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INTUITIONISTIC FALLING SHADOWS APPLIED TO COMMUTATIVE IDEALS IN BCK-ALGEBRAS

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Abstract. The notion of commutative falling intuitionistic fuzzy ideal of a BCK-algebra is introduced and related properties are investigated. We verify that every commutative intuitionistic fuzzy ideal is a commutative falling intuitionisticfuzzy ideal, and provide example to show that a commutative falling intuitionistic fuzzy ideal is not a commutative intuitionistic fuzzy ideal. Relations between a falling intuitionistic fuzzy ideal and a commutative falling intuitionistic fuzzy ideal are considered, and a condition for a falling intuitionistic fuzzy ideal to be a commutative falling intuitionistic fuzzy ideal is provided.

1. Introduction

Goodman [4] pointed out the equivalence of a fuzzy set and a class of random sets in the study of a unified treatment of uncertainty modeled by means of combining probability and fuzzy set theory. Wang and Sanchez [18] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. The mathematical structure of the theory of falling shadows is formulated in [17]. Tan et al. [15, 16] established a theoretical approach to define a fuzzy inference relation and fuzzy set operations based on the theory of falling shadows. Commutative ideals in BCK-algebras are studied by Meng [12], and its fuzzification is

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discussed by Jun and Roh [10]. As a generalization of a fuzzy commutative ideal in BCK-algebras, Jun et al. [8] studied its intuitionistic fuzzy version. Based on the theory of falling shadows and fuzzy sets, Jun and Kang [6] introduced a falling fuzzy commutative ideal in a BCK-algebra, and studied related properties. The notion of an intuitionistic random set and an intuitionistic falling shadow are introduced by Jun et al. [11]. Using these notions, they introduced the concept of falling intuitionistic subalgebras and falling intuitionistic ideals in BCI/BCK-algebras, and investigated related properties. They also discussed relations between falling intuitionistic subalgebras and falling intuitionistic ideals, and established a characterization of falling intuitionistic ideal.

As a generalization of the paper [6], in this paper, we introduce the commutative falling intuitionistic fuzzy ideal of a BCK-algebra, and investigate related properties. We show that every commutative intuitionistic fuzzy ideal is a commutative falling intuitionistic fuzzy ideal, and provide example to show that a commutative falling intuitionistic fuzzy ideal is not a commutative intuitionistic fuzzy ideal. We provide relations between a falling intuitionistic fuzzy ideal and a commutative falling intuitionistic fuzzy ideal. We consider a condition for a falling intuitionistic fuzzy ideal to be a commutative falling intuitionistic fuzzy ideal.

2. Preliminaries

In this section, we describe the basic concepts used in this article.

A set X with a special element 0 and a binary operation $*$ is called a *BCI-algebra* if

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) If $x * y = 0$, and $y * x = 0$, then $x = y$,

for all $x, y, z \in X$. A BCI-algebra X with the following identity:

(V) $0 * x = 0$ for all $x \in X$

is said to be a *BCK-algebra.* In any BCI/BCK-algebra X, we have

$$x * 0 = x, \tag{2.1}$$

$$x \leq y \;\Rightarrow\; x * z \leq y * z,\ z * y \leq z * x \tag{2.2}$$

$$(x * y) * z = (x * z) * y, \tag{2.3}$$

$$(x * z) * (y * z) \leq x * y \tag{2.4}$$

for all $x, y, z \in X$, where $x \le y$ if and only if $x * y = 0$. Let S be a nonempty subset of a BCI/BCK-algebra X. Then S is called a *subalgebra* of X if $x*y \in S$ for all $x, y \in S$. A subset J of a BCI/BCK-algebra X is called an *ideal* of X if it satisfies:

$$0 \in J, \tag{2.5}$$

$$y \in J,\ x * y \in J \ \Rightarrow\ x \in J \tag{2.6}$$

for all $x, y \in X$. Let J be a subset of a BCK-algebra X. Then J is called a *commutative ideal* of X if the condition (2.5) is valid and

$$z \in J,\ (x * y) * z \in J \ \Rightarrow\ x * (y * (y * x)) \in J \tag{2.7}$$

for all $x, y, z \in X$. Recall that any commutative ideal is an ideal, and an ideal may not be a commutative ideal (see [13]).

We refer the reader to the books [5, 13] for further information regarding BCI/BCK-algebras.

Let X be a non-empty set. An *intuitionistic fuzzy set* in X (see [1]) is a structure of the form:

$$h := \{\langle x; h_\alpha(x), h_\beta(x)\rangle \mid x \in X, 0 \le h_\alpha(x) + h_\beta(x) \le 1\} \tag{2.8}$$

where $h_\alpha : X \to [0, 1]$ is a membership function and $h_\beta : X \to [0, 1]$ is a nonmembership function. For the sake of simplicity, we shall use the symbol $h = (h_\alpha, h_\beta)$ for the intuitionistic fuzzy set (2.8).

