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# CHARACTERIZATION OF JORDAN $\{g,h\}$-DERIVATIONS OVER MATRIX ALGEBRAS

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Abstract. In this article, we characterize $\{g,h\}$-derivation on the upper triangular matrix algebra $\mathcal{T}_n(C)$ and prove that every Jordan $\{g,h\}$-derivation over $\mathcal{T}_n(C)$ is a $\{g,h\}$-derivation under a certain condition, where $C$ is a 2-torsion free commutative ring with unity $1 \neq 0$. Also, we study $\{g,h\}$-derivation and Jordan $\{g,h\}$-derivation over full matrix algebra $\mathcal{M}_n(C)$.

## 1. Introduction

Throughout the article, $C$ represents a commutative ring with unity $1 \neq 0$. Recall that a ring $R$ is 2-*torsion free* if $2a = 0$ for some $a \in R$ implies $a = 0$. Herstein [15] initiated the research on Jordan derivation over prime rings in 1957 and proved that every Jordan derivation over a prime ring of characteristic not 2 is a derivation. In 1975, Cusack [8] established the same result for semiprime rings. Let $A$ be an algebra over a commutative ring $C$. A linear map $d : A \to A$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in A$ and $d$ is said to be a Jordan derivation if $d(x^2) = d(x)x + xd(x)$ for all $x \in A$. When $C$ is a 2-torsion free, Jordan derivation is a linear map $d : A \to A$ such that $d(x \circ y) = d(x) \circ y + x \circ d(y)$ for all $x, y \in A$, where $x \circ y = xy + yx$. A linear map $T : A \to A$ is a left (right) centralizer if $T(xy) = T(x)y$

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$(T(xy) = xT(y))$ for all $x, y \in A$ and $T$ is two-sided centralizer if it is both left and right centralizers.

In the last six decades, many authors studied Jordan derivation over rings and algebras in [2, 3, 5, 13, 21, 22, 23], showing that every Jordan derivation over undertaken ring or algebra is a derivation. For more results on Jordan derivations, see [1, 10, 9, 11]. Later, mathematicians introduced some generalizations of (Jordan) derivations, like (Jordan) left derivation, (Jordan) generalized derivation, and (Jordan) $P$-derivation. They established many results on those derivations over some rings and algebras; we refer [6, 7, 12, 14, 16, 18, 19, 20].

Recently, in 2016, Brešar [4] introduced $\{g, h\}$ derivation and Jordan $\{g, h\}$ derivation on algebras. We note that they have considered algebra over the field $\mathbb{F}$ with char($\mathbb{F}$)$\neq 2$. Inspired by these above works, we study $\{g, h\}$-derivation for the algebra over $C$. Towards this, we recall the definition of $\{g, h\}$ derivation and Jordan $\{g, h\}$ derivation.

Let $A$ be an algebra over $C$ and $f, g, h : A \to A$ be the linear maps. Then $f$ is said to be a $\{g, h\}$-derivation if

$$f(xy) = g(x)y + xh(y) = h(x)y + xg(y), \text{ for all } x, y \in A. \tag{1.1}$$

If $f = g = h$ in (1.1), then $f$ is a usual derivation. Now, $f$ is said to be a Jordan $\{g, h\}$-derivation if

$$f(x \circ y) = g(x) \circ y + x \circ h(y), \text{ for all } x, y \in A. \tag{1.2}$$

If $f = g = h$ in (1.2), then $f$ is a usual Jordan derivation. Hence, every $\{g, h\}$-derivation is a Jordan $\{g, h\}$-derivation but the converse is not true (Example 2.1 of [4]). It is also easy to see that if every Jordan $\{g, h\}$-derivation over $A$ is a $\{g, h\}$-derivation, then every Jordan derivation on $A$ is a derivation. In this connection, in 2016, Brešar established that every Jordan $\{g, h\}$-derivation of a semiprime algebra $A$ over a field $\mathbb{F}$ with char($\mathbb{F}$)$\neq 2$, is a $\{g, h\}$-derivation [4]. Also, we can find results related to Jordan $\{g, h\}$-derivation on triangular algebras in [17].

