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ON THE COMPUTATIONAL COMPLEXITY ASPECTS OF PERFECT ROMAN DOMINATION

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Abstract. A perfect Roman dominating function (PRDF) on a graph G is a function $f : V(G) \to \{0,1,2\}$ such that every vertex u with $f(u) = 0$ is adjacent to exactly one vertex v for which $f(v) = 2$. The weight of a PRDF f is the sum of the weights of the vertices under f. The perfect Roman domination number of G is the minimum weight of a PRDF in G. In this paper we study algorithmic and computational complexity aspects of the minimum perfect Roman domination problem (MPRDP). We first correct the proof of a result published in [Bulletin Iran. Math. Soc. 14(2020), 342–351], and using a similar argument, show that MPRDP is APX-hard for graphs with bounded degree 4. We prove that the decision problem associated to MPRDP is NP-complete for star convex bipartite graphs, and it is solvable in linear time for bounded tree-width graphs. We also show that the perfect domination problem and perfect Roman domination problem are not equivalent in computational complexity aspects. Finally we propose an integer linear programming formulation for MPRDP.

1. Introduction

Graph Theory Notations: For notations and definitions not given here we refer to [15]. We consider simple and finite graphs $G = (V, E)$, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set. The *order* of G, denoted $|V(G)| = n$, is the number of vertices in G and the

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size of G, denoted $|E(G)| = m$, is the number of edges in G. For any two vertices $x, y \in V(G)$, x and y are *adjacent* if the edge $xy \in E(G)$. The *open neighborhood* of a vertex v is the set $N(v) = \{u \in V : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The open neighborhood of a set $D \subseteq V$ is the set $N(D) = \bigcup_{v\in D} N(v)$ and the closed neighborhood of D is the set $N[D] = N(D) \cup D$. The *degree* of a vertex v, denoted by $\deg(v)$ (or $\deg_G(v)$), is the cardinality of its open neighborhood, that is, $\deg_G(v) = |N(v)|$. A vertex with exactly one neighbor is called a *pendant vertex* (or *leaf* if the graph is a tree) and its neighbor is a *support vertex*. A support vertex with two or more leaf neighbors is called a *strong support vertex*. A vertex of degree zero is called an *isolated vertex*. We denote by Δ and δ, respectively, the *maximum degree* and *minimum degree* among the vertices of G. A vertex v of G is called *universal vertex* if $deg_G(v)$ is equal to Δ. An *induced subgraph* is a graph formed from a subset D of vertices of G and all of the edges in G connecting pairs of vertices in that subset, denoted by $\langle D\rangle$. An *independent set* is a set of vertices in which no two vertices are adjacent. A graph G is *bipartite* if $V(G)$ can be partitioned into two independent sets called *partite sets*. A bipartite graph G with partite sets X and Y is called *tree convex* if there exists a tree T with $V(T) = X$ such that, for each y in Y, the neighbors of y induce a subtree in T. When T is a star, G is called *star convex bipartite graph* [10].

Domination Theory Notations: A *dominating set* of a graph G is a subset D of vertices such that every vertex outside D has a neighbor in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality amongst all dominating sets of G. A *perfect dominating set* (PDS) in a graph G is a subset S such that for all vertices $v \in V(G)\setminus S$, $|N[v] \cap S| = 1$. The minimum cardinality of a perfect dominating set in G is called the *perfect domination number* of G and is denoted by $\gamma_p(G)$. A perfect dominating set of G of minimum cardinality is also called a γ_p-set of G. The concept of perfect dominating sets and its variations have received much attention; for example, see [15, 18]. Cockayne et al. [12] introduced the mathematical definition of Roman domination. This concept was subsequently developed very vastly, and to see the latest progress until 2020 we refer to [1, 2, 3, 6, 7, 8, 9]. A function $f : V \longrightarrow \{0, 1, 2\}$ is called a *Roman dominating function* or just an RDF for G if for every vertex $v \in V$ with $f(v) = 0$ there exists a vertex $u \in N(v)$ such that $f(u) = 2$. The *weight* of an RDF f is the sum $f(V) = \sum_{v\in V} f(v)$. The minimum weight of an RDF on G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. For an

RDF f in a graph G, we denote by V_i (or V_i^f to refer to f) the set of all vertices of G with label i under f. Thus an RDF f can be represented by a triple (V_0, V_1, V_2), and we can use the notation $f = (V_0, V_1, V_2)$.

