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INTRINSIC IDEALS OF DISTRIBUTIVE LATTICES

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Abstract. The concepts of intrinsic ideals and inlets are introduced in a distributive lattice. Intrinsic ideals are also characterized with the help of inlets. Certain equivalent conditions are given for an ideal of a distributive lattice to become intrinsic. Some equivalent conditions are derived for the quotient lattice, with respect to a congruence, to become a Boolean algebra. Some topological properties of the prime spectrum of intrinsic ideals of a distributive lattice are derived.

Introduction

Many authors introduced the concept of annihilators in the structures of rings as well as lattices and characterized several algebraic structures in terms of annihilators. T. P. Speed [9] and W. H. Cornish [4] made an extensive study of annihilators in distributive lattices. In [5], some properties of minimal prime ideals of distributive lattices are studied and some properties of dense elements and D-filters are studied in MS-algebras [8]. In [2], the notion of D-filters of pseudo-complemented semilattices was introduced. Later it was generalized by the author in MS-algebras [8]. In [7], the authors investigated certain important properties of prime D-filters of distributive lattices. In this paper, the author studied the properties of prime D-filters of a distributive lattice L, by using the lattice $\mathcal{RF}_\circ(L)$

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of all filters of the form $(a)^\circ$, where

$$(a)^\circ = \{x \in L \mid a \vee x \in D\}$$

In this paper, some properties of minimal prime D-filters of distributive lattices are derived with respect to congruences. Certain topological properties of the prime spectrum of D-filters of distributive lattices.

The main aim of this paper is to introduce the notion of intrinsic ideals and to study certain properties of these ideals with the help of prime ideals of distributive lattices. A sufficient condition is derived for an ideal (resp. prime ideal) of a distributive lattice to become an intrinsic ideal. It is proved that the class of all intrinsic ideals forms a distributive lattice and the intrinsic ideals are characterized. Some equivalent conditions are derived for every ideal of a distributive lattice to become an intrinsic ideal. The concept of inlets of distributive lattices is introduced and then proved that the class of all inlets of a distributive lattice forms a sublattice to the lattice of all intrinsic ideals. A congruence is introduced on a distributive lattice and then a set of equivalent conditions is given for the respective quotient lattice to become a Boolean algebra.

Certain basic properties of prime intrinsic ideals of a distributive lattice are investigated. Some preliminary topological properties of prime intrinsic ideals of a distributive lattice are studied. A set of equivalent conditions is given for the prime spectrum of intrinsic ideals of a distributive lattice to become a Hausdorff space. A necessary and sufficient condition is given for the prime spectrum of intrinsic ideals to become a regular space.

1. Preliminaries

The reader is referred to [1], [6], [7], and [9] for the elementary notions and notations of distributive lattices. However some of the preliminary definitions and results of [1] and [7] are presented for the ready reference of the reader.

Definition 1.1. [1] An algebra $(L, \wedge, \vee)$ of type $(2, 2)$ is called a distributive lattice if for all $x, y, z \in L$, it satisfies the following properties $(1), (2), (3)$ and (4) along with (5) or $(5')$

(1) $x \wedge x = x$, $x \vee x = x$,
(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$,
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$,
(4) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$,
(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

(5′) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

A non-empty subset A of a lattice L is called an ideal (filter) of L if $a \vee b \in A (a \wedge b \in A)$ and $a \wedge x \in A (a \vee x \in A)$ whenever $a, b \in A$ and $x \in L$. The set $\mathcal{I}(L)$ of all ideals of $(L, \vee, \wedge, 0)$ forms a complete distributive lattice as well as the set $\mathcal{F}(L)$ of all filters of $(L, \vee, \wedge, 1)$ forms a complete distributive lattice. A proper ideal (filter) M of a lattice is called *maximal* if there exists no proper ideal (filter) N such that $M \subset N$.

The set $(a] = \{x \in L \mid x \leq a\}$ is called a *principal ideal* generated by a and the set of all principal ideals is a sublattice of $\mathcal{I}(L)$. Dually the set $[a) = \{x \in L \mid a \leq x\}$ is called a *principal filter* generated by a and the set of all principal filters is a sublattice of $\mathcal{F}(L)$. A proper ideal (proper filter) P of a lattice L is called *prime* if for all $a, b \in L$, $a \wedge b \in P$ $(a \vee b \in P)$ then $a \in P$ or $b \in P$. Every maximal ideal (filter) is prime.

For any element a of a distributive lattice L, the *annihilator* [9] of a is defined as the set $(a)^* = \{ x \in L \mid x \wedge a = 0 \}$. An element a of a lattice L is called a *dense element* if $(a)^* = \{0\}$. The set D of all dense elements of a lattice L forms a filter of L. A distributive lattice L with 0 is called *quasi-complemented* [6] if to each $x \in L$ there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' \in D$.

Definition 1.2. [7] A filter F of a lattice L is called a D-filter if $D \subseteq F$.

The set D of all dense elements of a distributive lattice is the smallest D-filter of the lattice. For any subset A of a distributive lattice L, define $A^\circ = \{x \in L \mid a \vee x \in D \text{ for all } a \in A\}$. For any subset A of L, A° is a D-filter of L. For any $a \in L$, we simply represent $(\{a\})^\circ$ by $(a)^\circ$. Then clearly $(1)^\circ = L$. It is also obvious that $(0)^\circ = D$ and $D \subseteq (x)^\circ$ for all $x \in L$.

Proposition 1.3. [7] *Let L be a distributive lattice. For any $a, b, c \in L$,*

(1) $a \leq b$ *implies* $(a)^\circ \subseteq (b)^\circ$,
(2) $(a \wedge b)^\circ = (a)^\circ \cap (b)^\circ$,
(3) $(a \vee b)^{\circ\circ} = (a)^{\circ\circ} \cap (b)^{\circ\circ}$,
(4) $(a)^\circ = L$ *if and only if* $a \in D$.

Throughout this article, all lattices are distributive lattices with 0 and dense elements unless otherwise mentioned.

2. Intrinsic ideals

In this section, the notion of intrinsic ideals is introduced. A characterization theorem of intrinsic ideals is given. It is proved that the class of all intrinsic ideals forms a complete distributive lattice. A set of equivalent conditions is given for an ideal to become intrinsic.

Definition 2.1. For any non-empty subset A of a lattice L, the set $A^{\perp}$ is defined as $A^{\perp} = \{x \in L \mid (x)^{\circ} \subseteq (a)^{\circ} \text{ for some } a \in A\}$.

It can be easily seen that $D^{\perp} = L$. In case of $A = \{a\}$, we simply denote $\{a\}^{\perp}$ by $(a)^{\perp}$ where $(a)^{\perp} = \{x \in L \mid (x)^{\circ} \subseteq (a)^{\circ}\}$. Moreover, $(0)^{\perp} = \{x \in L \mid (x)^{\circ} = D\}$.

Lemma 2.2. *For any two non-empty subsets A and B of a lattice L,*

(1) $A \subseteq A^{\perp}$,
(2) $A \subseteq B$ *implies* $A^{\perp} \subseteq B^{\perp}$,
(3) $A^{\perp\perp} = A^{\perp}$.

Proof. (1) and (2) are clear.