Given an intuitionistic fuzzy set $h = (h_\alpha, h_\beta)$ in a set X and $\alpha, \beta \in [0, 1]$, we consider the following sets:

$$U_\in(h; \alpha) := \{x \in X \mid h_\alpha(x) \ge \alpha\},$$
$$L_\in(h; \beta) := \{x \in X \mid h_\beta(x) \le \beta\}.$$

We say $U_\in(h; \alpha)$ and $L_\in(h; \beta)$ are *intuitionistic fuzzy $\in$-subsets.*

An intuitionistic fuzzy set $h = (h_\alpha, h_\beta)$ in a BCI/BCK-algebra X is called an *intuitionistic fuzzy subalgebra* of X (see [7]) if it satisfies:

$$(\forall x, y \in X) \begin{pmatrix} h_\alpha(x * y) \ge \min\{h_\alpha(x), h_\alpha(y)\} \\ h_\beta(x * y) \le \max\{h_\beta(x), h_\beta(y)\} \end{pmatrix}. \tag{2.9}$$

An intuitionistic fuzzy set $h = (h_\alpha, h_\beta)$ in a BCI/BCK-algebra X is called an *intuitionistic fuzzy ideal* of X (see [7]) if it satisfies:

$$(\forall x \in X) \left(h_\alpha(0) \ge h_\alpha(x), h_\beta(0) \le h_\beta(x) \right). \tag{2.10}$$

$$(\forall x, y \in X) \begin{pmatrix} h_\alpha(x) \ge \min\{h_\alpha(x * y), h_\alpha(y)\} \\ h_\beta(x) \le \max\{h_\beta(x * y), h_\beta(y)\} \end{pmatrix}. \tag{2.11}$$

An intuitionistic fuzzy set $h = (h_\alpha, h_\beta)$ in a BCK-algebra X is called a *commutative intuitionistic fuzzy ideal* of X (see [8]) if it satisfies (2.10) and

$$(\forall x, y \in X) \left(\begin{array}{l} h_\alpha(x * (y * (y * x))) \geq \min\{h_\alpha((x * y) * z), h_\alpha(z)\} \\ h_\beta(x * (y * (y * x))) \leq \max\{h_\beta((x * y) * z), h_\beta(z)\} \end{array} \right). \tag{2.12}$$

Let X be a BCI/BCK-algebra. For each $x \in X$ and $D \in 2^X$, let

$$\bar{x} := \{C \in 2^X \mid x \in C\}, \tag{2.13}$$

and

$$\bar{D} := \{\bar{x} \mid x \in D\}. \tag{2.14}$$

Consider an ordered pair $(2^X, \mathcal{B})$. If $\mathcal{B}$ is a σ-field in 2^X and $\bar{X} \subseteq \mathcal{B}$, we say that $(2^X, \mathcal{B})$ is a *hyper-measurable structure* on X.

Let $(\mho, \mathcal{A}, P)$ be a probability space and let $(2^X, \mathcal{B})$ be a hyper-measurable structure on X. We say that a couple $\xi := (\xi_\alpha, \xi_\beta)$ is an *intuitionistic random set* on X, where ξ_α and ξ_β are mappings from $\mho$ to 2^X which are $\mathcal{A}$-$\mathcal{B}$ measurables, that is,

$$(\forall C \in \mathcal{B}) \left(\begin{array}{l} \xi_\alpha^{-1}(C) = \{\varepsilon_\alpha \in \mho \mid \xi_\alpha(\varepsilon_\alpha) \in C\} \in \mathcal{A} \\ \xi_\beta^{-1}(C) = \{\varepsilon_\beta \in \mho \mid \xi_\beta(\varepsilon_\beta) \in C\} \in \mathcal{A} \end{array} \right). \tag{2.15}$$

Given an intuitionistic random set $\xi := (\xi_\alpha, \xi_\beta)$ on X, consider functions:

$$\tilde{H}_\alpha : X \to [0,1],\ x_\alpha \mapsto P(\varepsilon_\alpha \mid x_\alpha \in \xi_\alpha(\varepsilon_\alpha)),$$
$$\tilde{H}_\beta : X \to [0,1],\ x_\beta \mapsto 1 - P(\varepsilon_\beta \mid x_\beta \in \xi_\beta(\varepsilon_\beta)).$$

Then $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ is an intuitionistic fuzzy set on X, and we call it the *intuitionistic falling shadow* of the intuitionistic random set $\xi := (\xi_\alpha, \xi_\beta)$, and $\xi := (\xi_\alpha, \xi_\beta)$ is called an *intuitionistic cloud* of $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$.

