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ON DETERMINING THE DISTANCE SPECTRUM OF A CLASS OF DISTANCE INTEGRAL GRAPHS

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Abstract. The distance eigenvalues of a connected graph G are the eigenvalues of its distance matrix $D(G)$. A graph is called distance integral if all of its distance eigenvalues are integers. Let n and k be integers with $n > 2k, k \geq 1$. The bipartite Kneser graph $H(n,k)$ is the graph with the set of all k and $n-k$ subsets of the set $[n] = \{1, 2, ..., n\}$ as vertices, in which two vertices are adjacent if and only if one of them is a subset of the other. In this paper, we determine the distance spectrum of $H(n,1)$. Although the obtained result is not new [12], but our proof is new. The main tool that we use in our work is the orbit partition method in algebraic graph theory for finding the eigenvalues of graphs. We introduce a new method for determining the distance spectrum of $H(n,1)$ and show how a quotient matrix can contain all distance eigenvalues of a graph.

1. Introduction and Preliminaries

In this paper, a graph $G = (V, E)$ is considered as an undirected simple graph where $V = V(G) = \{v_1, \dots, v_n\}$ is the vertex-set and $E = E(G) = \{e_1, \dots, e_m\}$ is the edge-set. For all the terminology and notation not defined here, we follow [1, 2, 3].
The *distance* between the vertices v_i and v_j, denoted by $d(v_i, v_j)$, is defined as the length of a shortest path between v_i and v_j. The *distance matrix* of G, denoted by $D(G)$, is the $n \times n$ matrix whose (i, j)-entry

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is equal to $d(v_i, v_j)$ for $1 \leq i, j \leq n$. The characteristic polynomial of $D(G)$ is defined $P_D(t) = P_{D(G)}(t) = Det(tI - D(G))$, where I is the $n \times n$ identity matrix. It is called the *distance characteristic polynomial* of G. Since $D(G)$ is a real symmetric matrix, all its eigenvalues, called *distance eigenvalues* of G, are real. The spectrum of $D(G) = D$ is denoted by $Spec(D) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and indexed such that $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$, is called the *distance spectrum* of G. If the eigenvalues of D are ordered by $\lambda_1 > \lambda_2 > \cdots > \lambda_r$, and their multiplicities are $m_1, m_2, \ldots, m_r$, respectively, then we write,

$$Spec(D) = \binom{\lambda_1, \lambda_2, \ldots, \lambda_r}{m_1, m_2, \ldots, m_r} \text{ or } Spec(D) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_r^{m_r}\}.$$

From matrix theory, since D is an irreducible, non-negative, real and symmetric matrix [1, 2, 3], it follows that λ_1 is a simple eigenvalue and satisfies $\lambda_1 \geq |\lambda_i|$, for $i = 2, 3, \ldots, n$, and there exists a positive eigenvector corresponding to λ_1. The largest eigenvalue λ_1 is called the *distance spectral radius* or *distance index* of the graph G.

Let n be a positive integer and $[n] = \{1, 2, \ldots, n\}$. Let V be the set of all k-subsets and $(n-k)$-subsets of $[n]$. The *bipartite Kneser* graph $H(n, k)$ has V as its vertex set, and two vertices A and B are adjacent if and only if $A \subset B$ or $B \subset A$. If $n = 2k$, then $H(n, k)$ is a null graph hence, we assume that $n \geq 2k + 1$. It follows from the definition of $H(n, k)$ that it has $2\binom{n}{k}$ vertices and the degree of each of its vertices is $\binom{n-k}{k} = \binom{n-k}{n-2k}$. Thus, $H(n, k)$ is a regular graph. It is clear that $H(n, k)$ is a bipartite graph. In fact, if $V_k = \{v \in V | |v| = k\}$ and $V_{n-k} = \{v \in V \mid |v| = n - k\}$, then $\{V_k, V_{n-k}\}$ is a partition of V and every edge of $H(n, k)$ has a vertex in V_k and a vertex in V_{n-k} and $|V_k| = |V_{n-k}|$. Also, it is easy to show that the graph $H(n, k)$ is a connected graph. Some of the symmetry properties of the graph $H(n, k)$ have been already determined in [5, 7, 8, 9]. In particular, the following facts can be found in [5, 7].

