Journal of Algebraic Systems
Vol. 11, No. 1, (2023), pp 37-43

ACENTRALIZERS OF GROUPS OF ORDER p^3

Z. MOZAFAR AND B. TAERI*

Abstract. Suppose that G is a finite group. The acentralizer $C_G(\alpha)$ of an automorphism α of G, is defined as the subgroup of fixed points of α, that is $C_G(\alpha) = \{g \in G \mid \alpha(g) = g\}$. In this paper we determine the acentralizers of groups of order p^3, where p is a prime number.

1. Introduction

Our notation is standard and taken mainly from [4]. In particular $\mathbb{Z}_n$, D_n and Q_n denote the cyclic group of integers modulo n, the dihedral group of order $2n$ and the dicyclic group of order $4n$, respectively. Let G be a group. The group of automorphisms of G is denoted by $\mathrm{Aut}(G)$. The symbol $G = K \rtimes H$ indicates that G is a split extension (semidirect product) of a normal subgroup K of G by a complement H.
If $\alpha \in \mathrm{Aut}(G)$, then the acentralizer of α in G which is defined as

$$C_G(\alpha) = \{g \in G \mid \alpha(g) = g\}$$

is a subgroup of G. In particular if $\alpha = \tau_a$ is the inner automorphism of G induced by $a \in G$, then $C_G(\tau_a) = C_G(a)$ is the centralizer of a in G. We denote the set of acentralizers of G by $\mathrm{Acent}(G)$, that is

$$\mathrm{Acent}(G) = \{C_G(\alpha) \mid \alpha \in \mathrm{Aut}(G)\}.$$

The group G is called n-acentralizer, if $|\mathrm{Acent}(G)| = n$.
Note that since the acentralizer of identity automorphism is G,

DOI: 10.22044/JAS.2022.11069.1547.
MSC(2010): Primary: 20D45; Secondary: 20D25.
Keywords: Automorphism; Centralizer; Acentralizer; Finite group.
Received: 6 August 2021, Accepted: 24 June 2022.
*Corresponding author.

$G \in \mathrm{Acent}(G)$. It is obvious that G is 1-acentralizer if and only if $|G| \leq 2$. Nasrabadi and Gholamian [3] characterized the n-acentralizer groups, $n \in \{2, 3, 4, 5\}$. Seifizadeh et al. [5] characterized n-acentralizer groups, where $n \in \{6, 7, 8\}$, and obtained a lower bound on the number of acentralizer subgroups for p-groups, where p is a prime number. They showed that if $p \neq 2$, then there is no n-acentralizer p-group, where $n \in \{6, 7\}$. Moreover, if $p = 2$, then there is no 6-acentralizer p-group. In [2] we showed that if G is a finite abelian p-group of rank 2, where p is an odd prime, then the number of acentralizers of G is equal to the number of subgroups of G. Also we obtained acentralizers of infinite two-generator abelian groups.

In this paper we compute the acentraizers of finite groups of order p^3, where p is a prime number.

2. Main results

In this section we find the number of acentralizers of groups order p^3, where p is a prime number. From the fundamental theorem of abelian groups we know that there are three non-isomorphic abelian groups of order p^3, namely $\mathbb{Z}_{p^3}$, $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ and $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. The acentralizers of such groups are determined in [2]. In fact,

$$|\mathrm{Acent}(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)| = p^8 - p^5 + 1,$$

and if p is an odd prime number, then $|\mathrm{Acent}(\mathbb{Z}_{p^3})| = 4$ and

$$|\mathrm{Acent}(\mathbb{Z}_{p^2} \times \mathbb{Z}_p)| = 2p + 4.$$

Also, $|\mathrm{Acent}(\mathbb{Z}_8)| = 3$ and $|\mathrm{Acent}(\mathbb{Z}_4 \times \mathbb{Z}_2)| = 5$.

Thus we must consider non-abelian groups of order p^3. It is well-known that there are exactly two non-isomorphic non-abelian groups of order p^3, where p is an odd prime (see page 178 of [1] or pages 59-64 of [6]):

$$\begin{aligned} G_1 &:= \langle a, b \mid a^{p^2} = b^p = 1,\ ba = a^{p+1}b \rangle \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p \\ G_2 &:= \langle a, b, c \mid a^p = 1,\ b^p = 1,\ c^p = 1,\ ba = ab,\ ca = abc,\ cb = bc \rangle \\ &\cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p. \end{aligned}$$

Also D_4 and Q_2 are non-isomorphic 2-groups of order 2^3. It easy to see that $|\mathrm{Acent}(D_4)| = |\mathrm{Acent}(Q_2)| = 5$ (see for example, Theorem 3.6 of [3]).

