\end{aligned}$$

We can see for $\overline{\lambda}$ other properties holds. □

4. Conclusion

The concept of H_v-structures were introduced by Vougiouklis at the fourth AHA congress (1990)[10]. The concept of an H_v -structure constitutes a generalization of the well-known algebraic hyperstructures. Numerous applications of hyperstructures are presented, especially those that were found and studied in the last fifteen years. By this hyperstructure, we can study chemical reactions as mathematical models [4]. In this paper, we introduce H-sets on H_v-groups and tensor products on H_v-groups that is crucially important in homological algebra. In a future study, by the concept H_v-group we will consider and classify mathematical and chemical properties of these H_v-groups.

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H-SETS AND APPLICATIONS ON H_v-GROUPS

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H- مجموعه‌ها و کاربردشان در H_v-گروه‌ها

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در این مقاله، مفهوم H-مجموعه‌ها روی H_v-گروه‌ها را معرفی کرده، برخی از ویژگی‌های آن‌ها را مورد بررسی قرار داده و مثال‌های متعددی نیز ارائه شده است. در این رابطه، مفاهیم روابط منظم، منظم قوی و همریختی معرفی می‌شود. همچنین، قضیه‌های کلاسیک یکریختی گروه‌ها به قضیه‌های یکریختی در H-مجموعه‌ها روی H_v-گروه‌ها تعمیم داده می‌شود. در نهایت، با استفاده از این مفاهیم، حاصلضرب تانسوری روی H_v-گروه‌ها معرفی و ثابت می‌شود این حاصلضرب وجود دارد و در حد یکریختی منحصربه‌فرد است.

کلمات کلیدی: H_v-گروه، H-مجموعه چپ(راست)، حاصلضرب تانسوری، رابطه منظم، رابطه منظم قوی.