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JORDAN HIGHER DERIVATIONS, A NEW APPROACH

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ABSTRACT. Let $\mathcal{A}$ be a unital algebra over a 2-torsion free commutative ring $\mathcal{R}$ and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. We show taht every Jordan higher derivation $D = \{D_n\}_{n\in\mathbb{N}_0}$ from the trivial extension $\mathcal{A} \ltimes \mathcal{M}$ into itself is a higher derivation, if $PD_1(QXP)Q = QD_1(PXQ)P = 0$ for all $X \in \mathcal{A} \ltimes \mathcal{M}$, in which $P = (e, 0)$ and $Q = (e', 0)$ for some non-trivial idempotent elements $e \in \mathcal{A}$ and $e' = 1_{\mathcal{A}} - e$ satisfying the following conditions: $e\mathcal{A}e'\mathcal{A}e = \{0\}$, $e'\mathcal{A}e\mathcal{A}e' = \{0\}$, $e(l.ann_{\mathcal{A}}\mathcal{M})e = \{0\}$, $e'(r.ann_{\mathcal{A}}\mathcal{M})e' = \{0\}$ and $eme' = m$ for all $m \in \mathcal{M}$.

1. Introduction and preliminaries

Let $\mathcal{A}$ be a unital algebra over a commutative ring $\mathcal{R}$ and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. An $\mathcal{R}$-linear mapping $\delta : \mathcal{A} \to \mathcal{M}$ is called a *derivation* if it satisfies the leibniz rule $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$ and is called an *antiderivation* if $\delta(xy) = \delta(y)x + y\delta(x)$ for all $x, y \in \mathcal{A}$. δ is called a *Jordan derivation* if $\delta(x^2) = \delta(x)x + x\delta(x)$ for all $x \in \mathcal{A}$. Obviously, every derivation or antiderivation is a Jordan derivation. However, the converse statement is not true in general (see [1]). It is natural and very interesting to find some conditions under which a Jordan derivation is a derivation or an antiderivation. Zhang and Yu [10] showed that every Jordan derivation of triangular algebras is a derivation, so every Jordan derivation from the algebra of all upper

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triangular matrices into itself is a derivation. Ghahramani [8] showed that every Jordan derivation of the trivial extensions of an algebra $\mathcal{A}$ by its bimodules, under some conditions, is the sum of a derivation and an antiderivation.

Let $\mathbb{N}_0$ be the set of all nonnegative integers. If we define a sequence d_n of linear mappings on $\mathcal{A}$ by $d_0 = I$ and $d_n = \frac{\delta^n}{n!}$, where I is the identity mapping on $\mathcal{A}$, then the Leibniz rule ensures us that d_n's satisfy the condition

$$d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y) \tag{1.1}$$

for each $x, y \in \mathcal{A}$ and each non-negative integer n. Such a sequence $d = \{d_n\}_{n\in\mathbb{N}_0}$ is called a *higher derivation*. d is called a *Jordan higher derivation* if for any $n \in \mathbb{N}_0$,

$$d_n(x^2) = \sum_{i+j=n} d_i(x)d_j(x) \tag{1.2}$$

for all $x \in \mathcal{A}$. Note that d_1 is a derivation (resp. Jordan derivation), if d is a higher derivation (resp. Jordan higher derivation).

Higher derivations were introduced by Hasse and Schmidt [9], and algebraists sometimes call them *Hasse-Schmidt derivations*. For an account on higher derivations the reader is referred to the book [3].

Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a commutative ring $\mathcal{R}$ and $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. The $\mathcal{R}$-algebra

$$Tri(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \Big| \quad a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

under the usual matrix operations is called a *triangular algebra*. Basic examples of triangular algebras are upper triangular matrix algebras and nest algebras (see [2], [4]).