Henning, Klostermeyer, and MacGillivray [17] introduced the concept of perfect Roman domination in graphs. A *perfect Roman dominating function* (PRDF) on a graph G is a function

$$f : V(G) \to \{0, 1, 2\}$$

satisfying the condition that every vertex u with $f(u) = 0$ is adjacent to exactly one vertex v for which $f(v) = 2$. The weight of a perfect Roman dominating function f is the sum of the its weights over the vertices of G and is denoted by $f(V)$. The *perfect Roman domination number* of G, denoted by $\gamma_{pR}(G)$, is the minimum weight of a PRDF of G. The concept of perfect Roman domination was further studied in, for example [4, 16]. Banerjee et al. [4] studied the algorithmic complexity of perfect Roman domination in graphs. They proved that the perfect Roman domination problem is NP-complete for chordal graphs, planar graphs, and bipartite graphs.

APX-hardness Notations: Let *APX* be the class of problems, for which a C-approximation algorithm exist. Let $I\Pi$ denotes the set of all instances of an optimization problem Π, $S\Pi(x)$ denotes the set of solutions of an instance x of problem Π, $m\Pi(x, y)$ denotes the measure of the objective function value for $x \in I\Pi$ and $y \in S\Pi(x)$ and $opt\Pi(x)$ denotes the optimal value of the objective function $x \in I\Pi$. The *L-reduction* is defined as follows. Given two NP optimization problems Π_1 and Π_2 and a polynomial time transformation f from instances of Π_1 to instances of Π_2 one can say that f is an *L-reduction* if there exists positive constants α and β such that for every instance x of Π_1:
1. $opt_{\Pi_2}(f(x)) \leq \alpha . opt_{\Pi_1}(x)$.
2. for every feasible solution y of $f(x)$ with objective value

$$m_{\Pi_2}(f(x), y) = c_2$$

in polynomial time one can find a solution y' of x with $m_{\Pi_1}(x, y') = c_1$ such that $|opt_{\Pi_1}(x) - c_1| \leq \beta . |opt_{\Pi_2}(f(x)) - c_2|$.

Here, $opt_{\Pi_1}(x)$ represents the size of an optimal solution for an instance x of an NP optimization problem Π_1.

Aims of the paper: In this paper we study algorithmic and computational complexity aspects of the minimum perfect Roman domination problem (MPRDP). The organization of the paper is as follows. In Section 2, we first correct the proof of a result published in [Bulletin Iran. Math. Soc. 14(3) 2020, 342–351], and using a similar argument, we show that the MPRDP is APX-hard for graphs with bounded degree 4. In Section 3, we prove that the decision problem associated to MPRDP is NP-complete even when restricted to star convex bipartite graphs. In Section 4, we show that the MPRDP can be solved in linear time for bounded tree-width graphs. In Section 5, we compare the computational complexity of perfect domination and perfect Roman domination, and prove that the perfect domination problem and perfect Roman domination problem are not equivalent in computational complexity aspects. In Section 6, we propose an integer linear programming formulation for MPRDP.

2. APX-hardnessresults

We first correct the proof of a theorem presented in [5] on the on APX-hardness of independent Roman domination.

2.1. **A Correction on an APX-hardness result of independent Roman domination.** An *independent Roman dominating function* ($IRDF$) on a graph G is an RDF f such that the vertices assigned positive values are independent. The weight of an $IRDF$ f is the value $f(V)$. The *independent Roman domination number* of G denoted by $i_R(G)$ is the minimum weight of an $IRDF$ on G. The minimum independent Roman domination problem ($MIRDP$) is to find an IRDF of minimum weigh in the input graph.