Proposition 1.1. *$H(n, k)$ is a vertex-transitive graph.*

Proposition 1.2. *$H(n, k)$ is a symmetric (or arc-transitive) graph.*

Proposition 1.3. *The connectivity of the bipartite Kneser graph $H(n, k)$ is maximum, namely, $\binom{n-k}{k}$.*

Proposition 1.4. *The bipartite keneser graph $H(n, 1)$ is a Cayley graph.*

Proposition 1.5. *The bipartite keneser graph $H(n, 1)$ is a distance-transitive graph.*

Proposition 1.6. *Let $H(n,k)$ be a bipartite kneser graph. Then*
$$Aut(H(n,k)) \cong Sym([n]) \times \mathbb{Z}_2,$$
where $\mathbb{Z}_2$ is the cyclic group of order 2.

In this paper, we are interested in the distance spectrum of $H(n,1)$. This graph is also known as the crown graph in literature. Although the distance spectrum of $H(n,1)$ has been already found [12], here we re-determine it by a new method which is completely different from what is appeared in [12]. The method that we use is the orbit partition method in algebraic graph theory. This method has been introduced in [3] for finding the adjacency eigenvalues of graphs. It is a powerful tool in spectral graph theory. Recently, using this method, the eigenvalues of the graph $L(B(n,1))$ [6] and the distance eigenvalues of the graph $B(n,k)$ [4] have been determined, where the graph $B(n,k)$ is a graph with the vertex-set $V = \{v \mid v \subset [n], |v| \in \{k, k+1\}\}$ and the edge-set $E = \{\{v,w\} \mid v,w \in V, v \subset w \text{ or } w \subset v\}$. Also, the distance eigenvalues of the line graph of the crown graph and a class of design graphs have been determined in [10, 11] by the orbit partition method.

Here we give a brief description of the orbit partition method. Let $A = (a_{ij})_{n\times n}$ be a real matrix. We recall that every eigenvector f with the eigenvalue θ for A is a real function such that
$$\textstyle\sum_{j=1}^{n} a_{ij} f(j) = \theta f(i),$$
for each $i \in [n]$. Let $G = (V,E)$ be a graph on n vertices and $D(G) = D$, be its distance matrix. Let $Aut(G)$ be the automorphism group of the graph G. Let $H \leq Aut(G)$ and
$$\pi = \{w_1^H = C_1, \ldots, w_m^H = C_m\}$$
be the orbit partition of H, where $\{w_1, \ldots, w_m\} \subset V$. Let
$$Q = Q_\pi = (q_{ij})_{m\times m}$$
be the matrix in which the rows and columns are indexed by π such that,
$$q_{ij} = \sum_{w\in C_j} d(v,w),$$
where v is a fixed element in the cell C_i. It is easy to check that this sum is independent of v, that is, if $u \in C_i$, then
$$q_{ij} = \textstyle\sum_{w\in C_j} d(v,w) = \sum_{w\in C_j} d(u,w).$$
Hence, the matrix Q is well defined. We call the matrix Q the *quotient matrix* of D over π. We claim that every eigenvalue of Q is an eigenvalue of the distance matrix D. In fact, we have the following fact.

Theorem 1.7. [10] *Let $G = (V,E)$ be a graph with the distance matrix D. Let π be an orbit partition of V and Q be a quotient matrix of D*

over π. Then, every eigenvalue of Q is an eigenvalue of the distance matrix D.

By Theorem 1.7, we can find some of the distance eigenvalues of the graph G, but we can not determine all the distance eigenvalues, since it is possible that G has a distance eigenvalue θ such that θ is not an eigenvalue of the quotient matrix Q over π. Next theorem provides a sufficient condition for avoiding this situation.

Theorem 1.8. [10] *Let $G = (V, E)$ be a graph with D as a distance matrix for G. Let $f \neq 0$ be an eigenvector with the eigenvalue θ for D. Let H be a subgroup of $Aut(G)$ and π be its orbit partition on V and Q be a quotient matrix of D over π. If θ is not an eigenvalue of Q, then the sum of the values of f on each cell of π is zero.*

In the next theorem, we give another condition that guarantees the eigenvalue θ to be an eigenvalue of Q.