First we find the acentralizers of G_1. Elementary calculations show that in G_1, for all integers x, y, n, with $1 \leq x \leq p^2 - 1$ and $1 \leq y \leq p - 1$, we have

$$b^y a^x = a^{x + xyp} b^y, \qquad (a^x b^y)^n = a^{nx + \frac{n(n-1)}{2} xyp} b^{ny}.$$

Lemma 2.1. (see pages 62-64 of [6]) Non-trivial subgroups of G_1 are $\langle a^p, b\rangle$, $\langle b\rangle$, $\langle a^p b^l\rangle$, $\langle ab^l\rangle$, where $0 \le l \le p-1$. In particular G_1 has $2p+4$ subgroups. The subgroups of order p are $\langle b\rangle$, $\langle a^p b^l\rangle$, where $0 \le l \le p-1$. The cyclic subgroups of order p^2 are $\langle ab^l\rangle$, where $0 \le l \le p-1$. Also $\langle a^p, b\rangle$ is the unique elementary abelian subgroup of order p^2.

Theorem 2.2. (see pages 25-29 of [6]) We have $|\mathrm{Aut}(G_1)| = p^3(p-1)$, in fact

$$\mathrm{Aut}(G_1) = \{\varphi_{i,j,m} \mid i \in \mathbb{Z}_{p^2},\ i \not\equiv 0 \pmod p,\ \ j, m \in \mathbb{Z}_p\},$$

where

$$\varphi_{i,j,m}(a^x b^y) = a^{xi+\frac{x(x-1)}{2}ijp+ypm} b^{xj+y} \tag{2.1}$$

for all $0 \le x \le p^2-1$ and $0 \le y \le p-1$.

Lemma 2.3. The identity subgroup is not an acentralizer for any automorphism of G_1.

Proof. Suppose, contrary that, there exists an automorphism $\varphi_{i,j,m}$ of G_1 such that $C_{G_1}(\varphi_{i,j,m}) = \{1\}$. Thus $\varphi_{i,j,m}$ fixes only the identity element. If p divides $i-1$, then since $a^{p^2} = b^p = 1$, from (2.1) we have

$$\varphi_{i,j,m}(a^p) = a^{pi+\frac{p(p-1)}{2}ijp} b^{pj} = a^{pi}\left(a^{p^2}\right)^{\frac{(p-1)ij}{2}} = a^p,$$

which is a contradiction. Now if p does not divide $i-1$, then the equation $(i-1)x \equiv -m \pmod p$ has a solution (see for example Proposition 4 on page 10 of [1]). Thus there exists $0 < k < p-1$ such that $p \mid k(i-1)+m$. But since

$$\varphi_{i,j,m}(a^{kp}b) = a^{kpi+\frac{kp(kp-1)}{2}ijp+mp} b^{kpj+1} = a^{kp}ba^{p(k(i-1)+m)} \ne a^{kp}b,$$

we see that $p \nmid k(i-1)+m$, which is a contradiction. Thus the identity subgroup cannot be an acentralizer. □

Theorem 2.4. Every non-identity subgroup of

$$G = G_1 := \langle a, b \mid a^{p^2} = b^p = 1, ba = a^{p+1}b\rangle$$

is an acentralizer of an automorphism of G. In particular,

$$|\mathrm{Acent}(G)| = 2p+3.$$

Proof. In what follows we use (2.1) frequently without explicit reference. For any non-identity subgroup H of G we find an automorphism $\varphi_{i,j,m}$ of G such that $H = C_G(\varphi_{i,j,m})$. Recall that the integer i must be chosen so that $i \in \mathbb{Z}_{p^2}$ and $i \not\equiv 0 \pmod p$, that is $i = kp + r$, for some integers k and r with $0 \le k \le p-1$ and $1 \le r \le p-1$.

Let $H := \langle a^p, b\rangle$. We put $i = p+1$ and $j = m = 0$. Then

$$\varphi_{i,j,m}(a^p) = a^{p(p+1)} = a^p$$

and $\varphi_{i,j,m}(b) = b$. Thus $H \leq C_G(\varphi_{i,j,m})$. Now since $a^{p+1}b \in G - H$ and $\varphi_{i,j,m}(a^{p+1}b) = a^{(p+1)(p+1)}b \neq a^{p+1}b$, we see that $C_G(\varphi_{i,j,m})$ is a proper subgroup of G. Since $|G| = p^3$ and $|H| = p^2$, it follows that $C_G(\varphi_{i,j,m}) = H$.