In [5] it was shown that $MIRDP$ is Apx-hard for graphs with maximum degree 4. They used an L-reduction from MINIMUM INDEPENDENT DOMINATING SET-3 (MIDS-3) problem (to find a minimum independent dominating set of a graph with maximum degree 3) which has been proved as APX-complete [11]. However the proof given in [5] is not correct. The major mistake is that in the given L-reduction the $IRDF$ g is a feasible solution and not necessarily i_R-function, so the set $D = \{v_i|\ g(v_i) = 2\ or\ g(a_i) = 2\ \}$ is not necessarily an IDS. To see a counterexample, let $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_3v_5, v_4v_5\}$, G' be the graph constructed from G, as described in [5] and g be an $IRDF$ defined by $g(x) = 1$ if $x \in \{v_1, a_2, a_4, a_5, d_1, e_1, f_1\}$, $g(x) = 2$ if $x \in \{v_3, b_1, c_2, c_3, c_4, c_5\}$ and $g(x) = 0$ otherwise, and observe that $\{v_i|g(v_i) = 2\ or\ g(a_i) = 2\}$ is not an IDS.

We fix this part of the proof as follows. We follow the proof given in [5]. Thus $i_R(G') = 3n+i(G)$ and $opt_{MIDP}(G') \leq 13.opt_{MIDS-3}(G)$. Let g be a feasible solution for G' and f be an optimal solution for it. Let $D = \{v_i \in V|\ g(v_i) = 2 \quad or \quad g(v_i) = 1\}$. Clearly D is an independent set. If D is not a dominating set for G, then there exists a vertex v_{j_1} such that v_{j_1} is not dominated by D. Let $D_1 = D \cup \{v_{j_1}\}$. If D_1 is not a dominating set for G, then there exists a vertex v_{j_2} such that v_{j_2} is not dominated by D_1, and let $D_2 = D_1 \cup \{v_{j_2}\}$. Continuing this process, we obtain a set $D_k = D \cup \{v_{j_1}, v_{j_2}, ..., v_{j_k}\}$ that dominates all vertices of G. Note that D_k is an IDS for G. For every vertex $v_i \in V(G)$, if $v_i \in D_k$ then $g(v_i) + g(a_i) + g(b_i) + g(c_i) + g(d_i) + g(e_i) + g(f_i) \geq 4$, while if $v_i \in V - D_k$ then

$$g(v_i) + g(a_i) + g(b_i) + g(c_i) + g(d_i) + g(e_i) + g(f_i) \geq 3.$$

Thus, $g(V') \geq 4|D_k| + 3(n - |D_k|) = 3n + |D_k|$. Consequently, $|D_k| \leq g(V') - 3n$. Since $i_R(G') = 3n + i(G)$, we have

$$\begin{aligned} |D_k| - i(G) &\leq g(V') - 3n - i(G) \\ &= g(V') - i_R(G') \\ &\leq 1 \times (g(V') - i_R(G')). \end{aligned}$$

Thus, $\beta = 1$.

2.2. **APX-hardness of perfect Roman domination.** We now prove the APX-hardness of minimum perfect Roman domination.

Theorem 2.1. *MPRDP is APX-hard for graphs with maximum degree 4.*

Proof. The proof is similar to APX-hardness of independent Roman domination given in [5] (Theorem 22), and so we omit the details. Let $G = (V, E)$ be an arbitrary instance of the MINIMUM PERFECT DOMINATING SET-3 (MPDS-3) problem, where $V = \{v_1, v_2, ..., v_n\}$, and let G' be the graph is constructed in [5]. It can be seen that $\gamma_{pR}(G') = 3n + \gamma_p(G)$ and $opt_{MPRDP}(G') \leq 13.opt_{MIDS-3}(G)$. Let g_0 be a feasible solution for G' and f be an optimal solution for it. Let $D_0 = \{v_i \in V(G) | g_0(v_i) = 1 \text{ or } g_0(v_i) = 2\}$. If D_0 is a PDS for G then $g_0(v_i) + g_0(a_i) + g_0(b_i) + ... + g_0(f_i) \geq 4$ for each $v_i \in D_0$, while $g_0(v_i) + g_0(a_i) + g_0(b_i) + ... + g_0(f_i) \geq 3$ for each $v_i \notin D_0$, and so $g_0(V') \geq 4|D_0| + (n - |D_0|) \times 3 = 3n + |D_0|$. This implies that $|D_0| - \gamma_p(G) \leq g_0(V') - 3n - \gamma_p(G)$. If D_0 is not a PDS for G, then there exists a vertex v_{j_1} such that v_{j_1} is not dominated by D_0. Let $D_1 = D_0 \cup \{v_{j_1}\}$. If D_1 is not a PDS for G, then there exist a vertex v_{j_2} such that v_{j_2} is not dominated by D_1, and let $D_2 = D_1 \cup \{v_{j_2}\}$. Continuing this process, we will reach a set $D_k = D \cup \{v_{j_1}, v_{j_2}, ..., v_{j_k}\}$ that dominates all vertices of G. It is straightforward to see that