(3) Since $A \subseteq A^{\perp}$, by (2), we get $A^{\perp} \subseteq A^{\perp\perp}$. Conversely, let $x \in A^{\perp\perp}$. Then $(x)^{\circ} \subseteq (a)^{\circ}$ for some $a \in A^{\perp}$. Since $a \in A^{\perp}$, we get $(a)^{\circ} \subseteq (b)^{\circ}$ for some $b \in A$. Hence $(x)^{\circ} \subseteq (a)^{\circ} \subseteq (b)^{\circ}$ and $b \in A$. Thus $x \in A^{\perp}$. Therefore $A^{\perp\perp} \subseteq A^{\perp}$. $\square$

Lemma 2.3. *Let L be a lattice. For any $a, b \in L$, we have the following:*

(1) $(a)^{\perp} = ((a])^{\perp}$,
(2) $(a)^{\perp\perp} = (a)^{\perp}$,
(3) $a \leq b$ *implies* $(a)^{\perp} \subseteq (b)^{\perp}$,
(4) $a \in (b)^{\perp}$ *implies* $(a)^{\perp} \subseteq (b)^{\perp}$,
(5) $(a)^{\perp} \cap (b)^{\perp} = (a \wedge b)^{\perp}$,
(6) $(a)^{\circ} = (b)^{\circ}$ *if and only if* $(a)^{\perp} = (b)^{\perp}$,
(7) $(a)^{\perp} = (b)^{\perp}$ *implies* $(a \wedge c)^{\perp} = (b \wedge c)^{\perp}$ *for any* $c \in L$,
(8) $(a)^{\perp} = (b)^{\perp}$ *implies* $(a \vee c)^{\perp} = (b \vee c)^{\perp}$ *for any* $c \in L$.

Proof. (1) Since $\{a\} \subseteq (a]$, we have $(a)^{\perp} \subseteq ((a])^{\perp}$. Conversely, let $x \in ((a])^{\perp}$. Then $(x)^{\circ} \subseteq (b)^{\circ}$ for some $b \in (a]$. Since $b \in (a]$, we get $(b)^{\circ} \subseteq (a)^{\circ}$. Hence $(x)^{\circ} \subseteq (b)^{\circ} \subseteq (a)^{\circ}$. Thus $x \in (a)^{\perp}$. Therefore $((a])^{\perp} \subseteq (a)^{\perp}$, which gives that $(a)^{\perp} = ((a])^{\perp}$.

(2) It clear by (1) and Lemma 2.2(3).

(3) Suppose $a \leq b$. Then $(a] \subseteq (b]$. Hence

$$(a)^{\perp} = (a]^{\perp} \subseteq (b]^{\perp} = (b)^{\perp}.$$

(4) Let $a \in (b)^{\perp}$. Then $(a] \subseteq (b)^{\perp}$. Therefore, by (2), we get $(a)^{\perp} \subseteq (b)^{\perp\perp} = (b)^{\perp}$.

(5) Let $a, b \in L$. Since $a \wedge b \leq a, b$, by (3), we have

$$(a \wedge b)^{\perp} \subseteq (a)^{\perp} \cap (b)^{\perp}.$$

Conversely, let $x \in (a)^{\perp} \cap (b)^{\perp}$. Then $(x)^{\circ} \subseteq (a)^{\circ}$ and $(x)^{\circ} \subseteq (b)^{\circ}$. Hence $(x)^{\circ} \subseteq (a)^{\circ} \cap (b)^{\circ} = (a \wedge b)^{\circ}$. Thus $x \in (a \wedge b)^{\perp}$. Therefore $(a)^{\perp} \cap (b)^{\perp} \subseteq (a \wedge b)^{\perp}$. Hence $(a)^{\perp} \cap (b)^{\perp} = (a \wedge b)^{\perp}$.

(6) Assume that $(a)^{\circ} = (b)^{\circ}$. Then clearly $(a)^{\perp} = (b)^{\perp}$. Conversely, assume that $(a)^{\perp} = (b)^{\perp}$. Since $a \in (a)^{\perp} = (b)^{\perp}$, we get $(a)^{\circ} \subseteq (b)^{\circ}$. Similarly, we can obtain that $(b)^{\circ} \subseteq (a)^{\circ}$. Therefore $(a)^{\circ} = (b)^{\circ}$.

(7) Assume that $(a)^{\perp} = (b)^{\perp}$. Then

$$(a \wedge c)^{\perp} = (a)^{\perp} \cap (c)^{\perp} = (b)^{\perp} \cap (c)^{\perp} = (b \wedge c)^{\perp}.$$

(8) Assume that $(a)^{\perp} = (b)^{\perp}$. Then by (6), we get $(a)^{\circ} = (b)^{\circ}$. Hence

$$\begin{aligned}(a \vee c)^{\circ\circ} &= (a)^{\circ\circ} \cap (c)^{\circ\circ} \\ &= (b)^{\circ\circ} \cap (c)^{\circ\circ} \\ &= (b \vee c)^{\circ\circ}\end{aligned}$$

Hence $(a \vee c)^{\circ} = (b \vee c)^{\circ}$. By (6), we get $(a \vee c)^{\perp} = (b \vee c)^{\perp}$. □

Lemma 2.4. *For any ideal I of a lattice L, $I^{\perp}$ is an ideal of L such that $I \subseteq I^{\perp}$.*

Proof. Clearly $I \subseteq I^{\perp}$. Let $x, y \in I^{\perp}$. Then $(x)^{\circ} \subseteq (a)^{\circ}$ and $(y)^{\circ} \subseteq (b)^{\circ}$ for some $a, b \in I$. Then

$$(a \vee b)^{\circ\circ} = (a)^{\circ\circ} \cap (b)^{\circ\circ} \subseteq (x)^{\circ\circ} \cap (y)^{\circ\circ} = (x \vee y)^{\circ\circ}.$$

Hence $(x \vee y)^{\circ} \subseteq (a \vee b)^{\circ}$ and $a \vee b \in I$. Therefore $x \vee y \in I^{\perp}$. Again, let $x \in I^{\perp}$ and $y \leq x$. Then $(y)^{\circ} \subseteq (x)^{\circ} \subseteq (a)^{\circ}$ for some $a \in I$. Thus $y \in I^{\perp}$. Therefore $I^{\perp}$ is an ideal of L such that $I \subseteq I^{\perp}$. □

Lemma 2.5. *For any two ideals I and J of a lattice L, we have*

(1) $I^{\perp} \cap J^{\perp} = (I \cap J)^{\perp}$,
(2) $(I \vee J)^{\perp} = (I^{\perp} \vee J^{\perp})^{\perp}$.

Proof. (1) Clearly $(I \cap J)^{\perp} \subseteq I^{\perp} \cap J^{\perp}$. Conversely, let $x \in I^{\perp} \cap J^{\perp}$. Then $(x)^{\circ} \subseteq (a)^{\circ}$ and $(x)^{\circ} \subseteq (b)^{\circ}$ for some $a \in I$ and $b \in J$. Hence $(x)^{\circ} \subseteq (a)^{\circ} \cap (b)^{\circ} = (a \wedge b)^{\circ}$ and $a \wedge b \in I \cap J$. Thus $x \in (I \cap J)^{\perp}$. Therefore $I^{\perp} \cap J^{\perp} \subseteq (I \cap J)^{\perp}$.

(2) Since $I \vee J \subseteq I^{\perp} \vee J^{\perp}$, we get $(I \vee J)^{\perp} \subseteq (I^{\perp} \vee J^{\perp})^{\perp}$. Conversely, let $x \in (I^{\perp} \vee J^{\perp})^{\perp}$. Then $(x)^{\circ} \subseteq (y)^{\circ}$ for some $y \in I^{\perp} \vee J^{\perp}$. Hence $y = a \vee b$ for some $a \in I^{\perp}$ and $b \in J^{\perp}$. Since $a \in I^{\perp}$, there exists $i \in I$ such that $(a)^{\circ} \subseteq (i)^{\circ}$. Since $b \in J^{\perp}$, there exists $j \in J$ such that $(b)^{\circ} \subseteq (j)^{\circ}$. Hence $(i)^{\circ\circ} \subseteq (a)^{\circ\circ}$ and $(j)^{\circ\circ} \subseteq (b)^{\circ\circ}$. Thus

$$(i \vee j)^{\circ\circ} = (i)^{\circ\circ} \cap (j)^{\circ\circ} \subseteq (a)^{\circ\circ} \cap (b)^{\circ\circ} = (a \vee b)^{\circ\circ}.$$

Hence $(x)^{\circ} \subseteq (y)^{\circ} = (a \vee b)^{\circ} \subseteq (i \vee j)^{\circ}$ where $a \vee b \in I \vee J$. Thus $x \in (I \vee J)^{\perp}$. Therefore $(I^{\perp} \vee J^{\perp})^{\perp} \subseteq (I \vee J)^{\perp}$. □

Proposition 2.6. *For any ideal I of a lattice L, the following conditions are equivalent:*

(1) $I^{\perp} = L$;
(2) $I^{\perp} \cap D \neq \emptyset$;
(3) $I \cap D \neq \emptyset$.