For example, consider a probability space $(\mho, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$ with a Borel field $\mathcal{A}$ on $[0,1]$ and the usual Lebesgue measure m. Let $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ be an intuitionistic fuzzy set in X. Then a couple $\xi := (\xi_\alpha, \xi_\beta)$ in which

$$\xi_\alpha : [0,1] \to 2^X, \alpha \mapsto U_\in(\tilde{H}; \alpha),$$
$$\xi_\beta : [0,1] \to 2^X, \beta \mapsto L_\in(\tilde{H}; \beta)$$

is an intuitionistic random set and $\xi := (\xi_\alpha, \xi_\beta)$ is an intuitionistic cloud of $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$. We will call $\xi := (\xi_\alpha, \xi_\beta)$ defined above as the *intuitionistic cut-cloud* of $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$.

For a probability space $(\Omega, \mathcal{A}, P)$, let $\xi := (\xi_\alpha, \xi_\beta)$ be an intuitionistic random set on a BCI/BCK-algebra X. Then the intuitionistic falling shadow $\tilde{F} := (\tilde{F}_\alpha, \tilde{F}_\beta)$ of $\xi := (\xi_\alpha, \xi_\beta)$ is called a *falling intuitionistic subalgebra* (resp., *falling intuitionistic ideal*) of X if $\xi_\alpha(\varepsilon_\alpha)$ and $\xi_\beta(\varepsilon_\beta)$ are subalgebras (resp., ideals) of X for all ε_α, $\varepsilon_\beta \in \Omega$ (see [9, 11]).

3. Commutative falling intuitionistic fuzzy ideals

In this section, we introduce the notion of a commutative falling intuitionistic fuzzy ideal of a BCK-algebra and related properties are investigated. Also, we provide some examples to show the relation between a falling intuitionistic fuzzy ideal and a commutative falling intuitionistic fuzzy ideal.

In what follows, let X be a BCK-algebra unless otherwise specified.

Definition 3.1. Given a probability space $(\Omega, \mathcal{A}, P)$ and an intuitionistic fuzzy random set $\xi := (\xi_\alpha, \xi_\beta)$ on X, the intuitionistic fuzzy falling shadow $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ of $\xi := (\xi_\alpha, \xi_\beta)$ is called a *commutative falling intuitionistic fuzzy ideal* of X if $\xi_\alpha(\varepsilon_\alpha)$ and $\xi_\beta(\varepsilon_\beta)$ are commutative ideals of X for all ε_α, $\varepsilon_\beta \in \Omega$.

Example 3.2. Consider a set $X = \{0, 1, 2, 3, 4\}$ and define the binary operation $*$ by Table 1.

Table 1. Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

Then $(X; *, 0)$ is a BCK-algebra (see [13]). Consider

$$(\Omega, \mathcal{A}, P) = ([0, 1], \mathcal{A}, m)$$

and let $\xi := (\xi_\alpha, \xi_\beta)$ be an intuitionistic fuzzy random set on X which is given as follows:

$$\xi_\alpha : [0, 1] \to \mathcal{P}(X),\ x \mapsto \begin{cases} \{0, 3\} & \text{if } x \in [0, 0.25), \\ \{0, 4\} & \text{if } x \in [0.25, 0.55), \\ \{0, 1, 2\} & \text{if } x \in [0.55, 0.85), \\ \{0, 3, 4\} & \text{if } x \in [0.85, 1], \end{cases}$$

and

$$\xi_\beta : [0,1] \to \mathcal{P}(X),\ x \mapsto \begin{cases} \{0\} & \text{if } x \in (0.9, 1], \\ \{0,3\} & \text{if } x \in (0.7, 0.9], \\ \{0,4\} & \text{if } x \in (0.5, 0.7], \\ \{0,1,2,3\} & \text{if } x \in (0.3, 0.5], \\ X & \text{if } x \in [0, 0.3]. \end{cases}$$

Then $\xi_\alpha(t)$ and $\xi_\beta(t)$ are commutative ideals of X for all $t \in [0,1]$. Hence the intuitionistic fuzzy falling shadow $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ of $\xi := (\xi_\alpha, \xi_\beta)$ is a commutative falling intuitionistic fuzzy ideal of X, and it is given as follows:

$$\tilde{H}_\alpha(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.3 & \text{if } x \in \{1,2\}, \\ 0.4 & \text{if } x = 3, \\ 0.45 & \text{if } x = 4, \end{cases}$$

and

$$\tilde{H}_\beta(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.5 & \text{if } x \in \{1,2,4\}, \\ 0.3 & \text{if } x = 3. \end{cases}$$