Let $H := \langle ab^l \rangle$, where $0 \leq l \leq p-1$. Put $i = lp+1$, $j = 0$ and $m = p-1$. Then $\varphi_{i,j,m}(ab^l) = a^{pl+1+lp(p-1)}b^l = ab^l$ and so $H \leq C_G(\varphi_{i,j,m})$. Since $a^{p+1}b \in G - H$ and

$$\varphi_{i,j,m}(a^{p+1}b) = a^{(p+1)(pl+1)+p(p-1)}b = a^{pl+1}b \neq a^{p+1}b,$$

we see that $C_G(\varphi_{i,j,m})$ is a proper subgroup of G. Since $|G| = p^3$ and $|H| = p^2$, it follows that $C_G(\varphi_{i,j,m}) = H$.

Let $H := \langle b \rangle$. Then $\varphi_{2,0,0}(b) = b$. Since $\varphi_{2,0,0}(a^w b^t) = a^{2w}b^t \neq a^w b^t$ ($0 \leq w, t \leq p-1$, $(w,t) \neq (0,1)$), we have $C_G(\varphi_{i,0,0}) = \langle b \rangle$.

Now let $H := \langle a^p b^l \rangle$, where $0 \leq l \leq p-1$. First suppose that $l = 0$. If we put $i = p+1$ and $j = m = 1$, then have

$$\varphi_{i,j,m}(a^p) = a^{p(p+1)+\frac{p^2(p-1)}{2}(p+1)}b^p = a^p.$$

Thus $H = \langle a^p \rangle \leq C_G(\varphi_{i,j,m})$. We see that $\varphi_{i,j,m}(b) = a^p b \neq b$ and $\varphi_{i,j,m}(ab^t) = a^{(p+1)+tp}b^{1+t} \neq ab^t$, for all $0 \leq t \leq p-1$. Since $b, ab^t \in G - H$, for all $0 \leq t \leq p-1$, it follows that $C_G(\varphi_{i,j,m})$ is not of order p^2 and so $C_G(\varphi_{i,j,m}) = H$. Now suppose that $1 \leq l \leq p-1$. Let $i = 2$, $j = 1$ and $m \neq 0$ such that $lm \equiv -1 \pmod{p}$ (such m exists by Proposition 4 on page 10 of [1]). Then

$$\varphi_{i,j,m}(a^p b^l) = a^{p(2+lm)+(p-1)p^2}b^{l+p} = a^p b^l$$

and therefore $H = \langle a^p b^l \rangle \leq C_G(\varphi_{i,j,m})$. We see that

$$\varphi_{i,j,m}(a^p) = a^{2p+(p-1)p^2}b^p \neq a^p,\ \varphi_{i,j,m}(b) = a^{mp}b \neq b$$

and $\varphi_{i,j,m}(ab^t) = a^{2+tmp}b^{1+t} \neq ab^t$, for all $0 \leq t \leq p-1$. Since $a^p, b, ab^t \in G - H$, for all $0 \leq t \leq p-1$, it follows that

$$C_G(\varphi_{i,j,m}) = \langle a^p b^l \rangle.$$

Thus we showed that every non-identity subgroup is the acentralizer of an automorphism of G, and therefore $|\mathrm{Acent}(G)| = 2p+3$. □

Now we obtain the acentralizers of G_2. Elementary calculations show that, for all integers x, y, z, n, where $1 \leq x, y, z \leq p-1$, we have

$$c^z a^x = a^x b^{xz} c^z, \qquad (a^x c^z)^n = a^{nx} b^{xz\frac{n(n-1)}{2}} c^{nz}.$$

Lemma 2.5. (pages 59-62 of [6]) Non-trivial subgroups of G_2 are $\langle a, b\rangle$, $\langle b, a^u c\rangle$, $\langle b\rangle$, $\langle ab^v\rangle$, $\langle a^u b^v c\rangle$, where $0 \leq u, v \leq p-1$. In particular G_2 has $p^2 + 2p + 4$ subgroups. The subgroups of order p are $\langle b\rangle$, $\langle ab^v\rangle$, $\langle a^u b^v c\rangle$, where $0 \leq u, v \leq p-1$. The elementary abelian subgroups of order p^2 are $\langle a, b\rangle$ and $\langle b, a^u c\rangle$, where $0 \leq u \leq p-1$.