Proof. (1) $\Rightarrow$ (2): It is clear.

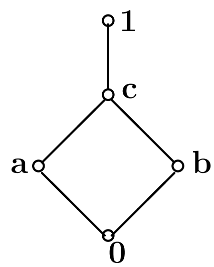
(2) $\Rightarrow$ (3): Assume that $I^{\perp} \cap D \neq \emptyset$. Choose $x \in I^{\perp} \cap D$. Since $x \in D$, by Proposition 1.3(4), we get $(x)^{\circ} = L$. Hence $L = (x)^{\circ} \subseteq (a)^{\circ}$ for some $a \in I$. Thus $(a)^{\circ} = L$, which means that $a \in D$. Hence $a \in I \cap D$. Therefore $I \cap D \neq \emptyset$.

(3) $\Rightarrow$ (1): Assume that $I \cap D \neq \emptyset$. Choose $a \in I \cap D$. For this $a \in I \cap D$, we get $(x)^{\circ} \subseteq (a)^{\circ} = L$ for all $x \in L$. Therefore $I^{\perp} = L$. $\square$

The concept of intrinsic ideals is now introduced in the following.

Definition 2.7. An ideal I of a lattice L is called *intrinsic* if $I = I^{\perp}$.

Example 2.8. Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given in the following figure:



Clearly L contains the dense elements c and 1. Note that $(0)^{\circ} = D; (a)^{\circ} = \{b, c, 1\}; (b)^{\circ} = \{a, c, 1\}$ and $(c)^{\circ} = (1)^{\circ} = L$. Hence $(0)^{\perp} = \{0\}; (a)^{\perp} = \{0, a\}; (b)^{\perp} = \{0, b\}$ and $(c)^{\perp} = (1)^{\perp} = L$. Consider the ideal $I = \{0, a\}$. Clearly $I^{\perp} = \{0, a\} = I$. Therefore I is an intrinsic ideal of L.

For any ideal I of a lattice L with $I \cap D = \emptyset$, $I^{\perp}$ is a proper intrinsic ideal of L because of Proposition 2.6. It can also be seen that the intrinsic ideals are closed elements with respect to the closure operation $^{+}$ defined on ideals of lattices.

Proposition 2.9. *Let L be a lattice. If M is maximal in the class of all ideals which are not meeting D, then M is an intrinsic ideal.*

Proof. Let M be an ideal which is maximal with respect to the property of $M \cap D = \emptyset$. By Proposition 2.6, we get $M^{\perp} \cap D = \emptyset$ and $M^{\perp} \neq L$. Thus $M^{\perp}$ is a proper ideal of L such that $M \subseteq M^{\perp}$. By the maximality of M, we get $M = M^{\perp}$. Therefore M is an intrinsic ideal of L. $\square$

Let us denote the class of all intrinsic ideals of the lattice L by $\mathcal{N}(L)$. Then it is clear that $\mathcal{N}(L)$ need not be a sublattice of the distributive lattice $\mathcal{I}(L)$ of all ideals of L. However, in the following, we derive that $\mathcal{N}(L)$ forms a distributive lattice on its own.

Theorem 2.10. *For any lattice L, the set $\mathcal{N}(L)$ of all intrinsic ideals of L forms a distributive lattice with greatest element L.*

Proof. For any $I, J \in \mathcal{N}(L)$, define the operations $\cap$ and $\sqcup$ on $\mathcal{N}(L)$ as follows:

$$I \cap J = (I \cap J)^{\perp} \text{ and } I \sqcup J = (I^{\perp} \vee J^{\perp})^{\perp} = (I \vee J)^{\perp}$$

Clearly $(I \cap J)^{\perp}$ is the infimum of I and J in $\mathcal{N}(L)$. Also $(I \vee J)^{\perp}$ is an upper bound of I and J. By Lemma 2.5(2), we have

$$(I^{\perp} \vee J^{\perp})^{\perp} = (I \vee J)^{\perp}.$$

Suppose $K \in \mathcal{N}(L)$ such that $I \subseteq K$ and $J \subseteq K$. Let $x \in (I \vee J)^{\perp}$. Then $(x)^{\circ} \subseteq (i \vee j)^{\circ}$ for some $i \in I \subseteq K$ and $j \in J \subseteq K$. Since $i \vee j \in K$, we get $x \in K^{\perp} = K$. Therefore $(I \vee J)^{\perp}$ is the supremum of both I and J in $\mathcal{N}(L)$. Then it can be easily verified that $\langle \mathcal{N}(L), \cap, \sqcup \rangle$ is a distributive lattice. Clearly L is the greatest element of the lattice $\langle \mathcal{N}(L), \cap, \sqcup \rangle$. $\square$

In the following, the class of all intrinsic ideals of a lattice are characterized.

Theorem 2.11. *For any ideal I of a lattice L, the following are equivalent:*

(1) *I is intrinsic;*
(2) *for any $x \in L$, $x \in I$ if and only if $(x)^{\perp} \subseteq I$;*
(3) *for any $x, y \in L$, $(x)^{\circ} = (y)^{\circ}$ and $x \in I$ imply that $y \in I$;*
(4) *for any $x, y \in L$, $(x)^{\perp} = (y)^{\perp}$ and $x \in I$ imply that $y \in I$;*
(5) $F = \bigcup\limits_{x \in I} (x)^{\perp}$.

Proof. (1) $\Rightarrow$ (2): Assume that I is intrinsic. Let $x \in I$ and $a \in (x)^{\perp}$. Then $(a)^{\circ} \subseteq (x)^{\circ}$ and $x \in I$. Therefore $a \in I^{\perp} = I$. Therefore $(x)^{\perp} \subseteq I$. Converse is clear.

(2) $\Rightarrow$ (3): Assume the condition (2). Let $x, y \in L$ be such that $(x)^{\circ} = (y)^{\circ}$. Suppose $x \in I$. Since $(x)^{\circ} = (y)^{\circ}$, we get $(x)^{\perp} = (y)^{\perp}$. Hence $y \in (y)^{\perp} = (x)^{\perp} \subseteq I$.

(3) $\Rightarrow$ (4): By Lemma 2.3(6), it is clear.

(4) $\Rightarrow$ (5): Assume the condition (4). For $x \in I$, we have $(x] \subseteq (x)^{\perp}$ and hence $I = \bigcup_{x \in I}(x] \subseteq \bigcup_{x \in I}(x)^{\perp}$. On the other hand, let $y \in (x)^{\perp}$ for some $x \in I$. Then we get $(y)^{\perp} \subseteq (x)^{\perp}$. Thus

$$(y)^{\perp} = (y)^{\perp} \cap (x)^{\perp} = (x \wedge y)^{\perp}.$$

Since $x \wedge y \in I$, by condition (4), we get $y \in I$. Hence $(x)^{\perp} \subseteq I$. Thus $\bigcup_{x \in I}(x)^{\perp} \subseteq I$. Therefore $I = \bigcup_{x \in I}(x)^{\perp}$.