For a probability space $(\Omega, \mathcal{A}, P)$, let $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ be an intuitionistic fuzzy falling shadow of an intuitionistic fuzzy random set $\xi := (\xi_\alpha, \xi_\beta)$. For $x \in X$, let

$$\Omega(x;\xi_\alpha) := \{\varepsilon_\alpha \in \Omega \mid x \in \xi_\alpha(\varepsilon_\alpha)\},$$
$$\Omega(x;\xi_\beta) := \{\varepsilon_\beta \in \Omega \mid x \in \xi_\beta(\varepsilon_\beta)\}.$$

Then $\Omega(x;\xi_\alpha), \Omega(x;\xi_\beta) \in \mathcal{A}$ (see [11]).

Proposition 3.3. *Let $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ be an intuitionistic fuzzy falling shadow of the intuitionistic fuzzy random set $\xi := (\xi_\alpha, \xi_\beta)$ on X. If $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ is a commutative falling intuitionistic fuzzy ideal of X, then*

$$\begin{pmatrix} \Omega((x*y)*z;\xi_\alpha) \cap \Omega(z;\xi_\alpha) \subseteq \Omega(x*(y*(y*x));\xi_\alpha) \\ \Omega((x*y)*z;\xi_\beta) \cap \Omega(z;\xi_\beta) \subseteq \Omega(x*(y*(y*x));\xi_\beta) \end{pmatrix}, \tag{3.1}$$

$$\begin{pmatrix} \Omega(x*(y*(y*x));\xi_\alpha) \subseteq \Omega((x*y)*z;\xi_\alpha) \\ \Omega(x*(y*(y*x));\xi_\beta) \subseteq \Omega((x*y)*z;\xi_\beta) \end{pmatrix}. \tag{3.2}$$

for all $x, y, z \in X$.

Proof. Let $\varepsilon_\alpha \in \Omega((x*y)*z;\xi_\alpha) \cap \Omega(z;\xi_\alpha)$ and

$$\varepsilon_\beta \in \Omega((x*y)*z;\xi_\beta) \cap \Omega(z;\xi_\beta)$$

for all $x, y, z \in X$. Then

$$(x * y) * z \in \xi_\alpha(\varepsilon_\alpha) \text{ and } z \in \xi_\alpha(\varepsilon_\alpha),$$
$$(x * y) * z \in \xi_\beta(\varepsilon_\beta) \text{ and } z \in \xi_\beta(\varepsilon_\beta).$$

Since $\xi_\alpha(\varepsilon_\alpha)$ and $\xi_\beta(\varepsilon_\beta)$ are commutative ideals of X, the condition (2.7) implies that

$$x * (y * (y * x)) \in \xi_\alpha(\varepsilon_\alpha) \cap \xi_\beta(\varepsilon_\beta)$$

and so that $\varepsilon_\alpha \in \Omega(x * (y * (y * x)); \xi_\alpha)$ and $\varepsilon_\beta \in \Omega(x * (y * (y * x)); \xi_\beta)$. Hence (3.1) is true. Now let $\varepsilon_\alpha \in \Omega(x * (y * (y * x)); \xi_\alpha)$ and

$$\varepsilon_\beta \in \Omega(x * (y * (y * x)); \xi_\beta)$$

for all $x, y, z \in X$. Then $x * (y * (y * x)) \in \xi_\alpha(\varepsilon_\alpha) \cap \xi_\beta(\varepsilon_\beta)$. Note that

$$\begin{aligned} ((x * y) * z) * (x * (y * (y * x))) &= ((x * y) * (x * (y * (y * x)))) * z \\ &\leq ((y * (y * x)) * y) * z \\ &= ((y * y) * (y * x)) * z \\ &= (0 * (y * x)) * z \\ &= 0 * z \\ &= 0, \end{aligned}$$

which yields

$$((x * y) * z) * (x * (y * (y * x))) = 0 \in \xi_\alpha(\varepsilon_\alpha) \cap \xi_\beta(\varepsilon_\beta).$$

Since $\xi_\alpha(\varepsilon_\alpha)$ and $\xi_\beta(\varepsilon_\beta)$ are commutative ideals and hence ideals of X, it follows that $(x * y) * z \in \xi_\alpha(\varepsilon_\alpha) \cap \xi_\beta(\varepsilon_\beta)$. Hence $\varepsilon_\alpha \in \Omega((x * y) * z; \xi_\alpha)$ and $\varepsilon_\beta \in \Omega((x * y) * z; \xi_\beta)$. Therefore (3.2) is valid. □