$$|D_k| \leq g_0(V') - 3n.$$

Thus $|D_k| - \gamma_p(G) \leq g_0(V') - 3n - \gamma_p(G)$. Consequently, $\beta = 1$, and the proof is completed. □

3. NP-completeness on star convex bipartite graphs

The decision problem associated to the perfect domination defined as follows.
PERFECT ROMAN DOMINATION PROBLEM (PRDP)
INSTANCE: A graph $G = (V, E)$ and a positive integer k.
QUESTION: Does G have a PRDF of weight at most k?

In this section, we show that PRDP is NP-complete for star convex bipartite graphs by giving a polynomial time reduction from a well-known NP-complete problem, Exact-3-Cover (X3C) which is known to be NP-complete in [14]. The Exact-3-Cover problem is defined as follows.
EXACT-3-COVER (X3C)
INSTANCE: A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X.
QUESTION: Is there a sub collection C' of C such that every element of X appears in exactly one member of C'?

Theorem 3.1. *PRDP is NP-complete for star convex bipartite graphs.*

Proof. Given a graph G and a function f, whether f is a PRDF of size at most k can be checked in polynomial time. Thus PRDP is a member of NP. Now we show that PRDP is NP-hard by transforming an instance $\langle X, C\rangle$ of $X3C$, where $X = \{x_1, x_2, ..., x_{3q}\}$ and $C = \{C_1, C_2, ..., C_t\}$, to an instance $\langle G, k\rangle$ of PRDP that was presented in [5].

Let $A = \{v_0\} \cup \{x_i : 1 \leq i \leq 3q\} \cup \{b_i : 1 \leq i \leq t\}$, $B = V \backslash A$. The subgraph induced by A is a star with vertex v_0 as central vertex and the neighbors of each element of B induce a subtree of the star. Therefore G is a star convex bipartite graph and can be constructed from the given instance $\langle X, C\rangle$ of $X3C$ in polynomial time. The constructed graph and the associated star is shown in the Figure 1. Next we show that $X3C$ has a solution if and only if G has a PRDF with weight at most $2t + q + 1$.

Suppose C' is a solution for $X3C$ with $|C'| = q$. Let f be a function defined by $f(v) = 2$ if $v \in C' \cup \{b_i : c_i \notin C'\}$, $f(v) = 1$ if $v \in \{a_i : c_i \in C'\} \cup \{v_0\}$ and $f(v) = 0$ otherwise. It can be easily verified that f is a PRDF of G and $f(V) = 2q+2(t-q)+q+1 = 2t+q+1$. Conversely, suppose that G has a PRDF g with weight $2t + q + 1$. Clearly, each path $a_i - b_i - c_i$ requires a weight of at least 2. The number of such a paths is t, so this makes the weight at least $2t$.

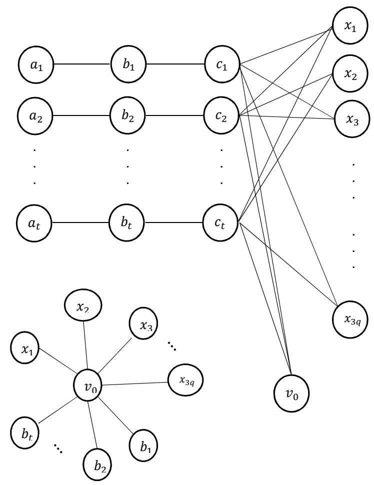

FIGURE 1. An illustration to the construction of star graph from an instance of X3C.

If $q = 1$ then $X = \{x_1, x_2, x_3\}$, $t = 1$ and we have only one subset of three members of X, so $C = \{C_1\}$. In this case if g is a PRDF with weight $2t + q + 1 = 4$ then $g(c_1) = 2$. Note that if $g(c_1) \neq 2$ then $g(V) \geq 6$ and clearly $D = \{C_1\}$ is a solution of $X3C$.