Theorem 1.9. [10] *Let $G = (V, E)$ be a vertex-transitive graph with the distance matrix D. Let H be a subgroup of $Aut(G)$ with the orbit partition π on V such that π has a singleton cell $\{x\}$. Let $Q = Q_\pi$ be a quotient matrix of D over π. Then the set of distinct eigenvalues of D is equal to the set of distinct eigenvalues of Q.*

In the next section, we will see how these facts enable us to find the distance spectrum of graph $H(n, 1)$.

2. Main results

Let $n \geq 3$ be an integer. We recall that the graph $H(n, 1)$ is a bipartite graph with the vertex set $V = V_1 \cup V_{n-1}$, where V_1 is the set of 1-subsets of $[n]$ and V_{n-1} is the set of $(n-1)$-subsets of $[n]$, in which two vertices X and Y are adjacent if and only if $X \subset Y$ or $Y \subset X$. Now it follows that this graph has $2n$ vertices and the degree of each of its vertices is $n - 1$. It is easy to show that $H(n, 1)$ is a connected vertex-transitive graph [5, 8].

At first, we compute the distance between any two vertices in $H(n, 1)$.

Proposition 2.1. *Let $n \geq 3$ be an integer and $G = H(n, 1)$. Then the distance between any two distinct vertices in this graph is $1, 2$ or 3.*

Proof. We consider three cases below:

(i) Let $A, B \in V_1$. Hence there are $x, y \in [n]$ such that $A = \{x\}$ and $B = \{y\}$. Let $C \subset [n]$ be such that $|C| = n - 1$ and $x, y \in C$. Thus $A \leftrightarrow C$ and $B \leftrightarrow C$, and hence $d(A, B) = 2$.

(ii) Let $A, B \in V_{n-1}$. Thus $|A \cap B| = n - 2$. Let $c \in A \cap B$ and $C = \{c\}$. Hence $A \leftrightarrow C \leftrightarrow B$ and thus $d(A, B) = 2$.

(iii) Now let $A \in V_1$ and $B \in V_{n-1}$ are non adjacent. There is a vertex C in V_1 such that C is adjacent to B. Now by (i) we have $d(A,B)$=3 □

Remark 2.2. Let S_n be the symmetric group on the set $[n]$. For each $\omega \in S_n$, let

$$\hat{\omega} : V(H(n,1)) \longrightarrow V(H(n,1))$$

be the mapping defined by the rule $\hat{\omega}(A) = \{\omega(a)|a \in A\}$ for every $A \in V(H(n,1))$. Let $\hat{S}_n = \{\hat{\omega}|\omega \in S_n\}$. It is easy to check that $\hat{\omega}$ is an automorphism of the graph $H(n,1)$ and $\hat{S}_n$ is a group isomorphic with S_n. Thus we have $\hat{S}_n \leq Aut(H(n,1))$.

We now construct a distance orbit partitions for $H(n,1)$.

Lemma 2.3. *Consider the graph $H(n,1)$ with the vertex-set*

$$V = V_1 \cup V_{n-1}$$

and the distance matrix D. Then $\Delta = \{V_1, V_{n-1}\}$ is a distance orbit partition for $V(H(n,1))$ with the quotient matrix

$$D_\Delta = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix}$$

over Δ, where $\delta_{11} = 2(n-1), \delta_{12} = n+2, \delta_{21} = n+2, \delta_{22} = 2(n-1)$.

Proof. It is easy to check that the sets V_1 and V_{n-1} are orbits of the group $\hat{S}_n$ on the set $V(H(n,1))$. Therefore, Δ is a distance orbit partition for the graph $H(n,1)$. Let $A \in V_1$ and $B \in V_{n-1}$, be fixed vertices of the graph $H(n,1)$. Then from the definition of the matrix D_Δ and Proposition 2.1, we have,

$$\begin{aligned}
\delta_{11} &= \textstyle\sum_{S \in V_1} d(S,A) = 0 + \sum_{S \in V_1 \setminus \{A\}} 2 = 2(n-1), \\
\delta_{12} &= \textstyle\sum_{S \in V_{n-1}} d(S,A) = \sum_{S \in V_{n-1}, A \subseteq S} 1 + \sum_{S \in V_{n-1}, A \nsubseteq S} 3 = (n-1)+3 \\
&= n+2, \\
\delta_{21} &= \textstyle\sum_{S \in V_1} d(S,B) = \sum_{S \subseteq B, S \in V_1} 1 + \sum_{S \nsubseteq B, S \in V_1} 3 = (n-1)+3 = n+2, \\
\delta_{22} &= \textstyle\sum_{S \in V_{n-1}} d(S,B) = 0 + \sum_{S \in V_{n-1} \setminus \{B\}} 2 = 0 + 2(n-1) = 2(n-1).
\end{aligned}$$