(5) $\Rightarrow$ (1): Assume the condition (5). Clearly $I \subseteq I^{\perp}$. Conversely, let $x \in I^{\perp}$. Then $(x)^{\circ} \subseteq (a)^{\circ}$ for some $a \in I$. Thus $x \in (a)^{\perp}$ for some $a \in I$. Hence $x \in \bigcup_{a \in I}(a)^{\perp} = I$. Thus $I^{\perp} \subseteq I$. Therefore I is an intrinsic ideal. □

Proposition 2.12. *Let L be a lattice. If P is minimal in the class of prime ideals with $P \cap D = \emptyset$ and is containing a given intrinsic ideal, then P is an intrinsic ideal.*

Proof. Let I be an intrinsic ideal and P a prime ideal of L such that $P \cap D = \emptyset$ and $I \subseteq P$. Suppose P is not an intrinsic ideal. Then there exists elements $x, y \in L$ such that $(x)^{\perp} = (y)^{\perp}$, $x \in P$ and $y \notin P$. Consider $F = (L - P) \vee [x \wedge y)$. Then $I \cap F = \emptyset$. Otherwise choose $a \in I \cap F$. Then $a = r \wedge s$ for some $r \in L - P$ and $s \in [x \wedge y)$. Then

$$\begin{aligned} r \wedge s &= r \wedge (s \vee (x \wedge y)) \\ &= (r \wedge s) \vee (r \wedge x \wedge y). \end{aligned}$$

Hence $r \wedge x \wedge y = (r \wedge s) \wedge (r \wedge x \wedge y) = (r \wedge s) \wedge (x \wedge y) \in I$, because of $r \wedge s \in I$. Since $(x)^{\perp} = (y)^{\perp}$, by Lemma 2.3(6), we get $(r \wedge y)^{\perp} = (r \wedge x \wedge y)^{\perp}$. Since I is a intrinsic ideal and $r \wedge x \wedge y \in I$, we get $r \wedge y \in I \subseteq P$. Hence $r \in P$ or $y \in P$, which is a contradiction. Hence $I \cap F = \emptyset$. Thus there exists a prime ideal Q such that $F \cap Q = \emptyset$ and $I \subseteq Q$. Since $F \cap Q = \emptyset$, we get $Q \subseteq L - F \subseteq P$ because of $L - P \subseteq F$. Since $x \wedge y \in F$, we get $x \wedge y \notin Q$. Hence $x \wedge y \in P$ and $x \wedge y \notin Q$. Thus $I \subseteq Q \subset P$. Since $P \cap D = \emptyset$, we must gave $Q \cap D = \emptyset$. Thus P is not a minimal in the class of all prime ideals with $P \cap D = \emptyset$ and containing I, which is a contradiction. Therefore P is an intrinsic ideal. □

Corollary 2.13. *If $\{0\}$ is an intrinsic ideal of a lattice L, then every minimal prime ideal of L is an intrinsic ideal.*

Proof. Let P be a minimal prime ideal of L. Suppose $x \in P \cap D$ for some $x \in L$. Since P is minimal, there exists $0 \neq y \notin P$ such that $x \wedge y = 0$. Hence $x \notin D$, which is a contradiction. Therefore $P \cap D = \emptyset$. Since $\{0\} \subseteq P$, by Proposition 2.12, P is intrinsic. □

Proposition 2.14. *Following assertions are equivalent in a lattice L:*

(1) *for $a, b \in L$, $(a)^\circ = (b)^\circ$ implies $a = b$;*
(2) *for $a, b \in L$, $(a)^\perp = (b)^\perp$ implies $a = b$;*
(3) *every ideal I with $I \cap D = \emptyset$ is intrinsic;*
(4) *every prime ideal P with $I \cap D = \emptyset$ is intrinsic.*

Proof. (1) $\Rightarrow$ (2): By Lemma 2.3(6), it is clear.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1): Assume that condition (4) holds. Let $a, b \in L - D$ be such that $(a)^\circ = (b)^\circ$. By Lemma 2.3(7), we get $(a)^\perp = (b)^\perp$. Suppose $a \neq b$. Without loss of generality, assume that $(a] \cap [b) = \emptyset$. Since $a \notin D$, we get $(a] \cap D = \emptyset$. Hence $(a] \cap \{[b) \vee D\} = \{(a] \cap [b)\} \vee \{(a] \cap D\} = \emptyset$. Then there exists a prime ideal P such that $(a] \subseteq P$ and $P \cap \{[b) \vee D\} = \emptyset$. Then $P \cap D = \emptyset$. By condition (4), P is intrinsic. Since P is intrinsic and $a \in P$, we get $b \in P$, which is a contradiction. Therefore $a = b$. $\square$

3. Inlets of lattices

In this section, the notion of inlets is introduced in lattices. The notion of weakly quasi-complemented lattices is introduced and then weakly quasi-complemented lattices are characterized with the help of the lattice of inlets of a lattice.

Definition 3.1. Let L be a lattice and $x \in L$. Then the ideal of the form $(x)^\perp$ is called an *inlet* of L.

Since $(0)^\circ = D$, we can observe that $(0)^\perp = \{x \in L \mid (x)^\circ = D\}$. In the following result, we give a set of equivalent conditions for an inlet of a lattice to become proper.

Proposition 3.2. *Let L be a lattice and $a \in L$. Then the following are equivalent:*

(1) *$(a)^\perp = L$;*
(2) *$(a)^\perp \cap D \neq \emptyset$;*
(3) *$a \in D$.*

Proof. Routine verification. $\square$

In the following lemma, some more basic properties of inlets can be observed.

Lemma 3.3. *Let L be a lattice. For any $a, b \in L$, the following properties hold:*

(1) *$a \vee b = 1$ implies $(a)^\perp \vee (b)^\perp = L$,*

(2) *For any* $a \notin D$, $(a)^\circ \cap (a)^\perp = \emptyset$,
(3) $(a \vee b)^\perp = ((a)^\perp \vee (b)^\perp)^\perp$.

Proof. (1) Let $a, b \in L$ be such that $a \vee b = 1$. Then

$$L = (1] = (a \vee b] = (a] \vee (b] \subseteq (a)^\perp \vee (b)^\perp.$$

Therefore $(a)^\perp \vee (b)^\perp = L$.

(2) Let $a \in L$ be such that $a \notin D$. Suppose $x \in (a)^\circ \cap (a)^\perp$. Then, we get $(x)^{\circ\circ} \subseteq (a)^\circ$ and $(x)^\circ \subseteq (a)^\circ$. Hence $a \in (a)^{\circ\circ} \subseteq (x)^{\circ\circ} \subseteq (a)^\circ$. Thus $a = a \vee a \in D$, which is a contradiction. Therefore $(a)^\circ \cap (a)^\perp = \emptyset$.

(3) It is clear by Lemma 2.5(2). □

Obviously each inlet is an intrinsic ideal and hence for any two inlets $(x)^\perp$ and $(y)^\perp$ their supremum in $\mathcal{N}(L)$ is given by

$$(x)^\perp \sqcup (y)^\perp = ((x] \vee (y])^\perp = ((x \vee y])^\perp = (x \vee y)^\perp$$

Also their infimum in $\mathcal{N}(L)$ is $(x)^\perp \cap (y)^\perp = (x \wedge y)^\perp$.

Theorem 3.4. *For any lattice L, the class $\mathcal{N}_+(L)$ of all inlets is a lattice $\langle \mathcal{N}_+(L), \cap, \sqcup \rangle$ and sublattice to the distributive lattice $\langle \mathcal{N}(L), \cap, \sqcup, L \rangle$ of all intrinsic ideals of L. Moreover, $\mathcal{N}_+(L)$ has the same greatest element $L = (d)^\perp; d \in D$ as $\mathcal{N}(L)$ while $\mathcal{N}_+(L)$ has the smallest element $(s)^\perp$ if and only if L has an element s of the form $(s)^\circ = D$.*

Proof. Clearly $(\mathcal{N}_+(L), \cap, \sqcup)$ is a sublattice to the distributive lattice $(\mathcal{N}(L), \cap, \sqcup)$. It is remaining to prove the statement concerning the smallest element of $\mathcal{N}_+(L)$. Suppose $(s)^\perp$ is the smallest element of $\mathcal{N}_+(L)$. Let $x \in (s)^\circ$. Then $x \vee s \in D$. Now, for any $x \in L$

$$(x)^\perp = (x)^\perp \sqcup (s)^\perp = (x \vee s)^\perp = L$$

which gives that $x \in D$. Hence $(s)^\circ \subseteq D$. Therefore $(s)^\circ = D$. Conversely, suppose that L has an element s such that $(s)^\circ = D$. Let $x \in (s)^\perp$. Then $(x)^\circ \subseteq (s)^\circ = D \subseteq (a)^\circ$ for all $a \in L$. Hence $x \in (a)^\perp$ for all $a \in L$. Thus $(s)^\perp \subseteq (a)^\perp$ for all $a \in L$. Hence $(s)^\perp$ is the smallest element of $\mathcal{N}_+(L)$. □

In any lattice L, it is a well known fact that the quotient algebra $L/\theta = \{[x]_\theta \mid x \in L\}$, where $[x]_\theta$ is the congruence class of x with respect to θ, is a distributive lattice with respect to the operations given by

$$[x]_\theta \cap [y]_\theta = [x \wedge y]_\theta \text{ and } [x]_\theta \vee [y]_\theta = [x \vee y]_\theta$$

Proposition 3.5. *Let L be a lattice. Define a relation θ on L by*

$$(x, y) \in \theta \textit{ if and only if } (x)^\perp = (y)^\perp$$

for all $x, y \in L$. Then θ is a congruence on L where $(0)^{\perp}$ is the smallest congruence class and D is the unit congruence class of L/θ. Furthermore, ker θ is an intrinsic ideal of L.