The manuscript is organized as follows: First we give an example of $\{g, -g\}$-derivation over $\mathcal{T}_2(C)$. Motivated by that example, we prove for a zero map to be a $\{g, h\}$-derivation, where $g$ and $h$ has to be equal but opposite sign (Theorem 2.2) and $g$ turns out to be a two-sided centralizer. Also, we prove every Jordan $\{g, -g\}$-derivation over $\mathcal{T}_n(C)$ has to be a zero map (Theorem 2.5) with some restriction on $C$. That restriction on $C$ is necessary given by Example 2.6. Further, we prove that every Jordan $\{g, h\}$-derivation on $\mathcal{T}_n(C)$ is a $\{g, h\}$-derivation under certain assumptions over the maps and $C$ (Theorem 2.7). If the assumptions in Theorem 2.7 are not considered, then the theorem may not hold (Example 2.13 and 2.14). Also, we establish that $\{g, h\}$-derivation is the only Jordan $\{g, h\}$-derivation on $\mathcal{M}_n(C)$ under only the torsion restrictions of $C$ (Theorem 3.2). In this case, we do not assume the relation between maps as for $\mathcal{T}_n(C)$. Before going further, let $e_{ij}$ denote the matrix whose $(i, j)$-th entry is 1 and 0 elsewhere.

## 2. Jordan $\{g, h\}$-derivation on $\mathcal{T}_n(C)$

Let $\mathcal{T}_n(C)$ be the algebra of $n \times n$ upper triangular matrices over $C$. In this section, we characterize $\{g, h\}$-derivation and Jordan $\{g, h\}$-derivation over $\mathcal{T}_n(C)$.

**Example 2.1.** We start here by an example of a $\{g, h\}$-derivation over $\mathcal{T}_2(C)$. Define $g : \mathcal{T}_2(C) \to \mathcal{T}_2(C)$ by $g(A) = A$, for all $A \in \mathcal{T}_2(C)$. Then 0 is a $\{g, -g\}$-derivation.

We want to see whether the converse of the example is true. The following theorem will answer this question.

**Theorem 2.2.** *If 0 is a $\{g, h\}$-derivation over $\mathcal{T}_n(C)$, $n \geq 2$, then $h = -g$ and $g(A) = \alpha A$, for some $\alpha \in C$. In particular, $g$ is a two-sided centralizer.*

*Proof.* Let

$$g(e_{ij}) = \sum_{1 \leq m \leq p \leq n} g_{mp}^{(ij)} e_{mp} \ , \ \text{where } g_{mp}^{(ij)} \in C, \tag{2.1}$$

and

$$h(e_{ij}) = \sum_{1 \leq m \leq p \leq n} h_{mp}^{(ij)} e_{mp} \ , \ \text{where } h_{mp}^{(ij)} \in C. \tag{2.2}$$

Since $0 = g(e_{ii})e_{ii} + e_{ii}h(e_{ii}) = h(e_{ii})e_{ii} + e_{ii}g(e_{ii})$,

$$\begin{aligned} g_{li}^{(ii)} &= h_{li}^{(ii)} = 0 \text{ for } l = \{1, \ldots, i-1\}, \\ g_{il}^{(ii)} &= h_{il}^{(ii)} = 0 \text{ for } l = \{i+1, \ldots, n\}, \\ g_{ii}^{(ii)} &+ h_{ii}^{(ii)} = 0, \text{ for all } i \in \{1, \ldots, n\}. \end{aligned} \tag{2.3}$$

Let $i \neq j$. From the relation

$$g(e_{ii})e_{jj} + e_{ii}h(e_{jj}) = 0 = h(e_{ii})e_{jj} + e_{ii}g(e_{jj}),$$

we have

$$\begin{aligned} & g_{lj}^{(ii)} = h_{lj}^{(ii)} = 0 \text{ for } l = \{1, \ldots, j\}, \\ & \text{for all } i, \ j \in \{1, \ldots, n\}. \end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we have

$$g(e_{ii}) = g_{ii}^{(ii)} e_{ii} \text{ and } h(e_{ii}) = h_{ii}^{(ii)} e_{ii} = -g_{ii}^{(ii)} e_{ii}.$$