Let $\mathcal{A}$ be a unital algebra over $\mathcal{R}$ and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. $\mathcal{A} \times \mathcal{M}$ as an $\mathcal{R}$-module together with the algebra product defined by:

$$(a, m).(b, n) = (ab, an + mb) \quad (a, b \in \mathcal{A}, \quad m, n \in \mathcal{M})$$

is an $\mathcal{R}$-algebra with unity $1 = (1_{\mathcal{A}}, 0)$ and zero $0 = (0, 0)$, which is called the *trivial extension* of $\mathcal{A}$ by $\mathcal{M}$ and denoted by $\mathcal{A} \ltimes \mathcal{M}$. Trivial extensions have been extensively studied in the algebra and analysis.

Let $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra over $\mathcal{R}$. Denote by $\mathcal{A} \oplus \mathcal{B}$ the direct sum of $\mathcal{A}$ and $\mathcal{B}$ as $\mathcal{R}$-algebra, and view $\mathcal{M}$ as an $(\mathcal{A} \oplus \mathcal{B})$-bimodule with the module actions given by

$$(a, b).m = am, \quad m.(a, b) = mb \quad (a \in \mathcal{A}, \quad m \in \mathcal{M}, \quad b \in \mathcal{B}).$$

Then $Tri(\mathcal{A},\mathcal{M},\mathcal{B})$ is isomorphic to $(\mathcal{A}\oplus\mathcal{B})\ltimes\mathcal{M}$ as an $\mathcal{R}$-algebra. So triangular algebras are examples of trivial extensions.

Ghahramani [8] has shown that under some mild conditions, every Jordan derivation on $\mathcal{A}\ltimes\mathcal{M}$ is a derivation. Erfanian Attar and Ebrahimi Vishki [6] gave characterizations of (Jordan) derivations on $\mathcal{A}\ltimes\mathcal{M}$. Note that a Jordan derivation on a trivial extension algebra may not be a derivation in general. To see an example the reader can refer to [5].

The following notations will be used in this paper.

Let $\mathcal{A}$ be an $\mathcal{R}$-algebra and $\mathcal{M}$ be an $\mathcal{A}$-bimodule, define the left annihilator of $\mathcal{M}$ and the right annihilator of $\mathcal{M}$ as follows:

$$l.ann_{\mathcal{A}}\mathcal{M}=\{a\in\mathcal{A}:a\mathcal{M}=\{0\}\},$$

$$r.ann_{\mathcal{A}}\mathcal{M}=\{a\in\mathcal{A}:\mathcal{M}a=\{0\}\}.$$

2. **Main result**

Let us first recall some basic facts concerning Jordan higher derivations on an associative algebra. Many different kinds of higher derivations have been studied in commutative and noncommutative rings (see [7] and the references therein).

Lemma 2.1. *Let $\mathcal{A}$ be an associative algebra over a 2-torsion free commutative ring $\mathcal{R}$ and $D=\{D_n\}_{n\in\mathbb{N}_0}$ be a Jordan higher derivation from $\mathcal{A}$ into itself. Then for all $x,y,z\in\mathcal{A}$ and each $n\in\mathbb{N}_0$, we have*

(a) $D_n(xy+yx)=\sum_{i+j=n}D_i(x)D_j(y)+D_i(y)D_j(x)$,
(b) $D_n(xyx)=\sum_{i+j+k=n}D_i(x)D_j(y)D_k(x)$,
(c) $D_n(xyz+zyx)=\sum_{i+j=n}D_i(x)D_j(y)D_k(z)+D_i(z)D_j(y)D_k(x)$.

Note that the converse holds only in the case where $\mathcal{R}$ is 2-torsion free (that is, $2x=0$ implies $x=0$ for any $x\in\mathcal{A}$).