Hence, we have

$$D_\Delta = \begin{pmatrix} 2(n-1) & n+2 \\ n+2 & 2(n-1) \end{pmatrix}.$$

□

It is easy to see that each of the functions (vectors) $f_1 = (1,1)^t$ and $f_2 = (1,-1)^t$ are the eigenvectors for the matrix D_Δ with the eigenvalues $3n$ and $n-4$, respectively. Hence, from Theorem 1.7, we have the following result.

Proposition 2.4. *Both of the numbers $3n$ and $n-4$ are distance eigenvalues of $H(n,1)$.*

Remark 2.5. Concerning Proposition 2.4, the eigenvector corresponding to the eigenvalue $3n$ for the distance matrix D of $H(n,1)$ is the function e_1, defined by the rule $e_1(v)=1$, for every

$$v \in V_1 \cup V_{n-1} = V(H(n,1)).$$

Also, the eigenvector corresponding to the eigenvalue $n-4$ for the distance matrix D of $H(n,1)$ is the function e_2, defined by the rule $e_2(v)=1$, for every $v\in V_1$, and $e_2(v)=-1$, for every $v\in V_{n-1}$ [10, Theorem 2.1].

Let $G_1=\{\sigma\in S_n|\sigma(1)=1\}$ be the stabilizer subgroup of $1\in[n]$ in the symmetric group S_n. Thus $\hat{G_1}=\{\hat{\sigma}|\sigma\in G_1\}\leq\hat{S_n}\leq Aut(H(n,1))$. It is easy to see that $V_{1,1}$,$V_{1,n-1}$,$V_{0,1}$ and $V_{0,n-1}$ are orbits of $\hat{G_1}$ on V, where

$$\begin{aligned} V_{1,1} &= \{S\in V_1|1\in S\}=\{\{1\}\},\\ V_{1,n-1} &= \{S\in V_{n-1}|1\in S\},\\ V_{0,1} &= \{S\in V_1|1\notin S\},\\ V_{0,n-1} &= \{S\in V_{n-1}|1\notin S\}. \end{aligned}$$

Lemma 2.6. *The partition $\Delta_1=\{V_{1,1},V_{1,n-1},V_{0,1},V_{0,n-1}\}$ is also a distance orbit partition for the vertex-set of $H(n,1)$ with the quotient matrix,*

$$D_{\Delta_1}=\begin{pmatrix} 0 & n-1 & 2(n-1) & 3\\ 1 & 2(n-2) & n+1 & 2\\ 2 & n+1 & 2(n-2) & 1\\ 3 & 2(n-1) & n-1 & 0 \end{pmatrix}. \qquad (*)$$

Proof. Note that $|V_{1,1}|=1$, $|V_{1,n-1}|=n-1$, $|V_{0,1}|=n-1$ and $|V_{0,n-1}|=1$. Let the distance quotient matrix be the following matrix,

$$D_{\Delta_1}=\begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} & \delta_{14}\\ \delta_{21} & \delta_{22} & \delta_{23} & \delta_{24}\\ \delta_{31} & \delta_{32} & \delta_{33} & \delta_{34}\\ \delta_{41} & \delta_{42} & \delta_{43} & \delta_{44} \end{pmatrix}.$$

Let $A_1\in V_{1,1}$, $A_2\in V_{1,n-1}$, $A_3\in V_{0,1}$ and $A_4\in V_{0,n-1}$ be fixed vertices. Now we have,