Let $i < j$ and $j \neq k$. By the relation

$$g(e_{ij})e_{kk} + e_{ij}h(e_{kk}) = 0 = h(e_{ij})e_{kk} + e_{ij}g(e_{kk}),$$

we get

$$\begin{aligned} &g_{lk}^{(ij)} = h_{lk}^{(ij)} = 0 \text{ for } l = \{1, \ldots, k\}, \\ &\text{for all } i,\ j,\ k \in \{1, \ldots, n\}. \end{aligned} \tag{2.5}$$

As the relation $g(e_{ii})e_{ij} + e_{ii}h(e_{ij}) = 0 = h(e_{ii})e_{ij} + e_{ii}g(e_{ij})$ holds,

$$\begin{aligned} &g_{il}^{(ij)} = h_{il}^{(ij)} = 0, \text{ for } l = \{i, i+1, \ldots, j-1, j+1, \ldots, n\}, \\ &g_{ii}^{(ii)} + h_{ij}^{(ij)} = 0, \\ &h_{ii}^{(ii)} + g_{ij}^{(ij)} = 0, \text{ for all } i,\ j \in \{1, \ldots, n\}. \end{aligned} \tag{2.6}$$

Since $g(e_{ij})e_{jj} + e_{ij}h(e_{jj}) = 0 = h(e_{ij})e_{jj} + e_{ij}g(e_{jj})$,

$$\begin{aligned} &g_{lj}^{(ij)} = h_{lj}^{(ij)} = 0, \text{ for } l = \{1, 2, \ldots, i-1, i+1, \ldots, j\}, \\ &g_{ij}^{(ij)} + h_{jj}^{(jj)} = 0, \\ &h_{ij}^{(ij)} + g_{jj}^{(jj)} = 0, \text{ for all } i,\ j \in \{1, \ldots, n\}. \end{aligned} \tag{2.7}$$

From (2.5), (2.6) and (2.7), we have $g(e_{ij}) = g_{ij}^{(ij)} e_{ij}$, $h(e_{ij}) = h_{ij}^{(ij)} e_{ij}$ and

$$g_{ij}^{(ij)} = g_{ii}^{(ii)} = g_{jj}^{(jj)} = -h_{ij}^{(ij)} = -h_{ii}^{(ii)} = -h_{jj}^{(jj)}$$

for all $i < j$ and $i,\ j \in \{1, 2, \ldots, n\}$. Let $\alpha = g_{ii}^{(ii)} \in C$. Then $g(A) = \alpha A$ and $h(A) = -\alpha A$, for all $A \in \mathcal{T}_n(C)$. Hence, $h = -g$ and it can easily be proved that $g$ is a two-sided centralizer. □

**Example 2.3.** Now, we give an example of a nonzero $\{g, h\}$-derivation on $\mathcal{T}_2(C)$ where $g = h$. Define $f,\ g : \mathcal{T}_2(C) \to \mathcal{T}_2(C)$ by
$f\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} 2a_{11} & a_{12} \\ 0 & 2a_{22} \end{pmatrix}$ and $g\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$,
respectively. Then $f$ is $\{g, g\}$-derivation on $\mathcal{T}_2(C)$.

**Example 2.4.** This is an example of a $\{g, h\}$-derivation $f$ on $\mathcal{T}_2(C)$ for nonzero different $f$, $g$ and $h$. Define $f,\ g,\ h : \mathcal{T}_2(C) \to \mathcal{T}_2(C)$ by
$f\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} 3a_{11} & 2a_{12} \\ 0 & 3a_{22} \end{pmatrix}$, $g\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$
and $h\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} 2a_{11} & a_{12} \\ 0 & 2a_{22} \end{pmatrix}$, respectively. Then $f$ is a $\{g, h\}$-derivation on $\mathcal{T}_2(C)$.

After these examples, we want to see the existence of a nonzero Jordan $\{g, -g\}$-derivation over $\mathcal{T}_n(C)$. The answer is in the next theorem. The following theorem is the converse of Theorem 2.2 with some torsion conditions on $C$.