Given a probability space $(\Omega, \mathcal{A}, P)$, let

$$\mathcal{F}(X) := \{f \mid f : \Omega \to X \text{ is a mapping}\}. \tag{3.3}$$

Define a binary operation $\circledast$ on $\mathcal{F}(X)$ as follows:

$$(\forall \omega \in \Omega)\,((f \circledast g)(\omega) = f(\omega) * g(\omega)) \tag{3.4}$$

for all $f, g \in \mathcal{F}(X)$. Then $(\mathcal{F}(X); \circledast, \theta)$ is a BCI/BCK-algebra (see [9]) where θ is given as follows:

$$\theta : \Omega \to X,\ \omega \mapsto 0.$$

Given a subset A of X and $g_\alpha, g_I, g_\beta \in \mathcal{F}(X)$, consider the followings:

$$A^g_\alpha := \{\varepsilon_\alpha \in \Omega \mid g_\alpha(\varepsilon_\alpha) \in A\},$$
$$A^g_\beta := \{\varepsilon_\beta \in \Omega \mid g_\beta(\varepsilon_\beta) \in A\}$$

and

$$\xi_\alpha : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \varepsilon_\alpha \mapsto \{g_\alpha \in \mathcal{F}(X) \mid g_\alpha(\varepsilon_\alpha) \in A\},$$
$$\xi_\beta : \Omega \to \mathcal{P}(\mathcal{F}(X)),\ \varepsilon_\beta \mapsto \{g_\beta \in \mathcal{F}(X) \mid g_\beta(\varepsilon_\beta) \in A\}.$$

Then A^g_α, $A^g_\beta \in \mathcal{A}$ (see [11]).

Theorem 3.4. *If K is a commutative ideal of X, then*

$$\xi_\alpha(\varepsilon_\alpha) = \{g_\alpha \in \mathcal{F}(X) \mid g_\alpha(\varepsilon_\alpha) \in K\},$$
$$\xi_\beta(\varepsilon_\beta) = \{g_\beta \in \mathcal{F}(X) \mid g_\beta(\varepsilon_\beta) \in K\}$$

are commutative ideals of $\mathcal{F}(X)$.

Proof. Assume that K is a commutative ideal of X. Since

$$\theta(\varepsilon_\alpha) = 0 \in K$$

and $\theta(\varepsilon_\beta) = 0 \in K$ for all $\varepsilon_\alpha, \varepsilon_\beta \in \Omega$, we have $\theta \in \xi_\alpha(\varepsilon_\alpha)$ and $\theta \in \xi_\beta(\varepsilon_\beta)$. Let f_α, g_α, $h_\alpha \in \mathcal{F}(X)$ be such that $(f_\alpha \circledast g_\alpha) \circledast h_\alpha \in \xi_\alpha(\varepsilon_\alpha)$ and $h_\alpha \in \xi_\alpha(\varepsilon_\alpha)$. Then

$$(f_\alpha(\varepsilon_\alpha) * g_\alpha(\varepsilon_\alpha)) * h_\alpha(\varepsilon_\alpha) = ((f_\alpha \circledast g_\alpha) \circledast h_\alpha)(\varepsilon_\alpha) \in K$$

and $h_\alpha(\varepsilon_\alpha) \in K$. Since K is a commutative ideal of X, it follows from (2.7) that

$$(f_\alpha \circledast (g_\alpha \circledast (g_\alpha \circledast f_\alpha)))(\varepsilon_\alpha) = f_\alpha(\varepsilon_\alpha) * (g_\alpha(\varepsilon_\alpha) * (g_\alpha(\varepsilon_\alpha) * f_\alpha(\varepsilon_\alpha))) \in K,$$

that is, $f_\alpha \circledast (g_\alpha \circledast (g_\alpha \circledast f_\alpha)) \in \xi_\alpha(\varepsilon_\alpha)$. Hence $\xi_\alpha(\varepsilon_\alpha)$ is a commutative ideal of $\mathcal{F}(X)$. Now, let f_β, g_β, $h_\beta \in \mathcal{F}(X)$ be such that

$$(f_\beta \circledast g_\beta) \circledast h_\beta \in \xi_\beta(\varepsilon_\beta)$$

and $h_\beta \in \xi_\beta(\varepsilon_\beta)$. Then

$$(f_\beta(\varepsilon_\beta) * g_\beta(\varepsilon_\beta)) * h_\beta(\varepsilon_\beta) = ((f_\beta \circledast g_\beta) \circledast h_\beta)(\varepsilon_\beta) \in K$$

and $h_\beta(\varepsilon_\beta) \in K$. Then

$$(f_\beta \circledast (g_\beta \circledast (g_\beta \circledast f_\beta)))(\varepsilon_\beta) = f_\beta(\varepsilon_\beta) * (g_\beta(\varepsilon_\beta) * (g_\beta(\varepsilon_\beta) * f_\beta(\varepsilon_\beta))) \in K,$$

and so $f_\beta \circledast (g_\beta \circledast (g_\beta \circledast f_\beta)) \in \xi_\beta(\varepsilon_\beta)$. Hence $\xi_\beta(\varepsilon_\beta)$ is a commutative ideal of $\mathcal{F}(X)$. This completes the proof. □