Next we assume that $q > 1$. It is evident that if $|\{c_i|g(c_i) = 2\}| > 1$ then $g(v_0) \neq 0$. Assume that $g(V) = 2t + q + 1$. We show that $|\{c_i|g(c_i) = 2\}| > 1$. Suppose that $|\{c_i|g(c_i) = 2\}| \leq 1$. Thus at most one of the c_i's has weight 2. If $|\{c_i|g(c_i) = 2\}| = 0$, then $g(c_i) \neq 2$ for each i. Since g is a PRDF, each path $a_i - b_i - c_i$ requires a weight of at least 2, and therefore on each sub graph $\langle\{a_i, b_i, c_i, x_{i_1}, x_{i_2}, x_{i_3}, v_0\}\rangle$ we have $g(a_i)+g(b_i)+g(c_i) \geq 2$, $g(x_{i_1}) \geq 1$, $g(x_{i_2}) \geq 1$, $g(x_{i_3}) \geq 1$ and $g(v_0) \geq 1$. Thus we arrive $g(V) \geq 2t+3q+g(v_0) \geq 2t+3q+1 > 2t+q+1$, a contradiction with $g(V) = 2t + q + 1$. Thus assume that

$$|\{c_i|g(c_i) = 2\}| = 1.$$

Without loss of generality, assume that the relevant branch is sub graph $\langle\{a_1, b_1, c_1, x_1, x_2, x_3\}\rangle$. Then the minimum weight of this branch is 3. Note that $g(c_1) = 2$ and $g(c_i) \neq 2$ for $i \neq 1$. Since each path $a_i - b_i - c_i$ requires a weight of at least 2, every x_i which is not joined to c_1 has a weight of at least 1. The number of such x_i's is equal to $3q-3$. Therefore $|\{x_i : g(x_i) \geq 1\}| = 3q-3$. Hence,

$$\begin{aligned} g(V) &\geq 3 + 2(t-1) + (3q-3) + g(v_0) \\ &\geq 2t + 1 + 3q - 3 + g(v_0) \\ &= 2t + 3q - 2 + g(v_0) > 2t + q + 1. \end{aligned}$$

Note that for $q \in N, q > 1$ we have $3q - 2 > q + 1$. This contradicts assumption $g(V) = 2t + q + 1$. Therefore $|\{c_i | g(c_i) = 2\}| > 1$. Now it is evident that $g(v_0) \neq 0$. We show next that $g(x_i) = 0$ for $1 \leq i \leq 3q$. Suppose that there exist m ($m \geq 1$) x_i's such that $g(x_i) \geq 1$. The number of x_i's with $g(x_i) = 0$ is $3q - m$. Since g is a PRDF, each x_i with $g(x_i) = 0$ should have exactly one neighbor c_j with $g(c_j) = 2$. So the number of c_j's required with $g(c_j) = 2$ is $\lceil \frac{3q-m}{3} \rceil$. For such branches $g(a_i) + g(b_i) + g(c_i) \geq 3$ and otherwise $g(a_i) + g(b_i) + g(c_i) \geq 2$. Hence

$$\begin{aligned} g(V) &\geq 3\lceil \frac{3q-m}{3} \rceil + 2(t - \lceil \frac{3q-m}{3} \rceil) + m + g(v_0) \\ &= 3q + 3\lceil \frac{-m}{3} \rceil + 2t - 2q - 2\lceil \frac{-m}{3} \rceil + m + g(v_0) \\ &= 2t + q + m + \lceil \frac{-m}{3} \rceil + g(v_0) \\ &\geq 2t + q + 1 + g(v_0) \\ &> 2t + q + 1. \end{aligned}$$

This is a contradiction. Therefore for each $x_i \in X$, $g(x_i) = 0$. Since each c_i has exactly three neighbors in X, so there exist q number of c_i's with weight 2 such that every element of X is adjacent with exactly one of the c_i's. Consequently, $C' = \{c_i : g(c_i) = 2\}$ is a solution for $X3C$. □