**Theorem 2.5.** *If $f$ is a Jordan $\{g, -g\}$-derivation over $\mathcal{T}_n(C)$ where $C$ is 2-torsion free, then $f = 0$.*

*Proof.* Let $i \le j$ and $i, j \in \{1, 2, \dots, n\}$. If $i = j$, from

$$f(e_{ii} \circ e_{ii}) = g(e_{ii}) \circ e_{ii} - e_{ii} \circ g(e_{ii}),$$

we have $2f(e_{ii}) = 0$. Therefore, $f(e_{ii}) = 0$, since $C$ is 2-torsion free. Now, if $i < j$, by using the relation

$$f(e_{ii} \circ e_{ij}) = g(e_{ii}) \circ e_{ij} - e_{ii} \circ g(e_{ij}) = -g(e_{ii}) \circ e_{ij} + e_{ii} \circ g(e_{ij})$$

and $C$ being 2-torsion free ring, we get $f(e_{ij}) = 0$. Hence, $f = 0$. □

**Example 2.6.** If we drop torsion condition of $C$, then Theorem 2.5 does not hold, which has shown in this example. Let $\mathbb{Z}_n$ be the ring of integers of modulo $n$. Then $\mathbb{Z}_4$ is not 2-torsion free. Define $f,\ g : \mathcal{T}_2(\mathbb{Z}_4) \to \mathcal{T}_2(\mathbb{Z}_4)$ by $f\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 2a_{11} + 2a_{22} \\ 0 & 0 \end{pmatrix}$ and $g\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 2a_{11} + a_{12} + 2a_{22} \\ 0 & a_{22} \end{pmatrix}$, respectively. Then $f$ is a nonzero Jordan $\{g, -g\}$-derivation on $\mathcal{T}_2(\mathbb{Z}_4)$.

Our next theorem tells that any Jordan $\{g, h\}$-derivation on $\mathcal{T}_n(C)$ becomes a $\{g, h\}$-derivation under certain conditions over the maps and on the underlying ring.

**Theorem 2.7.** *Let $f$ be a Jordan $\{g, h\}$-derivation on $\mathcal{T}_n(C)$, $C$ be a 2-torsion free ring, $n \ge 2$, with $f(e_{ii}) = g(e_{ii})e_{ii} + e_{ii}h(e_{ii})$ for all $i = 1, 2, \dots, n$. Then $f$ is a $\{g, h\}$-derivation.*

To prove the theorem, we first prove several related lemmas. Let $f : A \to A$ be a Jordan $\{g, h\}$-derivation.

**Lemma 2.8.** *Let $a \in A$ such that $f(a^2) = g(a)a + ah(a)$. Then $f(a^2) = h(a)a + ag(a)$.*

*Proof.* From (1.2), we have $f(a \circ a) = h(a) \circ a + a \circ g(a)$. Therefore, $f(a^2) = f(a \circ a) - f(a^2) = h(a)a + ag(a)$. □

**Lemma 2.9.** *Let $a, b \in A$ such that*

$$f(ab) = g(a)b + ah(b) = h(a)b + ag(b).$$

*Then $f(ba) = g(b)a + bh(a) = h(b)a + bg(a)$.*

*Proof.* From (1.2), $f(x \circ y) = g(x) \circ y + x \circ h(y)$, for all $x, y \in A$. Therefore,

$$\begin{aligned} f(ba) &= f(b \circ a) - f(ab) \\ &= g(b) \circ a + b \circ h(a) - h(a)b - ag(b) \\ &= g(b)a + bh(a). \end{aligned}$$

Similarly,

$$\begin{aligned} f(ba) &= f(a \circ b) - f(ab) \\ &= g(a) \circ b + a \circ h(b) - g(a)b - ah(b) \\ &= h(b)a + bg(a) \end{aligned}$$

□

Now, let $f : \mathcal{T}_n(C) \to \mathcal{T}_n(C)$ be a Jordan $\{g, h\}$-derivation.
First we prove

$$f(e_{ij}e_{kl}) = g(e_{ij})e_{kl} + e_{ij}h(e_{kl}) = h(e_{ij})e_{kl} + e_{ij}g(e_{kl}), \tag{2.8}$$

which is equivalent to

$$\begin{aligned} f(e_{kl}e_{ij}) &= g(e_{kl})e_{ij} + e_{kl}h(e_{ij}) \\ &= h(e_{kl})e_{ij} + e_{kl}g(e_{ij}) \text{ (by Lemma 2.9)}. \end{aligned} \tag{2.9}$$

Let $g(e_{ij})$ and $h(e_{ij})$ is of the form (2.1) and (2.2).