Theorem 2.6. (see pages 32-34 of [6]) We have

$$|\mathrm{Aut}(G_2)| = p^3(p^2-1)(p-1)$$

so that,

$$\begin{aligned}\mathrm{Aut}(G_2) &= \{\varphi_{(i,j,k),m,(q,r,s)} \mid i, j, k, m, q, r, s \in \mathbb{Z}_p,\\ &\quad m \not\equiv 0 \pmod p, \quad m \equiv si - kq \pmod p\}\end{aligned}$$

where, for all $0 \leq x, y, z \leq p-1$,

$$\varphi_{(i,j,k),m,(q,r,s)}(a^x b^y c^z) = a^{ix+qz} b^{xj+ym+zr+kqxz+\frac{x(x-1)}{2}ki+\frac{z(z-1)}{2}sq} c^{kx+sz}. \tag{2.2}$$

Theorem 2.7. Every subgroup of

$$G := G_2 = \langle a, b, c \mid a^p = 1, b^p = 1, c^p = 1, ba = ab, ca = abc, cb = bc\rangle$$

is an acentralizer of an automorphism of G. In particular,

$$|\mathrm{Acent}(G)| = p^2 + 2p + 4.$$

Proof. In what follows we use (2.2) frequently without explicit reference. For any subgroup H of G we find an automorphism $\varphi := \varphi_{(i,j,k),m,(q,r,s)}$ of G such that $H = C_G(\varphi)$.

Let $H := \langle a, b\rangle$. We put $i = m = s = q = 1$, $j = k = r = 0$. Then $\varphi(a) = a$ and $\varphi(b) = b$. Thus $H \leq C_G(\varphi)$. Since $abc \in G - H$ and $\varphi(abc) = a^2bc \neq abc$, it follows that $C_G(\varphi)$ is a proper subgroup of G and so $C_G(\varphi) = H$.

Let $H := \langle b, a^u c\rangle$, where $0 \leq u \leq p-1$. We put $i = j = m = s = 1$, $k = q = 0$ and $r = p - u$. Then we have $\varphi(b) = b$ and

$$\varphi(a^u c) = a^u b^{u+r} c = a^u c.$$

Thus $H \leq C_G(\varphi)$. Since $a \in G - H$ and $\varphi(a) = ab \neq a$, it follows that $C_G(\varphi)$ is a proper subgroup of G and so $C_G(\varphi) = H$.

Let $H := \langle b\rangle$. We put $m = 1$, $k = q = j = r = 0$ and choose $s, i \in \mathbb{Z}_p - \{0, 1\}$ such that $si \equiv 1 \pmod p$. Then we have $\varphi(b) = b$ and so $H \leq C_G(\varphi)$. Since

$$\varphi(a^w b^t) = a^{iw} b^t \neq a^w b^t \ (0 \leq w, t \leq p-1,\ (w, t) \neq (0, 1))$$

and $\varphi(a^w b^t c) = a^{iw} b^t c^s \neq a^w b^t c$ $(0 \leq w, t \leq p-1)$, it follows that H is not of order p^2. Hence $H = C_G(\varphi)$.

Now $H := \langle ab^v \rangle$, where $0 \leq v \leq p-1$. We put $i = 1$, $k = q = r = 0$, $s = m = 2$ and $j = p - v$. Then $\varphi(ab^v) = ab^{p+v} = ab^v$ and so $H \leq C_G(\varphi)$. Since

$$\varphi(a^u b^w) = a^u b^{u(p-v)+2w} \neq a^u b^w$$

$(0 \leq u, w \leq p-1$, $(u, w) \neq (1, v)$, $(u, w) \neq (0, 0))$ and

$$\varphi(a^u b^w c) = a^u b^{u(p-v)+2w} c^2 \neq a^u b^w c \ (0 \leq u, w \leq p-1),$$

it follows that H is not of order p^2. Hence $H = C_G(\varphi)$.

Let $H := \langle b^v c \rangle$, where $0 \leq v \leq p-1$. We put

$$i = m = 2,\ j = k = q = 0,\ s = 1$$

and $r = p - v$. Then $\varphi(b^v c) = b^{p+v} c = b^v c$ and so $H \leq C_G(\varphi)$. Since

$$\varphi(a^u b^w) = a^{2u} b^{2w} \neq a^u b^w \ (0 \leq u, w \leq p-1,\ (u, w) \neq (0, 0))$$

and

$$\varphi(a^u b^w c) = a^{2u} b^{p-v+2w} c \neq a^u b^w c \ (0 \leq u, w \leq p-1,\ (u, w) \neq (0, v)),$$

it follows that $C_G(\varphi)$ is not of order p^2. Hence $H = C_G(\varphi)$.