Theorem 2.2. *Let $\mathcal{A}$ be a unital algebra over a 2-torsion free commutative ring $\mathcal{R}$ and $\mathcal{M}$ be a unital $\mathcal{A}$-bimodule. Suppose that e is a non-trivial idempotent element in $\mathcal{A}$ and $e'=1_{\mathcal{A}}-e$ such that*

$$e\mathcal{A}e'\mathcal{A}e=\{0\},\quad e'\mathcal{A}e\mathcal{A}e'=\{0\},$$

$$e(l.ann_{\mathcal{A}}\mathcal{M})e=\{0\},\quad e'(r.ann_{\mathcal{A}}\mathcal{M})e'=\{0\},$$

and $eme'=m$ for all $m\in\mathcal{M}$. Let $P=(e,0)$ and $Q=(e',0)$.

If $D=\{D_n\}_{n\in\mathbb{N}_0}$ is a Jordan higher derivation from the trivial extension $\mathcal{A}\ltimes\mathcal{M}$ into itself such that $PD_1(QXP)Q=QD_1(PXQ)P=0$ for all $X\in\mathcal{A}\ltimes\mathcal{M}$, then D is a higher derivation.

Note that P and Q are idempotents of $\mathcal{A} \ltimes \mathcal{M}$ such that $P+Q=1$ and $PQ=0$. Also for any $X, Y \in \mathcal{A} \ltimes \mathcal{M}$, we have $PXQYP=0$ and $QXPYQ=0$. Since if $X=(a,m)$ and $Y=(b,n)$, then

$$PXQYP = (eae'be, eae'ne + eme'be) = 0$$

and similarly $QXPYQ=0$.

Since the triangular algebra $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is isomorphic to the trivial extension $(\mathcal{A} \oplus \mathcal{B}) \ltimes \mathcal{M}$, we have the following result.

Corollary 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a 2-torsion free commutative ring $\mathcal{R}$ and $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Then any Jordan higher derivation from triangular algebra $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ into itself, is a higher derivation.*

To prove Theorem 2.2 we need some lemmas.

Lemma 2.4. *For every $n \in \mathbb{N}$ we have $PD_n(P)P = 0$, $QD_n(Q)Q = 0$ and for every $n \in \mathbb{N}_0$ we have $PD_n(Q)P = 0$, $QD_n(P)Q = 0$.*

Proof. It follows from

$$D_1(P) = D_1(P^2) = D_1(P)P + PD_1(P) \tag{2.1}$$

that $PD_1(P)P = 0$. Suppose that $PD_m(P)P = 0$ for all $m < n$. From

$$D_n(P) = D_n(P)P + PD_n(P) + \sum_{\substack{i+j=n \\ i,j \geq 1}} D_i(P)D_j(P), \tag{2.2}$$

we have

$$PD_n(P)P = PD_n(P)P + PD_n(P)P + \sum_{\substack{i+j=n \\ i,j \geq 1}} PD_i(P)D_j(P)P.$$

It follows that

$$PD_n(P)P + \sum_{\substack{i+j=n \\ i,j \geq 1}} \Big(PD_i(P)PD_j(P)P + PD_i(P)QD_j(P)P \Big) = 0.$$

So we get $PD_n(P)P = 0$.

By induction on n, it follows from (1.2) that $D_n(I) = 0$ for all $n \in \mathbb{N}$. Thus $D_n(Q) = -D_n(P)$ and so

$$PD_n(Q)P = -PD_n(P)P = 0$$

for all $n \in \mathbb{N}$. Similarly we can show that $QD_n(Q)Q = 0$ and $QD_n(P)Q = 0$. $\square$

Lemma 2.5. *For every $n \in \mathbb{N}$, we have*

$$PD_n(P) = D_n(P)Q, \quad D_n(P)P = QD_n(P)$$
$$QD_n(Q) = D_n(Q)P, \quad D_n(Q)Q = PD_n(Q).$$

Proof. It follows from (2.2) that

$$PD_n(P) = PD_n(P)P + PD_n(P) + P \sum_{\substack{i+j=n \\ i,j\geq 1}} D_i(P)D_j(P).$$