Proof. From (7) and (8) of Lemma 2.3, θ is a congruence on L. Clearly $(0)^{\perp}$ is the smallest congruence class of L/θ. Let $x, y \in D$. By Proposition 3.2, we get $(x)^{\perp} = (y)^{\perp} = L$. Thus $(x, y) \in \theta$. Therefore D is a congruence class of L/θ. Now, let $a \in D$ and $x \in L$. Since D is a filter, we get $a \vee x \in D$. Since D is a congruence class with respect to θ, we get $[x]_{\theta} \vee [a]_{\theta} = [x \vee a]_{\theta} = D$. Therefore D is the unit congruence class of L/θ.

Clearly *ker* θ is an ideal of L. Let $x \in$ *ker* θ. Then $(x)^{\perp} = (0)^{\perp}$. Let $a \in (x)^{\perp}$. Then $(a)^{\perp} \subseteq (x)^{\perp} = (0)^{\perp}$. Since $0 \leq a$, we get $(0)^{\perp} \subseteq (a)^{\perp}$. Hence $(a)^{\perp} = (0)^{\perp}$, which means that $a \in$ *ker* θ. Hence $(x)^{\perp} \subseteq$ *ker* θ. Therefore *ker* θ is an intrinsic ideal of L. □

Definition 3.6. A lattice L is called *weakly quasi-complemented* if to each $x \in L$, there exists $y \in L$ such that $(x \wedge y)^{\perp} = (0)^{\perp}$ and $x \vee y \in D$.

Clearly every quasi-complemented lattice is weakly quasi-complemented and the converse is not true. We now characterize weakly quasi-complemented lattices with help of the lattice of inlets and the congruence θ.

Theorem 3.7. *The following conditions are equivalent in a lattice L:*

(1) *L is weakly quasi-complemented;*
(2) *$\mathcal{N}_{+}(L)$ is a Boolean algebra;*
(3) *L/θ is a Boolean algebra.*

Proof. (1) ⇒ (2): Assume that L is weakly quasi-complemented. Let $(x)^{\perp} \in \mathcal{N}_{+}(L)$. Then there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x' \in D$. Hence $(x)^{\perp} \cap (x')^{\perp} = (x \wedge x')^{\perp} = (0)^{\perp}$ and

$$(x)^{\perp} \sqcup (x')^{\perp} = (x \vee x')^{\perp} = L.$$

Therefore $\mathcal{N}_{+}(L)$ is a Boolean algebra.
(2) ⇒ (3): Assume that $\mathcal{N}_{+}(L)$ is a Boolean algebra. Let $[x]_{\theta} \in L/\theta$. Then $(x)^{\perp} \in \mathcal{N}_{+}(L)$. Hence there exists $(y)^{\perp} \in \mathcal{N}_{+}(L)$ such that $(x \wedge y)^{\perp} = (x)^{\perp} \cap (y)^{\perp} = (0)^{\perp}$ and $(x \vee y)^{\perp} = (x)^{\perp} \sqcup (y)^{\perp} = L$. Hence $x \wedge y \in [0]_{\theta}$ and $x \vee y \in D$. Thus $[x]_{\theta} \cap [y]_{\theta} = [x \wedge y]_{\theta} = [0]_{\theta}$ and $[x]_{\theta} \vee [y]_{\theta} = [x \vee y]_{\theta} = D$. Therefore L/θ is a Boolean algebra.
(3) ⇒ (1): Assume that L/θ is a Boolean algebra. Let $x \in L$. Then $[x]_{\theta} \in L/\theta$. Since L/θ is a Boolean algebra, there exists $[x']_{\theta} \in L/\theta$ such that $[x \wedge x']_{\theta} = [x]_{\theta} \cap [x']_{\theta} = [0]_{\theta}$ and

$$[x \vee x']_\theta = [x]_\theta \vee [x']_\theta = D.$$

Thus $(x \wedge x')^\perp = (0)^\perp$ and $x \vee x' \in D$. Therefore L is weakly quasi-complemented. $\square$

Theorem 3.8. *Every lattice L is epimorphic to the lattice $\langle \mathcal{N}_+(L), \sqcup, \cap \rangle$ of inlets.*

Proof. Define a mapping $\psi : L \longrightarrow \mathcal{N}_+(L)$ by $\psi(x) = (x)^\perp$ for all $x \in L$. Clearly ψ is well-defined. Let $a, b \in L$. Then

$$\psi(a \wedge b) = (a \wedge b)^\perp = (a)^\perp \cap (b)^\perp = \psi(a) \cap \psi(b).$$

By Lemma 3.3(3), we get

$$\psi(a \vee b) = (a \vee b)^\perp = ((a)^\perp \vee (b)^\perp)^\perp = (a)^\perp \sqcup (b)^\perp = \psi(a) \sqcup \psi(b).$$

Therefore ψ is a homomorphism. Clearly ψ is surjective. $\square$

4. Prime spectrum of intrinsic ideals

In this section, we discuss some algebraic properties of prime intrinsic ideals of a lattice. A set of equivalent conditions is given for the space of prime intrinsic ideals of a lattice to become a Hausdorff space.

Proposition 4.1. *Every maximal intrinsic ideal of a lattice is prime.*

Proof. Let M be a maximal intrinsic ideal of a lattice L. Let $x, y \in L$ be such that $x \notin M$ and $y \notin M$. Then $M \sqcup (x)^\perp = L$ and $M \sqcup (y)^\perp = L$. Now

$$\begin{aligned} L &= L \cap L \\ &= \{M \sqcup (x)^\perp\} \cap \{M \sqcup (y)^\perp\} \\ &= M \sqcup \{(x)^\perp \cap (y)^\perp\} \\ &= M \sqcup (x \wedge y)^\perp \end{aligned}$$

Suppose $x \wedge y \in M$. Since M is intrinsic, we get $(x \wedge y)^\perp \subseteq M$. Hence $M = L$, which is a contradiction. Therefore M is prime. $\square$

Theorem 4.2. *Let I be an intrinsic ideal and F a filter of a lattice L such that $I \cap F = \emptyset$. Then there exists a prime intrinsic ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Proof. Let I be an intrinsic ideal and F a filter of a L such that $I \cap F = \emptyset$. Consider

$$\sum = \{J \mid J \text{ is an intrinsic ideal, } I \subseteq J \text{ and } J \cap F = \emptyset\}.$$

Clearly $I \in \sum$. Clearly $\sum$ satisfies the hypothesis of Zorn's Lemma. Let M be a maximal element of $\sum$. Choose $x, y \in L$ such that $x \notin M$ and $y \notin M$. Then $M \subset M \vee (x] \subseteq M \vee (x)^{\perp}$ and

$$M \subset M \vee (y] \subseteq M \vee (y)^{\perp}.$$

By the maximality of M, we get $\{M \vee (x)^{\perp}\} \cap F \neq \emptyset$ and

$$\{M \vee (y)^{\perp}\} \cap F \neq \emptyset.$$

Choose $a \in \{M \vee (x)^{\perp}\} \cap F$ and $b \in \{M \vee (y)^{\perp}\} \cap F$. Then $a \wedge b \in F$ and

$$\begin{aligned} a \wedge b &\in \{M \vee (x)^{\perp}\} \cap \{M \vee (y)^{\perp}\} \\ &= M \vee \{(x)^{\perp} \cap (y)^{\perp}\} \\ &= M \vee (x \wedge y)^{\perp} \end{aligned}$$