Let $H := \langle a^u c \rangle$, where $1 \leq u \leq p-1$. We put $i = m = 2$, $k = j = r = 0$, $s = 1$ and $q = p - u$. Then $\varphi(a^u c) = a^{2u+p-u} c = a^u c$ and so $H \leq C_G(\varphi)$. Since

$$\varphi(a^w b^v) = a^{2w} b^{2v} \neq a^w b^v \ (0 \leq w, v \leq p-1,\ (w, v) \neq (0, 0))$$

and

$$\varphi(a^w b^v c) = a^{p-u+2w} b^{2v} c \neq a^w b^v c \ (0 \leq u, w \leq p-1,\ (w, v) \neq (u, 0)),$$

it follows that $C_G(\varphi)$ is not of order p^2. Hence $H = C_G(\varphi)$.

Let $H := \langle a^u b^v c \rangle$, where $1 \leq u, v \leq p-1$. We put $i = m = 2$, $s = 1$, $k = j = 0$, $r = p - v$ and $q = p - u$. Then

$$\varphi(a^u b^v c) = a^{2u+(p-u)} b^{2v+p-v} c = a^u b^v c$$

and therefore $H \leq C_G(\varphi)$. Since

$$\varphi(a^w b^t) = a^{2w} b^{2t} \neq a^w b^t \ (0 \leq w, t \leq p-1,\ (w, t) \neq (0, 0))$$

and $\varphi(a^w b^t c) = a^{p-u+2w} b^{p+2t-v} c \neq a^w b^t c$ $(0 \leq w, t \leq p-1)$, it follows that $C_G(\varphi)$ is not of order p^2. Hence $H = C_G(\varphi)$.

Finally we consider the identity subgroup. We put

$$m = s = i = q = 2,\ k = 1$$

and $r = j = 0$. Then $\varphi(1) = \varphi(a^p) = 1$ and for every $1 \neq x \in G$, $\varphi(x) \neq x$. Thus $C_G(\varphi) = 1$.

Therefore $|\mathrm{Acent}(G)| = p^2 + 2p + 4$. □

Acknowledgments

The authors gratefully thank two anonymous referees for careful reading of the article and their invaluable comments.

References

1. D. Dummit and M. Foote, *Abstract algebra*, 3rd Edition-Wiley, United States of America, 2004.
2. Z. Mozafar and B. Taeri, Acentralizers of abelian groups of rank 2, *Hacet. J. Math. Stat.*, **49** (2020), 273–281.
3. M. M. Nasrabadi and A. Gholamian, On finite n-acentralizers groups, *Comm. Algebra*, **43** (2015), 378–383.
4. D. J. S. Robinson, *A course in the theory of groups*, 2nd ed., New York: Springer-Verlag, 1996.
5. P. Seifizadeh, A. Gholamian, A. A. Farokhniaee and M. M. Nasrabadi, On the autocentralizer subgroups of finite p-groups, *Turkish J. Math.*, **44** (2020), 1802–1812.
6. G. S. Ståhl, J. Laine, and G. Behm, *On p-group of low power order*, Bachelors thesis, KTH Royal Institute of Technology, 2010.

Zahra Mozafar
Department of Mathematical Sciences, Isfahan University of Technology, P. O. Box 84156-83111, Isfahan, Iran.
Email: `z.mozafar@math.iut.ac.ir`

Bijan Taeri
Department of Mathematical Sciences, Isfahan University of Technology, P. O. Box 84156-83111, Isfahan, Iran.
Email: `b.taeri@iut.ac.ir`

Journal of Algebraic Systems

ACENTRALIZERS OF GROUPS OF ORDER p^3

Z. MOZAFAR AND B. TAERI

خودریختی-مرکزسازهای گروه‌های مرتبه p^3

زهرا مظفر[۱] و بیژن طائری[۲]

[۱,۲]دانشکده علوم ریاضی، دانشگاه صنعتی اصفهان، اصفهان، ایران

فرض کنید G یک گروه متناهی باشد. خودریختی-مرکزساز یک خودریختی α از G را برابر زیرگروه متشکل از نقاط ثابت α تعریف می‌کنیم، یعنی $C_G(\alpha) = \{g \in G \mid \alpha(g) = g\}$. در این مقاله خودریختی-مرکزسازهای گروه‌های مرتبه‌ی p^3، که در آن p یک عدد اول است، را محاسبه می‌کنیم.

کلمات کلیدی: خودریختی، مرکزساز، خودریختی-مرکزساز، گروه متناهی.