Theorem 3.5. *If we consider a probability space*

$$(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m),$$

then every commutative intuitionistic fuzzy ideal is a commutative falling intuitionistic fuzzy ideal.

Proof. Let $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ be a commutative intuitionistic fuzzy ideal of X. Then $U_\in(\tilde{H};\alpha)$, $I_\in(\tilde{H};\beta)$ and $L_\in(\tilde{H};\beta)$ are commutative ideals of X for all $\alpha \in (0,1]$ and $\beta \in [0,1)$. Hence a pair $\xi := (\xi_\alpha, \xi_\beta)$ in which

$$\xi_\alpha : [0,1] \to \mathcal{P}(X),\ \alpha \mapsto U_\in(\tilde{H};\alpha),$$
$$\xi_\beta : [0,1] \to \mathcal{P}(X),\ \beta \mapsto L_\in(\tilde{H};\beta)$$

is an intuitionistic fuzzy cut-cloud of $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$, and so $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ is a commutative falling intuitionistic fuzzy ideal of X. □

The converse of Theorem 3.5 is not true as shown by the next example.

Example 3.6. Consider a set $X = \{0,1,2,3,4\}$ and define the binary operation $*$ on X by Table 2.

Table 2. Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

Then $(X;*,0)$ is a BCK-algebra (see [13]). Consider

$$(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$$

and let $\xi := (\xi_\alpha, \xi_\beta)$ be an intuitionistic fuzzy random set on X and it is given as follows:

$$\xi_\alpha : [0,1] \to \mathcal{P}(X),\ x \mapsto \begin{cases} \{0,1\} & \text{if } x \in [0, 0.2), \\ \{0,2\} & \text{if } x \in [0.2, 0.55), \\ \{0,2,4\} & \text{if } x \in [0.55, 0.75), \\ \{0,1,2,3\} & \text{if } x \in [0.75, 1], \end{cases}$$

and

$$\xi_\beta : [0,1] \to \mathcal{P}(X),\ x \mapsto \begin{cases} \{0\} & \text{if } x \in (0.87, 1], \\ \{0,2\} & \text{if } x \in (0.76, 0.87], \\ \{0,4\} & \text{if } x \in (0.58, 0.76], \\ \{0,2,4\} & \text{if } x \in (0.33, 0.58], \\ X & \text{if } x \in [0, 0.33]. \end{cases}$$

Then $\xi_\alpha(t)$ and $\xi_\beta(t)$ are commutative ideals of X for all $t \in [0,1]$. Hence the intuitionistic fuzzy falling shadow $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ of $\xi := (\xi_\alpha, \xi_\beta)$ is a commutative falling intuitionistic fuzzy ideal of X, and it is given as follows:

$$\tilde{H}_\alpha(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.45 & \text{if } x = 1, \\ 0.8 & \text{if } x = 2, \\ 0.25 & \text{if } x = 3, \\ 0.2 & \text{if } x = 4, \end{cases}$$

and

$$\tilde{H}_\beta(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.67 & \text{if } x \in \{1,3\}, \\ 0.31 & \text{if } x = 2, \\ 0.24 & \text{if } x = 4. \end{cases}$$

But $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ is not a commutative intuitionistic fuzzy ideal of X since $(3 * 4) * 2 \in U_\in(\tilde{H}; 0.4)$ and $2 \in U_\in(\tilde{H}; 0.6)$, but

$$3 * (4 * (4 * 3)) = 3 \notin U_\in(\tilde{H}; 0.4).$$

We provide relations between a falling intuitionistic fuzzy ideal and a commutative falling intuitionistic fuzzy ideal.