Suppose $x \wedge y \in M$. Since M is intrinsic, we get $(x \wedge y)^{\perp} \subseteq M$. Hence $a \wedge b \in M$ and thus $a \wedge b \in M \cap F \neq \emptyset$, which is a contradiction. Therefore M is prime. $\square$

Corollary 4.3. *Let I be an intrinsic ideal of a lattice L and $x \notin I$. Then there exists a prime intrinsic ideal P of L such that $I \subseteq P$ and $x \notin P$.*

Corollary 4.4. *For any intrinsic ideal I of L, we have*

$$I = \bigcap\{P \mid P \text{ is a prime intrinsic ideal of } L \text{ such that } I \subseteq P\}$$

Corollary 4.5. *The intersection of all prime intrinsic ideals is equal to $(0)^{\perp}$.*

Proof. Since $(0)^{\perp}$ is the smallest intrinsic ideal, the proof follows immediately. $\square$

Let I be an intrinsic ideal and P be a prime intrinsic ideal of a lattice L such that $I \subseteq P$. Then P is called a minimal prime intrinsic ideal belonging to I if there exists no prime intrinsic ideal Q such that $I \subseteq Q \subset P$. A minimal prime intrinsic ideal belonging to $(0)^{\perp}$ is simply called minimal prime intrinsic. In the following theorem, a necessary and sufficient condition is derived for a prime intrinsic ideal of a lattice to become minimal.

Theorem 4.6. *Let I be an intrinsic ideal and P a prime intrinsic ideal of a lattice L such that $I \subseteq P$. Then P is a minimal prime intrinsic ideal belonging to I if and only if to each $x \in P$, there exists $y \notin P$ such that $x \wedge y \in I$.*

Proof. Let I and P be as in the statement. Assume that P is a minimal prime intrinsic ideal belonging to I. Since P is a proper intrinsic ideal, by Proposition 2.6, we get $P \cap D = \emptyset$. Then $L - P$ is a maximal filter with respect to the property that $(L-P) \cap I = \emptyset$. Let $x \in P$. Then clearly $L - P \subset (L-P) \vee [x)$. By the maximality of $L - P$, we get $\{(L-P) \vee [x)\} \cap I \neq \emptyset$. Choose $a \in \{(L-P) \vee [x)\} \cap I$. Then we get $a = r \wedge x$ for some $r \in L - P$ and $a \in I$. Therefore $r \wedge x = a \in I$ where $r \notin P$.

Conversely, assume that the condition holds. Suppose P is not a minimal prime intrinsic ideal belonging to I. Then there exists a prime intrinsic ideal Q of L such that $I \subseteq Q \subset P$. Choose $x \in P - Q$. Then, by the assumed condition, there exists $y \notin P$ such that $x \wedge y \in I \subseteq Q$. Since $x \notin Q$, it yields that $y \in Q \subset P$, which is a contradiction. Therefore P is a minimal prime intrinsic ideal belonging to I. □

Corollary 4.7. *A prime intrinsic ideal P of a lattice is minimal if and only if to each $x \in P$, there exists $y \notin P$ such that $x \wedge y \in (0)^{\perp}$.*

Proof. Replacing the intrinsic ideal I of Theorem 4.6 by the smallest intrinsic ideal $(0)^{\perp}$, the proof is an immediate consequence. □

For any lattice L, let us denote the class of all prime intrinsic ideals of L by $Spec^{\perp}(L)$. For any $A \subseteq L$, let

$$\mathcal{K}(A) = \{P \in Spec^{\perp}(L) \mid A \nsubseteq P\}$$

and for any $x \in L, \mathcal{K}(x) = \mathcal{K}(\{x\})$. Then we have the following observations which can be verified directly.

Lemma 4.8. *Let L be a lattice. For any $x, y \in L$, the following properties hold:*

(1) $\bigcup\limits_{x \in L} \mathcal{K}(x) = Spec^{\perp}(L)$,
(2) $\mathcal{K}(x) \cap \mathcal{K}(y) = \mathcal{K}(x \wedge y)$,
(3) $\mathcal{K}(x) \cup \mathcal{K}(y) = \mathcal{K}(x \vee y)$,
(4) $\mathcal{K}(x) = \emptyset$ *if and only if* $x \in (0)^{\perp}$,
(5) $\mathcal{K}(x) = Spec^{\perp}(L)$ *if and only if* $x \in D$.

From the above lemma, it can be easily observed that the collection $\{\mathcal{K}(x) | x \in L\}$ forms a base for a topology on $Spec^{\perp}(L)$ which is called a hull-kernel topology. Under this topology we have the following topological properties:

Theorem 4.9. *In any lattice L, the following properties hold:*

(1) *For any $x \in L$, $\mathcal{K}(x)$ is compact in $Spec^{\perp}(L)$,*
(2) *Let C be a compact open subset of $Spec^{\perp}(L)$. Then $C = \mathcal{K}(x)$*

for some $x \in L$,

(3) *$Spec^{\perp}(L)$ is a T_0-space,*

(4) *The map $x \mapsto \mathcal{K}(x)$ is a homomorphism from L onto the lattice of all compact open subsets of $Spec^{\perp}(L)$.*

Proof. (1) Let $x \in L$ and $A \subseteq L$ be such that $\mathcal{K}(x) \subseteq \bigcup\limits_{y \in A} \mathcal{K}(y)$. Let I be the ideal generated by the set A. Suppose $x \notin I^{\perp}$. By Corollary 4.7, there exits a prime intrinsic ideal P such that $I^{\perp} \subseteq P$ and $x \notin P$. Hence $P \in \mathcal{K}(x) \subseteq \bigcup\limits_{y \in A} \mathcal{K}(y)$. Therefore $y \notin P$ for some $y \in A$, which is a contradiction to that $y \in A \subseteq I \subseteq I^{\perp} \subseteq P$. Therefore $x \in I^{\perp}$. Then $x \in (a)^{\perp}$ for some $a \in I$. Since I is the ideal generated by A, we get $a = a_1 \vee a_2 \vee \ldots \vee a_n$ for some $a_1, a_2, \ldots, a_n \in A$. Hence $x \in (a)^{\perp} = (a_1 \vee a_2 \vee \ldots \vee a_n)^{\perp}$. Then clearly $\mathcal{K}(x) \subseteq \bigcup\limits_{i=1}^{n} \mathcal{K}(a_i)$, which is a finite subcover of $\mathcal{K}(x)$. Hence $\mathcal{K}(x)$ is compact in $Spec^{\perp}(L)$. Thus for each $x \in L$, $\mathcal{K}(x)$ is a compact open subset of $Spec^{\perp}(L)$.

(2) Let C be a compact open subset of $Spec^{\perp}(L)$. Since C is open, we get $C = \bigcup\limits_{a \in A} \mathcal{K}(a)$ for some $A \subseteq L$. Since C is compact, there exists $a_1, a_2, \ldots, a_n \in A$ such that

$$C = \bigcup_{i=1}^{n} \mathcal{K}(a_i) = \mathcal{K}\big(\bigvee_{i=1}^{n} a_i\big)$$

Therefore $C = \mathcal{K}(x)$ for some $x \in L$.

(3) Let P and Q be two distinct prime intrinsic ideals of L. Without loss of generality, assume that $P \nsubseteq Q$. Choose $x \in L$ such that $x \in P$ and $x \notin Q$. Hence $P \notin \mathcal{K}(x)$ and $Q \in \mathcal{K}(x)$. Therefore $Spec^{\perp}(L)$ is a T_0-space.