Thus

$$P \sum_{\substack{i+j=n \\ i,j\geq 1}} D_i(P)D_j(P) = 0. \tag{2.3}$$

Also it follows from (2.2) that

$$QD_n(P) = QD_n(P)P + Q \sum_{\substack{i+j=n \\ i,j\geq 1}} D_i(P)D_j(P).$$

Thus

$$Q \sum_{\substack{i+j=n \\ i,j\geq 1}} D_i(P)D_j(P) = QD_n(P) - QD_n(P)P = QD_n(P)Q = 0. \tag{2.4}$$

From (2.3) and (2.4) we obtain that

$$\sum_{\substack{i+j=n \\ i,j\geq 1}} D_i(P)D_j(P) = 0. \tag{2.5}$$

So

$$D_n(P) = D_n(P)P + PD_n(P).$$

Therefore we get

$$D_n(P)P = D_n(P) - PD_n(P) = QD_n(P),$$

$$PD_n(P) = D_n(P) - D_n(P)P = D_n(P)Q.$$

Similarly we can get that $QD_n(Q) = D_n(Q)P$ and $D_n(Q)Q = PD_n(Q)$. □

Lemma 2.6. *For every $n \in \mathbb{N}_0$ and any $X \in \mathcal{A} \ltimes \mathcal{M}$, we have*

$$PD_n(PXQ)P = 0, \quad PD_n(QXP)P = 0, \quad PD_n(QXQ)P = 0,$$

$$QD_n(PXP)Q = 0, \quad QD_n(PXQ)Q = 0, \quad QD_n(QXP)Q = 0.$$

Proof. By Lemma 2.1 (a) we have

$$\begin{aligned} PD_n(PXQ)P &= PD_n(PXQ+QPX)P \\ &= \sum_{i+j=n} \Big(PD_i(PX)D_j(Q)P + PD_i(Q)D_j(PX)P\Big) \\ &= \sum_{i+j=n} \Big(PD_i(PX)PD_j(Q)P + PD_i(PX)QD_j(Q)P \\ &+ PD_i(Q)PD_j(PX)P + PD_i(Q)QD_j(PX)P\Big) = 0. \end{aligned}$$

Also by Lemma 2.1 (b) we have

$$\begin{aligned} PD_n(QXQ)P &= \sum_{i+j+k=n} PD_i(Q)D_j(X)D_k(Q)P \\ &= \sum_{i+j+k=n} (PD_i(Q)D_j(X)PD_k(Q)P \\ &+ PD_i(Q)D_j(X)QD_k(Q)P) = 0. \end{aligned}$$

Similarly we get

$$\begin{aligned} PD_n(QXP)P &= QD_n(PXP)Q = QD_n(PXQ)Q \\ &= QD_n(QXP)Q = 0. \end{aligned}$$

□

Lemma 2.7. *Let $X \in \mathcal{A} \ltimes \mathcal{M}$. Then for each $n \in \mathbb{N}_0$,*

$$PD_n(QXP)Q = QD_n(PXQ)P = 0.$$

Proof. It is true for $n = 0$ and by assumption for $n = 1$. Let $n \geq 2$, then

$$\begin{aligned} PD_n(QXP)Q &= PD_n(QXP+PQX)Q \\ &= \sum_{i+j=n} PD_i(QX)D_j(P)Q + PD_i(P)D_j(QX)Q \\ &= \sum_{i+j=n} PD_i(QXP)D_j(P)Q + PD_i(P)D_j(QXP)Q \\ &+ \sum_{i+j=n} PD_i(QXQ)D_j(P)Q + PD_i(P)D_j(QXQ)Q \\ &= \sum_{i+j=n} PD_i(QXP)PD_j(P) + D_i(P)QD_j(QXP)Q \\ &+ PD_n(QXQP+PQXQ)Q = 0 \end{aligned}$$

Similarly we can show that $QD_n(PXQ)P = 0$. □

Lemma 2.8. *Let $X, Y \in \mathcal{A} \ltimes \mathcal{M}$. Then for each $n \in \mathbb{N}_0$,*

(a) $PD_n(PXPYP)P = \sum_{i+j=n} PD_i(PXP)D_j(PYP)P$,

(b) $QD_n(QXQYQ)Q = \sum_{i+j=n} QD_i(QXQ)D_j(QYQ)Q$.