4. MPRDP in bounded tree-width graphs.

In this section we focus on bounded tree-width graphs and show that the perfect Roman domination problem is solvable in linear time for bounded tree-width graphs. To achieve this goal, we need to state some notations and definitions. Let G be a graph, T a tree and v a family of vertex sets $V_t \subseteq V(G)$ indexed by the vertices t of T. The pair (T, v) is called a *tree-decomposition* of G if it satisfies the following three conditions: (i) $V(G) = \bigcup_{t \in V_T} V_t$, (ii) for every edge $e \in E(G)$ there exists a $t \in V(T)$ such that both ends of e lie in V_t, (iii) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever $t_1, t_2, t_3 \in V(T)$ and t_2 is on the path in T from t_1 to t_3. The *width* of (T, v) is the number $max\{|V_t| - 1 : t \in T\}$, and the *tree-width* $tw(G)$ of G is the minimum width of any tree-decomposition of G. By

Courcelle's Theorem, it is well known that every graph problem that can be described by counting monadic second-order logic (CMSOL) can be solved in linear-time in graphs of bounded tree-width, given a tree decomposition as input [13].

Theorem 4.1 (Courcelle's Theorem [13])**.** *Let P be a graph property expressible in CMSOL and k be a constant. Then, for any graph G of tree-width at most k, it can be checked in linear-time whether G has property P.*

We show that the PRDP can be expressed in CMSOL. For this purpose we use the following notations.
1. $adj(p,q)$ is the binary adjacency relation which holds if and only if, p,q are two adjacent vertices of G.
2. $inc(v,e)$ is the binary incidence relation which holds if and only if edge e is incident to vertex v in G.

Theorem 4.2. *Given a graph G and a positive integer k, the PRDP can be expressed in CMSOL.*

Proof. Let $f : V \to \{0,1,2\}$ be a function on a graph G and $V_i = \{v : f(v) = i\}$ for $i = 0,1,2$. The $CMSOL$ formula for the PRDF problem is expressed as follows.

$$\begin{aligned} &f - PRDF \equiv \exists V_0, V_1, V_2, \\ &\forall p(p \in V_1 \vee p \in V_2 \vee (p \in V_0 \wedge \exists! q \in V_2 \wedge adj(p,q))). \end{aligned}$$

$f - PRDF$ guarantees that for every vertex $p \in V$, either $p \in V_1$ or $p \in V_2$ or if $p \in V_0$ then there exist exactly one vertex $q \in V_2$ such that p is adjacent to q. Now, we can express $PRDP$ in $CMSOL$ as follows:

$$PRDP \equiv (f(V) \leq k) \wedge (f - PRDF).$$

□

Now, the following result is immediately obtained from Theorems 4.1 and 4.2.

Theorem 4.3. *PRDP can be solved in linear time for bounded tree-width graphs.*

5. Complexity difference in perfect dominating set problem and perfect Roman dominating problem

Consider the following decision problem associated to perfect domination set.

PERFECT DOMINATION DECISION PROBLEM (PDSP)

INSTANCE : A simple, undirected graph G and a positive integer k.
QUESTION : Does there exist a perfect dominating set of size at most k in G?

In this section, we discuss the difference in computational complexity between the problems PDSP and PRDP. Although problems PDSP and PRDP are types of domination problems, these two problems can differ in complexity. For some instances (for example K_n), both problems may be solved in a maximum of polynomial-time. In certain cases, there are classes of graphs for which the decision version of the problem of domination PRDP can be solved in polynomial-time, while the problem of PDSP for them is in the NP-complete class, and vice versa. Similar study has been made between domination and other domination parameters in [19, 20, 21]. We consider a new class of graphs, namely, GC graphs, in which the MPRDP can be solved trivially, whereas the PDSP is NP-complete. A graph is *GC graph* if it can be constructed from a connected graph $G = (V, E)$, with $V = \{v_1, v_2, ..., v_n\}$ by joining each vertex v_i to the vertex c_i of a graph G_i, where G_i is a graph with

$$V(G_i) = \{a_i, b_i, c_i, d_i, e_i\}$$

and $E(G_i) = \{a_ib_i, b_ic_i, c_id_i, d_ie_i, e_ia_i, b_id_i\}$, for $i = 1, 2, ..., n$. Figure 2 depicts a GC graph.