Theorem 3.7. *For a probability space $(\Omega, \mathcal{A}, P)$, let $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ be an intuitionistic fuzzy falling shadow of an intuitionistic fuzzy random set $\xi := (\xi_\alpha, \xi_\beta)$ on X. If $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ is a commutative falling intuitionistic fuzzy ideal of X, then it is a falling intuitionistic fuzzy ideal of X.*

Proof. Let $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ be a commutative falling intuitionistic fuzzy ideal of X. Then $\xi_\alpha(\varepsilon_\alpha)$ and $\xi_\beta(\varepsilon_\beta)$ are commutative ideals of X for all ε_α, $\varepsilon_\beta \in \Omega$. Thus $\xi_\alpha(\varepsilon_\alpha)$ and $\xi_\beta(\varepsilon_\beta)$ are ideals of X for all ε_α, $\varepsilon_\beta \in \Omega$. Therefore $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ is a falling intuitionistic fuzzy ideal of X. □

In the next example, we know that the converse of Theorem 3.7 is not true in general.

Example 3.8. Consider a set $X = \{0, 1, 2, 3, 4\}$ and define the binary operation $*$ on X by Table 3.
Then $(X; *, 0)$ is a BCK-algebra (see [13]). Consider

$$(\Omega, \mathcal{A}, P) = ([0,1], \mathcal{A}, m)$$

Table 3. Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	1	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

and let $\xi := (\xi_\alpha, \xi_\beta)$ be an intuitionistic fuzzy random set on X that is given by

$$\xi_\alpha : [0,1] \to \mathcal{P}(X),\ x \mapsto \begin{cases} \{0,3\} & \text{if } x \in [0, 0.27), \\ \{0,1,2,3\} & \text{if } x \in [0.27, 0.66), \\ \{0,1,2,4\} & \text{if } x \in [0.67, 1], \end{cases}$$

and

$$\xi_\beta : [0,1] \to \mathcal{P}(X),\ x \mapsto \begin{cases} \{0\} & \text{if } x \in (0.84, 1], \\ \{0,3\} & \text{if } x \in (0.76, 0.84], \\ \{0,1,2,4\} & \text{if } x \in (0.58, 0.76], \\ X & \text{if } x \in [0, 0.58]. \end{cases}$$

Then $\xi_\alpha(t)$ and $\xi_\beta(t)$ are ideals of X for all $t \in [0,1]$. Hence the intuitionistic fuzzy falling shadow $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ of $\xi := (\xi_\alpha, \xi_\beta)$ is a falling intuitionistic fuzzy ideal of X. But it is not a commutative falling intuitionistic fuzzy ideal of X because if $\alpha \in [0, 0.27)$ and $\beta \in (0.76, 0.84]$, then $\xi_\alpha(\alpha) = \{0,3\}$ and $\xi_\beta(\beta) = \{0,3\}$ are not commutative ideals of X, respectively.

Since every ideal is commutative in a commutative BCK-algebra, we have the following theorem.

Theorem 3.9. *For a probability space $(\Omega, \mathcal{A}, P)$, let $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ be an intuitionistic fuzzy falling shadow of an intuitionistic fuzzy random set $\xi := (\xi_\alpha, \xi_\beta)$ on a commutative BCK-algebra. If $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ is a falling intuitionistic fuzzy ideal of X, then it is a commutative falling intuitionistic fuzzy ideal of X.*

Corollary 3.10. *Let* $(\Omega, \mathcal{A}, P)$ *be a probability space. For any* BCK*-algebra* X *which satisfies one of the following assertions*

$$(\forall x, y \in X)(x \leq y \;\Rightarrow\; x \leq y * (y * x)), \tag{3.5}$$

$$(\forall x, y \in X)(x \leq y \;\Rightarrow\; x = y * (y * x)), \tag{3.6}$$

$$(\forall x, y \in X)(x * (x * y) = y * (y * (x * (x * y)))), \tag{3.7}$$

$$(\forall x, y, z \in X)(x, y \leq z, z * y \leq z * x \;\Rightarrow\; x \leq y), \tag{3.8}$$

$$(\forall x, y, z \in X)(x \leq z, z * y \leq z * x \;\Rightarrow\; x \leq y), \tag{3.9}$$

let $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ *be an intuitionistic fuzzy falling shadow of an intuitionistic fuzzy random set* $\xi := (\xi_\alpha, \xi_\beta)$ *on* X*. If* $\tilde{H} := (\tilde{H}_\alpha, \tilde{H}_\beta)$ *is a falling intuitionistic fuzzy ideal of* X*, then it is a commutative falling intuitionistic fuzzy ideal of* X*.*

4. Conclusion

The notion of commutative falling intuitionistic fuzzy ideal of a BCK-algebra is introduced and is verified that every commutative intuitionistic fuzzy ideal is a commutative falling intuitionistic fuzzy ideal and by example is showed that the converse may not be true. Then the relations between different kinds of falling intuitionistic fuzzy ideals are investigated, and is provided a condition to make a falling intuitionistic fuzzy ideal to be a commutative one.