(4) It can be obtained from (2) and (3) of Lemma 4.8. $\square$

Lemma 4.10. *The following properties hold in a lattice L:*

(1) *for any $x \in L$, $\mathcal{K}(x) = \mathcal{K}((x)^{\perp})$,*

(2) *for any ideal I of L, $\mathcal{K}(I) = \mathcal{K}(I^{\perp})$,*

(3) *for any intrinsic ideal I of L, $\mathcal{K}(I) = \bigcup\limits_{x \in I} \mathcal{K}((x)^{\perp})$.*

Proof. (1) Let $P \in \mathcal{K}(x) \cap Spec^{\perp}(L)$. Then $x \notin P$. Since P is intrinsic, we get $(x)^{\perp} \nsubseteq P$. Hence $P \in \mathcal{K}((x)^{\perp})$. Therefore $\mathcal{K}(x) \subseteq \mathcal{K}((x)^{\perp})$. Similarly, the other inclusion holds.

(2) Since $I \subseteq I^{\perp}$, we get $\mathcal{K}(I) \subseteq \mathcal{K}(I^{\perp})$. Conversely, let

$$P \in \mathcal{K}(I^{\perp}) \cap Spec^{\perp}(L).$$

Then $I^{\perp} \nsubseteq P$. Choose $x \in I^{\perp}$ and $x \notin P$. Then $(x)^{\circ} \subseteq (a)^{\circ}$ for some $a \in I$. Hence $x \in (x)^{\perp} \subseteq (a)^{\perp}$. If $P \notin \mathcal{K}(I)$, then $a \in I \subseteq P$. Since P is intrinsic, we get $x \in (x)^{\perp} \subseteq (a)^{\perp} \subseteq P$, which is a contradiction. Thus $P \in \mathcal{K}(I)$. Therefore $\mathcal{K}(I^{\perp}) \subseteq \mathcal{K}(I)$.

(3) Let $P \in \mathcal{K}(I) \cap Spec^{\perp}(L)$. Then $I \nsubseteq P$. Choose $x \in I$ such that $x \notin P$. Then $P \in \mathcal{K}(x)$. Since $x \in I$, we get $P \in \bigcup_{x\in I} \mathcal{K}(x)$. Hence $\mathcal{K}(I) \subseteq \bigcup_{x\in I} \mathcal{K}(x)$. Conversely, let $P \in \bigcup_{x\in I} \mathcal{K}(x)$. Then $P \in \mathcal{K}(x)$ for some $x \in I$. Then $x \notin P$ for some $x \in I$. Hence $I \nsubseteq P$. Thus $P \in \mathcal{K}(I)$. Therefore $\bigcup_{x\in I} \mathcal{K}(x) \subseteq \mathcal{K}(I)$. □

Theorem 4.11. *For any lattice L, the lattice $(\mathcal{N}(L), \sqcup, \cap)$ of all intrinsic ideals of L is isomorphic to the lattice of all open subsets in $Spec^{\perp}(L)$.*

Proof. Denote the class of all open subsets of the space $Spec^{\perp}(L)$ by $\Im$. Clearly $(\Im, \cap, \cup)$ is a lattice. Define $\varphi : \mathcal{N}(L) \longrightarrow \Im$ by $\varphi(I^{\perp}) = \mathcal{K}(I)$ for all $I \in \mathcal{N}(L)$. By Lemma 4.10(2), every open subset of $Spec^{\perp}(L)$ is of the form $\mathcal{K}(I)$ for some $I \in \mathcal{N}(L)$. Hence the mapping φ is onto. Let $I, J \in \mathcal{N}(L)$ and suppose $\varphi(I) = \varphi(J)$. If $I \neq J$, then there exists $x \in J$ such that $x \notin I$. By Corollary 4.3, there exists $P \in Spec^{\perp}(L)$ such that $I \subseteq P$ and $x \notin P$. Thus $P \in \mathcal{K}(x)$ for $x \in J$. By Lemma 4.10(3), we get $P \in \bigcup_{x\in J} \mathcal{K}(x) = \mathcal{K}(J)$. Since $\varphi(I) = \varphi(J)$, we get $\mathcal{K}(I) = \mathcal{K}(J)$. Hence $P \in \mathcal{K}(J) = \mathcal{K}(I)$. Thus $I \nsubseteq P$ which contradicts the choice of P. Hence $I = J$ and therefore φ is one-one.

For any $I, J \in \mathcal{N}(L)$, we have

$$\varphi(I \cap J) = \mathcal{K}(I \cap J) = \mathcal{K}(I) \cap \mathcal{K}(J) = \varphi(I) \cap \varphi(J).$$

Also

$$\begin{aligned}
\varphi(I \sqcup J) &= \mathcal{K}(I \sqcup J) \\
&= \mathcal{K}((I \vee J)^{\perp}) && \text{by Theorem 2.10} \\
&= \mathcal{K}(I \vee J) && \text{by Lemma 4.10(2)} \\
&= \mathcal{K}(I) \cup \mathcal{K}(J) && \text{by Lemma 4.8(3)} \\
&= \varphi(I) \cup \varphi(J).
\end{aligned}$$

Hence φ is a homomorphism. Therefore $\mathcal{N}(L)$ is isomorphic to $\Im$. □

For any $A \subseteq L$, denote $\mathcal{H}(A) = \{P \in Spec^{\perp}(L) \mid A \subseteq P\}$. Then clearly $\mathcal{H}(A) = Spec^{\perp}(L) - \mathcal{K}(A)$. Therefore $\mathcal{H}(A)$ is a closed set in $Spec^{\perp}(L)$. Also every closed set in $Spec^{\perp}(L)$ is of the form $\mathcal{H}(A)$ for some $A \subseteq L$. Then we have the following:

Theorem 4.12. *For any lattice L and $X \subseteq Spec^{\perp}(L)$, the closure of X is given by $\overline{X} = \mathcal{H}(\bigcap\limits_{P \in X}(P))$.*

Proof. Let $X \subseteq Spec^{\perp}(L)$ and $Q \in X$. Then $\bigcap\limits_{P\in X} P \subseteq Q$. Thus $Q \in \mathcal{H}(\bigcap\limits_{P\in X} P)$. Therefore $\mathcal{H}(\bigcap\limits_{P\in X} P)$ is a closed set containing X. Let C be any closed set in $Spec^{\perp}(L)$. Then $C = \mathcal{H}(A)$ for some $A \subseteq L$. Since $X \subseteq C = H(A)$, we get that $A \subseteq P$ for all $P \in X$. Hence $A \subseteq \bigcap\limits_{P\in X} P$. Therefore $H(\bigcap\limits_{P\in X} P) \subseteq H(A) = C$. Hence $H(\bigcap\limits_{P\in X} P)$ is the smallest closed set containing X. Therefore $\overline{X} = H(\bigcap\limits_{P\in X} P)$. $\square$

Theorem 4.13. *The following conditions are equivalent in a lattice L:*

1. *every prime intrinsic ideal is maximal;*
2. *every prime intrinsic ideal is minimal;*
3. *$Spec^{\perp}(L)$ is a T_1-space;*
4. *$Spec^{\perp}(L)$ is a Hausdorff space;*
5. *for any $x, y \in L$, there exists $z \in L$ such that $x \wedge z \in (0)^{\perp}$ and*
$$\mathcal{K}(y) \cap \{Spec^{\perp}(L) - \mathcal{K}(x)\} = \mathcal{K}(y \wedge z)$$

Proof. (1) $\Leftrightarrow$ (2): Since every maximal intrinsic ideal is prime, it is clear.

(2) $\Rightarrow$ (3): Assume that every prime intrinsic ideal is minimal. Let P and Q be two distinct prime intrinsic ideals of L. By (2), P and Q are minimal. Hence, we get $P \nsubseteq Q$ and $Q \nsubseteq P$. Choose $x \in P - Q$ and $y \in Q - P$. Then $Q \in \mathcal{K}(x) - \mathcal{K}(y)$ and $P \in \mathcal{K}(y) - \mathcal{K}(x)$. Therefore $Spec^{\perp}(L)$ is a T_1-space.