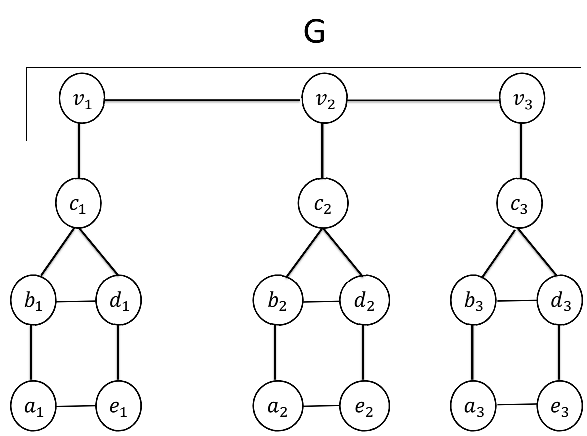

FIGURE 2. An illustration of the GC construction.

Theorem 5.1. *If G is a graph of order n and G' is a GC graph obtained from G, then $\gamma_{pR}(G') = 4n$.*

Proof. Let $G' = (V', E')$ be a GC graph constructed from G, and let $f : V \to \{0, 1, 2\}$ be a function on G', defined by $f(v) = 2$ if $v \in \{c_i : 1 \leq i \leq n\}$, $f(v) = 1$ if $v \in \{a_i, e_i : 1 \leq i \leq n\}$ and $f(v) = 0$ otherwise. Then f is a PRDF and $\gamma_{pR}(G') \leq 4n$. Next, we show that $\gamma_{pR}(G') \geq 4n$. Let g be a PRDF on graph G'. Clearly, $g(a_i) + g(e_i) \geq 2$ and $g(v_i) + g(b_i) + g(c_i) + g(d_i) \geq 2$ for $i = 1, 2, ..., n$, and so $g(v_i) + g(a_i) + g(b_i) + g(c_i) + g(d_i) + g(e_i) \geq 4$ for $i = 1, 2, ..., n$. Since g is arbitrary so $min\,\{g(V')\} \geq 4n$. Therefore $\gamma_{pR}(G') \geq 4n$. Hence $\gamma_{pR}(G') = 4n$. □

Lemma 5.2. *Let G' be a GC graph constructed from a graph $G = (V, E)$. Then $\gamma_p(G') = 2n + \gamma_p(G)$.*

Proof. Let D be a minimum perfect dominating set of G. It is clear that $D' = D \cup \{a_i, b_i | v_i \notin D\} \cup \{a_i, e_i | v_i \in D\}$ is a PDS for G' and $|D'| = |D| + 2n$. Thus $\gamma_p(G') \leq 2n + \gamma_p(G)$. Conversely, let D' be a minimum perfect dominating set of G'. Then at least vertices $\{a_i, b_i\}$ or $\{d_i, e_i\}$ or $\{a_i, e_i\}$ must be included in D'. Therefore $|D'| > 2n$. Observe that if a vertex like $v_j \in V(G)$ is not dominated by vertices D', then $c_j \in D'$; if $c_j \in D'$, then $b_i, d_i \in D'$, since D' is a PDS; and if D' has the minimum size, then $c_j \notin D'$. Let $D = D' \cap V(G)$. Note that D is a PDS for G and $|D'| \geq 3 \times |D| + (n - |D|) \times 2$ so $|D'| \geq 2n + |D| \geq 2n + \gamma_p(G)$. We conclude that

$$\gamma_p(G') = 2n + \gamma_p(G).$$

□

The following result is well known for the Perfect Domination Problem.

Theorem 5.3. *The PERFECT DOMINATION PROBLEM is NP-complete for general graphs.*

From Lemma 5.2 and Theorem 5.3 we have the following.