Proof. For any $X, Y, Z \in \mathcal{A} \ltimes \mathcal{M}$ and $n \in \mathbb{N}_0$ we have

$$\begin{aligned} PD_n(PXPYPZQ)Q &= \sum_{k+l=n} \Big(PD_k(PXPYP)D_l(PZQ)Q \\ &\quad + PD_k(PZQ)D_l(PXPYP)Q \Big) \\ &= \sum_{k+l=n} PD_k(PXPYP)D_l(PZQ)Q. \end{aligned}$$

On the other hand

$$\begin{aligned} PD_n(PXPYPZQ)Q &= \sum_{i+j+l=n} \Big(PD_i(PXP)D_j(PYP)D_l(PZQ)Q \\ &\quad + PD_i(PZQ)D_j(PYP)D_l(PXP)Q \Big) \\ &= \sum_{i+j+l=n} PD_i(PXP)D_j(PYP)D_l(PZQ)Q \\ &= \sum_{k+l=n} \sum_{i+j=k} PD_i(PXP)D_j(PYP)D_l(PZQ)Q. \end{aligned}$$

It follows from the above two equations that

$$\sum_{k+l=n} P\Big(D_k(PXPYP) - \sum_{i+j=k} D_i(PXP)D_j(PYP) \Big) D_l(PZQ)Q = 0 \tag{2.6}$$

for any $X, Y, Z \in \mathcal{A} \ltimes \mathcal{M}$ and $n \in \mathbb{N}_0$. Suppose that

$$X_k = D_k(PXPYP) - \textstyle\sum_{i+j=k} D_i(PXP)D_j(PYP).$$

It follows from (2.6) that

$$\sum_{k+l=n} PX_kPD_l(PZQ)Q = 0. \tag{2.7}$$

We show that $PX_kP = 0$ for all $k = 0, 1, \ldots, n$, as desired.

Trivially $PX_0P = 0$. Letting $n = 1$ in (2.7) we get

$$PX_0PD_1(PZQ)Q + PX_1PD_0(PZQ)Q = 0 \tag{2.8}$$

or equivalently $PX_1PZQ = 0$ for all $Z \in \mathcal{A} \ltimes \mathcal{M}$. Thus by Lemma 3.6 of [8] we get $PX_1P = 0$. Now assume that $PX_kP = 0$ for all $k \leq n-1$, then it follws from (2.7) that $PX_nP = 0$. Similarly we can prove the part (b). This completes the proof. $\square$

Lemma 2.9. *Let $X, Y \in \mathcal{A} \ltimes \mathcal{M}$. Then for each $n \in \mathbb{N}_0$, we have*

(a) $PD_n(PXPYP)Q = \sum_{i+j=n} PD_i(PXP)D_j(PYP)Q$,
(b) $QD_n(PXPYP)P = \sum_{i+j=n} QD_i(PXP)D_j(PYP)P$,
(c) $PD_n(QXQYQ)Q = \sum_{i+j=n} PD_i(QXQ)D_j(QYQ)Q$,
(d) $QD_n(QXQYQ)P = \sum_{i+j=n} QD_i(QXQ)D_j(QYQ)P$.