(3) $\Rightarrow$ (4): Assume that $Spec^{\perp}(L)$ is a T_1-space. Let P be a prime intrinsic ideal of L. By Theorem 4.12,

$$\{P\} = \overline{\{P\}} = \{Q \in Spec^{\perp}_{F}(L) \mid P \subseteq Q\}.$$

Therefore P is maximal. Thus every prime intrinsic ideal is a maximal intrinsic ideal. Since every maximal intrinsic ideal is prime, we get that every prime intrinsic ideal is a minimal prime intrinsic ideal. Let $P, Q \in Spec^{\perp}(L)$ be such that $P \neq Q$. Choose $x \in P$ and $x \notin Q$. Since P is minimal, there exists $y \notin P$ such that $x \wedge y \in (0)^{\perp}$. Thus $P \in \mathcal{K}(y), Q \in \mathcal{K}(x)$ and $\mathcal{K}(x) \cap \mathcal{K}(y) = \mathcal{K}(x \wedge y) = \emptyset$. Therefore $Spec^{\perp}(L)$ is a Hausdorff space.

(4) $\Rightarrow$ (5): Assume that $Spec^{\perp}(L)$ is Hausdorff. Hence $\mathcal{K}(a)$ is a compact subset of $Spec^{\perp}(L)$, for each $a \in L$. Then $\mathcal{K}(a)$ is a clopen subset of $Spec^{\perp}(L)$. Let $x, y \in L$ such that $x \neq y$. Then

$\mathcal{K}(y) \cap \{Spec^{\perp}(L) - \mathcal{K}(x)\}$ is a compact subset of the compact space $\mathcal{K}(y)$. Since $\mathcal{K}(y)$ is open in $Spec^{\perp}(L)$, we get $\mathcal{K}(y) \cap \{Spec^{\perp}(L) - \mathcal{K}(x)\}$ is a compact open subset of $Spec^{\perp}(L)$. Hence by Theorem 4.9(2), there exists $z \in L$ such that

$$\mathcal{K}(z) = \mathcal{K}(y) \cap \{Spec^{\perp}(L) - \mathcal{K}(x)\}$$

Therefore $\mathcal{K}(y) \cap \{Spec^{\perp}(L) - \mathcal{K}(x)\} = \mathcal{K}(y) \cap \mathcal{K}(z) = \mathcal{K}(y \wedge z)$. Also $\mathcal{K}(x \wedge z) = \mathcal{K}(x) \cap \mathcal{K}(z) = \emptyset$. Therefore $x \wedge z \in (0)^{\perp}$.

(5) $\Rightarrow$ (2): Let P be a prime intrinsic ideal of L. Choose $x, y \in L$ such that $x \in P$ and $y \notin P$. Then by condition (5), there exists $z \in L$ such that $x \wedge z \in (0)^{\perp}$ and

$$\mathcal{K}(y) \cap \{Spec^{\perp}(L) - \mathcal{K}(x)\} = \mathcal{K}(y \wedge z)$$

Then clearly $P \in \mathcal{K}(y) \cap \{Spec^{\perp}(L) - \mathcal{K}(x)\} = \mathcal{K}(y \wedge z)$. If $z \in P$, then $y \wedge z \in P$, which is a contradiction to $P \in \mathcal{K}(y \wedge z)$. Hence $z \notin P$. Thus for each $x \in P$, there exists $z \notin P$ such that $x \wedge z \in (0)^{\perp}$. Therefore P is a minimal prime intrinsic ideal. $\square$

For any lattice L, it is clear that $\mathcal{H}(A) = Spec^{\perp}(L) - \mathcal{K}(A)$ and hence $\mathcal{H}(A)$ is a closed set in $Spec^{\perp}(L)$. In the following result, a necessary and sufficient condition is derived for the space $Spec^{\perp}(L)$ to become regular.

Theorem 4.14. *For any lattice L, the space $Spec^{\perp}(L)$ is a regular space if and only if for any $P \in Spec^{\perp}(L)$ and $a \notin P$, there exist an ideal I of L and $b \in L$ such that $P \in \mathcal{K}(b) \subseteq \mathcal{H}(I) \subseteq \mathcal{K}(a)$.*

Proof. Assume that $Spec^{\perp}(L)$ is a regular space. Let $P \in Spec^{\perp}(L)$ and $a \notin P$ for some $a \in L$. Then $P \notin \mathcal{H}(a)$. Since $Spec^{\perp}(L)$ is a regular space, there exist two disjoint open sets G and H in $Spec^{\perp}(L)$ such that $P \in G$ and $\mathcal{H}(a) \subseteq H$. Therefore $Spec^{\perp}(L) - H \subseteq \mathcal{K}(a)$. Since $Spec^{\perp}(L) - H$ is a closed set, we get that

$$Spec^{\perp}(L) - H \;=\; \mathcal{H}(I)$$

for some intrinsic ideal I in L. Thus $\mathcal{H}(I) = Spec^{\perp}(L) - H \subseteq \mathcal{K}(a)$. Now $G \cap H = \emptyset$ will imply that $H \subseteq Spec^{\perp}(L) - G$. Since $Spec^{\perp}(L) - G$ is closed, we get $Spec^{\perp}(L) - G = \mathcal{H}(J)$ for some intrinsic ideal J of L. Since $P \in G$, we get $P \notin Spec^{\perp}(L) - G = \mathcal{H}(J)$ and hence $J \nsubseteq P$. Choose $b \in J$ such that $b \notin P$. Then $P \in \mathcal{K}(b)$. Let $T \in H$. Then $J \subseteq T$ because of $H \subseteq \mathcal{H}(J)$. Since $b \in J \subseteq T$, we get $T \in \mathcal{H}(b)$. Thus $H \subseteq \mathcal{H}(b)$. Hence by (1),

$$\mathcal{K}(b) = Spec^{\perp}(L) - \mathcal{H}(b) \subseteq Spec^{\perp}(L) - H = \mathcal{H}(I).$$

which means $\mathcal{K}(b) \subseteq \mathcal{H}(I)$. Thus for any $P \in Spec^{\perp}(L)$ and $a \notin P$, there exist an ideal I of L and $b \in L$ such that

$$P \in \mathcal{K}(b) \subseteq \mathcal{H}(I) \subseteq \mathcal{K}(a).$$

Conversely, assume that for any $P \in Spec^{\perp}(L)$ and $a \notin P$, there exist an ideal I of L and $b \in L$ such that $P \in \mathcal{K}(b) \subseteq \mathcal{H}(I) \subseteq \mathcal{K}(a)$. To show that the space $Spec^{\perp}(L)$ is regular, let $P \in Spec^{\perp}(L)$ and $\mathcal{H}(K)$ be any closed set of $Spec^{\perp}(L)$ such that $P \notin \mathcal{H}(K)$. Then $K \nsubseteq P$. Hence there exist $a \in K$ such that $a \notin P$. Thus $P \in \mathcal{K}(a)$. Since $a \notin P$, by the assumption, there exists an ideal I of L and $b \in L$ such that $P \in \mathcal{K}(b) \subseteq \mathcal{H}(I) \subseteq \mathcal{K}(a)$. Hence $\mathcal{K}(a) \cap \mathcal{H}(K) = \emptyset$, because of $K \in \mathcal{H}(a)$ for $a \in K$. Thus

$$\mathcal{H}(K) \subseteq Spec^{\perp}(L) - \mathcal{K}(a) \subseteq Spec^{\perp}(L) - \mathcal{H}(I).$$

Therefore

$$\mathcal{H}(K) \subseteq Spec^{\perp}(L) - \mathcal{K}(a) \subseteq Spec^{\perp}(L) - \mathcal{H}(I).$$

Also $\mathcal{K}(b) \cap \mathcal{K}(I) = \emptyset$. Thus there exist two disjoint open sets $\mathcal{K}(b)$ and $\mathcal{K}(I)$ such that $P \in \mathcal{K}(b)$ and $\mathcal{H}(K) \subseteq \mathcal{K}(I)$. Therefore $Spec^{\perp}(L)$ is a regular space. $\square$

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