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References

1. K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, **20** (1986), 87–96.
2. R. A. Borzooei and M. Aaly Kologani, Minimal prime ideals in hoops, *Journal of Algebraic Hyperstructures and Logical Algebras*, **2**(3) (2021), 109–120.
3. R. A. Borzooei, X. Zhang, F. Smarandache and Y. B. Jun, Commutative generalized neutrosophic ideals in BCK-algebras, *Symmetry*, **10**(8), (2018), Article ID: 350.
4. I. R. Goodman, Fuzzy sets as equivalence classes of random sets, in "*Recent Developments in Fuzzy Sets and Possibility Theory*"(R. Yager, Ed.), Pergamon, New York 1982, pp. 327–343.
5. Y. Huang, *BCI-Algebra,* Science Press, Beijing, 2006.
6. Y. B. Jun and M. S. Kang, Falling fuzzy commutative ideals of BCK-algebras, *J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.*, **18**(1) (2011), 41–50.

7. Y. B. Jun and K. H. Kim, Intuitionistic fuzzy ideals of BCK-algebras, *Internat. J. Math. Math. Sci.*, **24**(12) (2000), 839–849.
8. Y. B. Jun, D. S. Lee and C. H. Park, Intuitionistic fuzzy commutative ideals of BCK-algebras, *J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.*, **15**(1) (2008), 73–84.
9. Y. B. Jun and C. H. Park, Falling shadows applied to subalgebras and ideals of BCK/BCI-algebras, *Honam Math. J.*, **34**(2) (2012), 135–144.
10. Y. B. Jun and E. H. Roh, Fuzzy commutative ideals of BCK-algebras, *Fuzzy Sets and Systems*, **64** (1994), 401–405.
11. Y. B. Jun, S. Z. Song and K. J. Lee, Intuitionistic falling shadow theory with application in BCK/BCI-algebras, *Mathematics*, **6**(8) (2018), Article ID: 138.
12. J. Meng, Commutative ideals in BCK-algebras, *Pure Appl. Math. (in Chinese)*, **9** (1991), 49–53.
13. J. Meng and Y. B. Jun, BCK-algebras, Kyungmoonsa Co. Seoul, Korea 1994.
14. S. Z. Song, Implicative soju ideals of BCK-algebras, *Journal of Algebraic Hyperstructures and Logical Algebras*, **2**(3) (2021), 29–40.
15. S. K. Tan, P. Z. Wang and E. S. Lee, Fuzzy set operations based on the theory of falling shadows, *J. Math. Anal. Appl.*, **174** (1993), 242–255.
16. S. K. Tan, P. Z. Wang and X. Z. Zhang, Fuzzy inference relation based on the theory of falling shadows, *Fuzzy Sets and Systems*, **53** (1993), 179–188.
17. P. Z. Wang, *Fuzzy Sets and Falling Shadows of Random Sets,* Beijing Normal Univ. Press (In Chinese), People's Republic of China, 1985.
18. P. Z. Wang and E. Sanchez, Treating a fuzzy subset as a projectable random set, in: "*Fuzzy Information and Decision*" (M. M. Gupta, E. Sanchez, Eds.), Pergamon, New York, 1982, pp. 212–219.

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INTUITIONISTIC FALLING SHADOWS APPLIED TO COMMUTATIVE IDEALS IN BCK-ALGEBRAS

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به‌کاربردن سایه‌های سقوط شهودی برای ایده‌آل‌های جابه‌جایی در BCK-جبرها

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در این مقاله ابتدا مفهوم ایده‌آل فازی شهودی سقوط جابه‌جایی یک BCK-جبر معرفی شده و سپس ویژگی‌های مرتبط با آن‌ها را بررسی می‌کنیم. نشان می‌دهیم که هر ایده‌آل فازی شهودی جابه‌جایی، یک ایده‌آل فازی شهودی در حال سقوط است، و مثالی ارائه می‌کنیم تا نشان دهیم که یک ایده‌آل فازی شهودی در حال سقوط، یک ایده‌آل فازی شهودی جابه‌جایی نیست. روابط بین یک ایده‌آل فازی شهودی در حال سقوط و یک ایده‌آل فازی شهودی در حال سقوط جابه‌جایی بررسی می‌شوند، و شرطی برای اینکه یک ایده‌آل فازی شهودی در حال سقوط یک ایده‌آل فازی شهودی در حال سقوط جابه‌جایی باشد، ارائه می‌شود.

کلمات کلیدی: فازی شهودی، ایده‌آل فازی شهودی، ایده‌آل فازی شهودی جابه‌جایی.