Proof. Since $QD_n(PXP)Q = 0$, $QD_n(P)Q = 0$ for all $n \in \mathbb{N}_0$ and $PD_n(P)P = 0$ for all $n \in \mathbb{N}$, we get

$$\begin{aligned} 2PD_n(PXPYP)Q &= PD_n(PXPYP.P + P.PXPYP)Q \\ &= \sum_{k+l=n} (PD_k(PXPYP)D_l(P)Q \\ &+ PD_k(P)D_l(PXPYP)Q \\ &= \sum_{k+l=n} (PD_k(PXPYP)PD_l(P)Q) \\ &+ PD_n(PXPYP)Q. \end{aligned}$$

Therefore by Lemma 2.8 (a) we have

$$\begin{aligned} &PD_n(PXPYP)Q \\ &= \sum_{k+l=n} (PD_k(PXPYP)PD_l(P)Q) \\ &= \sum_{k+l=n} \sum_{i+j=k} PD_i(PXP)D_j(PYP)PD_l(P)Q \\ &= \sum_{i+j+l=n} PD_i(PXP)D_j(PYP)D_l(P)Q \\ &= \sum_{i+k=n} PD_i(PXP)\Big(\sum_{j+l=k} D_j(PYP)D_l(P)\Big)Q \\ &= \sum_{i+k=n} PD_i(PXP)\Big(2D_k(PYP) - \sum_{j+l=k} D_j(P)D_l(PYP)\Big)Q \\ &= 2\sum_{i+k=n} PD_i(PXP)D_k(PYP)Q \\ &- \sum_{i+j+l=n} PD_i(PXP)D_j(P)D_l(PYP)Q \\ &= 2\sum_{i+k=n} PD_i(PXP)D_k(PYP)Q - \sum_{i+l=n} PD_i(PXP)PD_l(PYP)Q \end{aligned}$$

$$= 2 \sum_{i+k=n} PD_i(PXP)D_k(PYP)Q - \sum_{i+l=n} PD_i(PXP)D_l(PYP)Q$$
$$= \sum_{i+k=n} PD_i(PXP)D_k(PYP)Q.$$

The other parts can be proved similarly. □

Lemma 2.10. *Let $X, Y \in \mathcal{A} \ltimes \mathcal{M}$. Then for each $n \in \mathbb{N}_0$, we have*

(a) $PD_n(PXQYQ)Q = \sum_{i+j=n} PD_i(PXQ)D_j(QYQ)Q$,
(b) $PD_n(PXPYQ)Q = \sum_{i+j=n} PD_i(PXP)D_j(PYQ)Q$,
(c) $QD_n(QXPYP)P = \sum_{i+j=n} QD_i(QXP)D_j(PYP)P$,
(d) $QD_n(QXQYP)P = \sum_{i+j=n} QD_i(QXQ)D_j(QYP)P$.

Proof. It follows from Lemmas 2.1 and 2.6 that

$$\begin{aligned} &PD_n(PXQYQ)Q \\ &= PD_n((PXQ)(QYQ) + (QYQ)(PXQ))Q \\ &= \sum_{i+j=n} (PD_i(PXQ)D_j(QYQ)Q + PD_i(QYQ)D_j(PXQ)Q) \\ &= \sum_{i+j=n} PD_i(PXQ)D_j(QYQ)Q. \end{aligned}$$

Other parts proved similarly. □

Proof of Theorem 2.2

Proof. For any $X \in \mathcal{A} \ltimes \mathcal{M}$ we have $X = PXP + PXQ + QXP + QXQ$, so by Lemmas 2.6 and 2.7 it follows that

$$\begin{aligned} D_n(X) = &PD_n(PXP)P + PD_n(PXP)Q + QD_n(PXP)P \\ &+ PD_n(PXQ)Q + QD_n(QXP)P + PD_n(QXQ)Q \\ &+ QD_n(QXQ)P + QD_n(QXQ)Q \end{aligned}$$

for all $X \in \mathcal{A} \ltimes \mathcal{M}$.