Theorem 5.4. *The PERFECT DOMINATION DECISION problem is NP-complete for GC graphs.*

6. Integer linear programming formulation for MPRDP

In this section we propose an integer linear programming (ILP) formulation for the MPRDP. Let $G = (V, E)$ be a simple undirected graph, with $|V| = n$, $|E| = m$ and $f : V \to \{0, 1, 2\}$ be a PRDF on G. The MPRDP can be modeled as Integer Linear Programming. This model uses three sets of binary variables. For each vertex $v \in V$, we define

$$a_v = \begin{cases} 1 & f(v) = 0 \\ 0 & otherwise \end{cases} \quad b_v = \begin{cases} 1 & f(v) = 1 \\ 0 & otherwise \end{cases} \quad c_v = \begin{cases} 1 & f(v) = 2 \\ 0 & otherwise. \end{cases}$$

So clearly, $f(V) = \sum_{v \in V(G)} (b_v + 2c_v)$. Therefore the ILP model of the $MPRDP$ can now be formulated as:

$$min(\sum_{v \in V(G)} (b_v + 2c_v)) \tag{6.1}$$

subject to

$$2b_v + c_v + (1 - b_v)(1 - c_v) \sum_{u \in N(v)} 2c_u = 2, \; v \in V(G) \tag{6.2}$$

$$a_v + b_v + c_v = 1, \; v \in V(G) \tag{6.3}$$

$$a_v, b_v, c_v \in \{0, 1\}, \; v \in V(G) \tag{6.4}$$

The objective function 6.1 minimizes the weight of a PRDF. The condition in 6.2, guarantees that every vertex labeled zero, is adjacent to exactly one vertex v for which $f(v) = 2$. The condition in 6.3, guarantees that exactly one label is assigned to every vertex and the condition in 6.4, ensures that the variables are binary in nature. Clearly, the number of variables is $3n$ and the number of constraints is $3n$.

7. Conclusion

In this paper, we have shown that PRDP is NP-complete for star convex bipartite graphs. Next, we have studied algorithmic and computational complexity aspects of the minimum perfect Roman domination problem and it is shown that MPRDP is linear time solvable for bounded tree-width graphs. For approximation point of view, we have shown that MPRDP is APX-hard for graphs with maximum degree 4. Also, by constructing a new class of graphs, we have shown that perfect domination problem and perfect Roman domination problem are not equivalent in computational complexity aspects. Investigating the algorithmic complexity of these problems for other subclasses of bipartite graphs remains open. Finally, we have proposed an ILP formulation for the MPRDP.

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ON THE COMPUTATIONAL COMPLEXITY ASPECTS OF PERFECT ROMAN DOMINATION

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جنبه‌های پیچیدگی محاسباتی احاطه گری رومی کامل

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تابع $f : V(G) \rightarrow \{۰, ۱, ۲\}$ روی گراف G را تابع احاطه‌گر رومی کامل ($PRDF$) نامیم هرگاه هر راس u با $f(u) = ۰$ با دقیقا یک راس مانند v با $f(v) = ۲$ مجاور باشد. عدد $\sum_{v\in V} f(v)$ وزن تابع f می‌باشد. مینیمم وزن از بین وزن تمام توابع احاطه‌گر رومی کامل را عدد احاطه‌گری رومی کامل گراف G می‌نامیم. در این مقاله جنبه‌های پیچیدگی محاسباتی مساله یافتن مینیمم عدد احاطه‌گری رومی کامل ($MPRDP$) را بررسی می‌کنیم. ابتدا اثبات قضیه‌ای از یک مقاله منتشر شده در بولتن انجمن ریاضی ایران را تصحیح کرده و با استفاده از رویکرد مشابه نشان می‌دهیم که مساله $MPRDP$ برای گراف‌های با حداکثر درجه ۴، APX -سخت می‌باشد. همچنین ثابت می‌کنیم مساله تصمیم‌گیری مربوط به آن، حتی وقتی روی گراف‌های دوبخشی محدب ستاره‌ای محدود می‌شویم، به کلاس NP -کامل تعلق دارد. علاوه بر این، ضمن ارائه یک مدل ILP برای مساله $MPRDP$، به مقایسه مسائل احاطه‌گری کامل و احاطه‌گری رومی کامل پرداخته و نشان می‌دهیم این دو مساله از لحاظ پیچیدگی محاسباتی با هم یکسان نمی‌باشند.

کلمات کلیدی: مجموعه احاطه گر، مجموعه احاطه گر کامل، تابع احاطه گر رومی، تابع احاطه گر رومی کامل، APX-سخت.