$\delta_{11}=\sum_{S\in V_{1,1}} d(S,A_1)=0;$

$\delta_{12} = \sum_{S\in V_{1,n-1}} d(S, A_1) = n-1;$
$\delta_{13} = \sum_{S\in V_{0,1}} d(S, A_1) = 2(n-1);$
$\delta_{14} = \sum_{S\in V_{0,n-1}} d(S, A_1) = 3;$
$\delta_{21} = \sum_{S\in V_{1,1}} d(S, A_2) = 1;$
$\delta_{22} = \sum_{S\in V_{1,n-1}} d(S, A_2) = 0 + \sum_{A_2\neq S\in V_{a,n-1}} 2 = 2(n-2);$
$\delta_{23} = \sum_{S\in V_{0,1}} d(S, A_2) = \sum_{S\in V_{0,1}, S\subseteq A_2} + \sum_{S\in V_{0,1}, S\nsubseteq A_2} 3 = n-2+3$
$= n+1;$
$\delta_{24} = \sum_{S\in V_{0,n-1}} d(S, A_2) = 2;$
$\delta_{31} = \sum_{S\in V_{1,1}} d(S, A_3) = 2;$
$\delta_{32} = \sum_{S\in V_{1,n-1}} d(S, A_3) = \sum_{S\in V_{1,n-1}, A_3\nsubseteq S} 3 + \sum_{S\in V_{1,n-1}, A_3\subseteq S}$
$= 3+n-2 = n+1;$
$\delta_{33} = \sum_{S\in V_{0,1}} d(S, A_3) = 0 + \sum_{A_3\neq S\in V_{0,1}} 2 = 2(n-2);$
$\delta_{34} = \sum_{S\in V_{0,n-1}} d(S, A_3) = 1;$
$\delta_{41} = \sum_{S\in V_{1,1}} d(S, A_4) = 3;$
$\delta_{42} = \sum_{S\in V_{1,n-1}} d(S, A_4) = 2(n-1);$
$\delta_{43} = \sum_{S\in V_{0,1}} d(S, A_4) = n-1;$
$\delta_{44} = \sum_{S\in V_{0,n-1}} d(S, A_4) = 0.$ □

It follows from Theorem 1.7, that every eigenvalue of the matrix D_{Δ_1} is a distance eigenvalue for $H(n,1)$. Hence, in the first step, we must find the eigenvalues of D_{Δ_1}.

Theorem 2.7. *The set of distance eigenvalues of the bipartite Kneser graph $H(n,1)$ is $E_1 = \{3n, n-4, 0, -4\}$.*

Proof. Consider the distance quotient matrix D_{Δ_1} defined in (∗). Let e_1 and e_2 be the functions that are defined in Remark 2.5. Hence e_1 is an eigenvector of graph $H(n,1)$ with the eigenvalue $3n$. Since the sum of the values of the function e_1 on each cell of the partition Δ_1 is not zero, hence by Theorem 1.8, the number $3n$ is an eigenvalue matrix D_{Δ_1}. From a similar argument, it follows that the number $n-4$ is also an eigenvalue of the distance quotient matrix D_{Δ_1}. Let R_i, $1 \leq i \leq 4$, be the i-th row of the matrix D_{Δ_1}. It is easy to check that $R_1 + R_4 = (3, 3n-3, 3n-3, 3)$=$R_2 + R_3$. Thus the matrix D_{Δ_1} is not non-singular. Hence $\lambda_0 = 0$ is an eigenvalue of the matrix D_{Δ_1}. Let λ be an eigenvalue of D_{Δ_1} distinct from each of $0, 3n, n-4$. Note that

$$trace(D_{\Delta_1}){=}4(n-2) = \text{the sum of the eigenvalues of } D_{\Delta_1}.$$

Hence, we have $4(n-2) = 0 + 3n + (n-4) + \lambda$. Now it follows that $\lambda = -4$.

On the other hand, Δ_1 is an orbit partition of the vertex-set of $H(n,1)$ with a singleton cell $V_{1,1} = \{1\}$. Now since $H(n,1)$ is a vertex-transitive graph, it follows from Theorem 1.9, that the set of distance eigenvalues of $H(n,1)$ is $E_1 = \{3n, n-4, 0, -4\}$. □