It is a consequence of Lemmas 2.8, 2.9, 2.10 and the facts

$$\begin{aligned} 0 &= PD_n((PXP)(QYQ) + (QYQ)(PXP))Q \\ &= \sum_{i+j=n} (PD_i(PXP)D_j(QYQ)Q + PD_i(QYQ)D_j(PXP)Q) \\ &= \sum_{i+j=n} PD_i(PXP)D_j(QYQ)Q \end{aligned}$$

and

$$0 = QD_n((QXQ)(PYP) + (PYP)(QXQ))P$$

$$= \sum_{i+j=n} (QD_i(QXQ)D_j(PYP)P + QD_i(PYP)D_j(QXQ)P)$$
$$= \sum_{i+j=n} QD_i(QXQ)D_j(PYP)P$$

that

$$D_n(XY) = \sum_{i+j=n} D_i(X)D_j(Y)$$

for all $X, Y \in \mathcal{A} \ltimes \mathcal{M}$. Therefore D is a higher derivation from $\mathcal{A} \ltimes \mathcal{M}$ into itself. □

Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a 2-torsion free commutative ring $\mathcal{R}$ and $\mathcal{A} \oplus \mathcal{B}$ be the direct sum of $\mathcal{A}$ and $\mathcal{B}$ as $\mathcal{R}$-algebras. Let $\mathcal{M}$ be an $(\mathcal{A} \oplus \mathcal{B})$-bimodule. If $e = (1_{\mathcal{A}}, 0)$, then $e' = (0, 1_{\mathcal{B}})$ and so $P = ((1_{\mathcal{A}}, 0), 0)$ and $Q = ((0, 1_{\mathcal{B}}), 0)$. Then the trivial extension $(\mathcal{A} \oplus \mathcal{B}) \ltimes \mathcal{M}$ satisfies all the requirements in Theorem 2.2. Let $D = \{D_n\}_{n\in\mathbb{N}_0}$ be a Jordan higher derivation on $(\mathcal{A}\oplus\mathcal{B}) \ltimes \mathcal{M}$, then D_1 is a Jordan derivation on it and so $PD_1(QXP)Q = QD_1(PXQ)P = 0$ for all $X \in (\mathcal{A} \oplus \mathcal{B}) \ltimes \mathcal{M}$. Therefore by Theorem 2.2, every Jordan higher derivation from $(\mathcal{A} \oplus \mathcal{B}) \ltimes \mathcal{M}$ into itself is a higher derivation.

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JORDAN HIGHER DERIVATIONS, A NEW APPROACH

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اشتقاق‌های بالاتر ژوردان: یک رویکرد جدید

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فرض کنید $\mathcal{A}$ یک جبر یک‌دار روی حلقه جابجایی ۲-پیچش آزاد $\mathcal{R}$ بوده و $\mathcal{M}$ یک $\mathcal{A}$-دومدول یک‌دار باشد. در این مقاله نشان می‌دهیم که هر اشتقاق بالاتر ژوردان $D=\{D_n\}_{n\in\mathbb{N}_0}$ از توسیع بدیهی $\mathcal{A}\ltimes\mathcal{M}$ به خودش یک اشتقاق بالاتر است، اگر به ازای هر $X\in\mathcal{A}\ltimes\mathcal{M}$، $PD_1(QXP)Q=QD_1(PXQ)P=0$، که در آن $P=(e,0)$ و $Q=(e',0)$ به ازای عناصر خودتوان غیربدیهی $e\in\mathcal{A}$ و $e'=1_{\mathcal{A}}-e$ که در شرایط $e\mathcal{A}e'\mathcal{A}e=\{0\}$، $e'\mathcal{A}e\mathcal{A}e'=\{0\}$، $e(l.ann_{\mathcal{A}}\mathcal{M})e=\{0\}$، $e'(r.ann_{\mathcal{A}}\mathcal{M})e'=\{0\}$ و به ازای هر $m\in\mathcal{M}$، $eme'=m$ صدق می‌کنند، تعریف شده‌اند.

کلمات کلیدی: اشتقاق بالاتر ژوردان، اشتقاق بالاتر، توسیع بدیهی، جبر مثلثی.