Theorem 2.8. *Let n be a positive integer. Then for the distance spectrum of $H(n,1)$ we have*

$$Spec(D(H(n,1)) = \{(3n)^1, (n-4)^1, 0^{n-1}, (-4)^{n-1}\}.$$

Proof. All distinct distance eigenvalues of $H(n,1)$ are obtained in Theorem 2.7. Now, we determine the multiplicities of these eigenvalues. Let D be the distance matrix of $H(n,1)$. In the first step, we calculate the diagonal entries of D^2. For $A \in V_1$, let $d^{(2)}(A,A)$ denote the diagonal entry of D^2 corresponding to A. Hence, we have

$$\begin{aligned} d^{(2)}(A,A) &= \sum_{S \in V(H(n,1))} d(A,S)d(S,A) \\ &= \sum_{S \in V(H(n,1))} d(A,S)^2 \\ &= \sum_{S \in V_{1,1}} d(A,S)^2 + \sum_{S \in V_{0,1}} d(A,S)^2 \\ &+ \sum_{S \in V_{1,n-1}} d(A,S)^2 + \sum_{S \in V_{0,n-1}} d(A,S)^2 \\ &= 1 \times 0 + (n-1) \times 4 + (n-1) \times 1 + 1 \times 9 \\ &= 5n + 4, \end{aligned}$$

where $V_{1,1}, V_{0,1}, V_{1,n-1}, V_{0,n-1}$ are the orbits which are defined before Lemma 2.6. Similarly, for $B \in V_{n-1}$, the diagonal entry of D^2 corresponding to B is

$$d^{(2)}(B,B) = 5n + 4.$$

Therefore, the trace of D^2 is

$$\begin{aligned} tr(D^2) &= |V_1| d^{(2)}(A,A) + |V_{n-1}| d^{(2)}(B,B) \\ &= n(5n+4) + n(5n+4) \\ &= 10n^2 + 8n. \end{aligned}$$

Now, assume that the multiplicities of $n-4$ and -4 are m_1 and m_2, respectively. Note that the distance spectral radius $3n$ is simple, i.e., with multiplicity 1. Thus, we have the following equation.

$$(3n)^2 + m_1(n-4)^2 + 16m_2 = tr(D^2) = 10n^2 + 8n. \quad (1)$$

Now, by using the trace of the distance matrix of the graph $H(n,1)$, we have:

$$3n+m_1(n-4)+m_2(-4) = tr(D) = 0 \Rightarrow m_1(n-4)-4m_2 = -3n. \quad (2)$$

By solving the following equations obtained in (1) and (2),

$$\begin{cases} m_1(n-4)^2 + 16m_2 = n^2 + 8n \\ m_1(n-4) - 4m_2 = -3n \end{cases},$$

we have $m_1 = 1$ and $m_2 = n - 1$.
Now, the multiplicity of 0 is

$$|V(H(n,1))| - 1 - m_1 - m_2 = 2n - 1 - 1 - n + 1 = n - 1.$$

□

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روشی برای تعیین طیف فاصله‌ای رده‌ای از گراف‌های فاصله صحیح

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مقادیر ویژه فاصله‌ای یک گراف همبند مانند G، برابر با مقادیر ویژه ماتریس فاصله این گراف یعنی $D(G)$ تعریف می‌شود. گراف G را فاصله صحیح نامیم، هرگاه همه مقادیر ویژه فاصله‌ای آن صحیح باشد. فرض کنید n و k اعداد صحیح و مثبتی باشند به طوری که $n > ۲k$ و $k \geq ۱$. در این صورت گراف کنسر دوبخشی $H(n,k)$ گرافی است که مجموعه رئوس آن، برابر با همه زیر مجموعه‌های k عضوی و $n-k$ عضوی از مجموعه $[n] = \{۱, ۲, \ldots, n\}$ می‌باشد و در این گراف دو رأس مجاورند اگر و فقط اگر یکی از این دو رأس زیرمجموعه دیگری باشد. در این مقاله، طیف فاصله‌ای گراف $H(n, ۱)$ را تعیین کرده‌ایم. اگرچه نتایج به دست آمده جدید نیستند [۱۲]، اما برهان ارائه شده جدید است. ابزار اصلی استفاده شده در این مقاله روش افراز مداری در نظریه جبری گراف است، که برای تعیین مقادیر ویژه گراف به کار می‌رود. نشان داده‌ایم که چگونه ماتریس خارج قسمت می‌تواند شامل همه مقادیر ویژه فاصله‌ای یک گراف باشد.

کلمات کلیدی: ماتریس فاصله، طیف فاصله‌ای، افراز مداری، گراف کنسر.