**Lemma 2.10.** *$f(e_{ii}e_{jj}) = g(e_{ii})e_{jj} + e_{ii}h(e_{jj}) = h(e_{ii})e_{jj} + e_{ii}g(e_{jj})$, for $i \neq j$.*

*Proof.* Let $i \neq j$. Without loss of generality, let $i < j$. Since $f$ is a Jordan $\{g, h\}$-derivation on $\mathcal{T}_n(C)$,

$$\begin{aligned} 0 &= f(e_{ii} \circ e_{jj}) \\ &= g(e_{ii}) \circ e_{jj} + e_{ii} \circ h(e_{jj}) \\ &= h(e_{ii}) \circ e_{jj} + e_{ii} \circ g(e_{jj}). \end{aligned} \tag{2.10}$$

From (2.10), we have

$$\begin{aligned} g_{1j}^{(ii)} &= g_{2j}^{(ii)} = \cdots = g_{i-1,j}^{(ii)} = g_{i+1,j}^{(ii)} = \cdots = g_{j-1,j}^{(ii)} = 0, \\ g_{jj}^{(ii)} &= h_{ii}^{(jj)} = 0 \text{ (since } C \text{ is 2-torsion free)}, \\ h_{i,i+1}^{(jj)} &= h_{i,i+2}^{(jj)} = \cdots = h_{i,j-1}^{(jj)} = h_{i,j+1}^{(jj)} = \cdots = h_{in}^{(jj)} = 0, \\ g_{ij}^{(ii)} &+ h_{ij}^{(jj)} = 0. \end{aligned} \tag{2.11}$$

Therefore, we get $f(e_{ii}e_{jj}) = 0 = g(e_{ii})e_{jj} + e_{ii}h(e_{jj})$, by using (2.11). Similarly, by using the second part of (2.10), we get

$$h(e_{ii})e_{jj} + e_{ii}g(e_{jj}) = 0.$$

Hence, $f(e_{ii}e_{jj}) = g(e_{ii})e_{jj} + e_{ii}h(e_{jj}) = h(e_{ii})e_{jj} + e_{ii}g(e_{jj})$. □

**Lemma 2.11.** *$f(e_{ii}e_{jk}) = g(e_{ii})e_{jk} + e_{ii}h(e_{jk}) = h(e_{ii})e_{jk} + e_{ii}g(e_{jk})$, for $j < k$.*

*Proof.* Our aim is to prove the following:

$$f(e_{ii}e_{jk}) = g(e_{ii})e_{jk} + e_{ii}h(e_{jk}) = h(e_{ii})e_{jk} + e_{ii}g(e_{jk}), \tag{2.12}$$

where $j < k$.

**Case 1.** Let $i = j$. Then (2.12) is equivalent to

$$f(e_{ij}e_{ii}) = g(e_{ij})e_{ii} + e_{ij}h(e_{ii}) = h(e_{ij})e_{ii} + e_{ij}g(e_{ii}),$$

where $i < j$.

For $i < j$,

$$\begin{aligned} f(e_{ij}) &= f(e_{ij} \circ e_{ii}) \\ &= g(e_{ij}) \circ e_{ii} + e_{ij} \circ h(e_{ii}) \\ &= h(e_{ij}) \circ e_{ii} + e_{ij} \circ g(e_{ii}), \\ \\ f(e_{ij}) &= f(e_{ij} \circ e_{jj}) \\ &= g(e_{ij}) \circ e_{jj} + e_{ij} \circ h(e_{jj}) \\ &= h(e_{ij}) \circ e_{jj} + e_{ij} \circ g(e_{jj}). \end{aligned} \tag{2.13}$$

From (2.13),

$$\begin{aligned} g_{1i}^{(ij)} = g_{2i}^{(ij)} = \cdots = g_{i-1,i}^{(ij)} = g_{ii}^{(ij)} = 0, \\ h_{1i}^{(ij)} = h_{2i}^{(ij)} = \cdots = h_{i-1,i}^{(ij)} = h_{ii}^{(ij)} = 0. \end{aligned} \tag{2.14}$$

From (2.10),

$$\begin{aligned} h_{jj}^{(ii)} = g_{ii}^{(jj)} = 0, \\ h_{j,j+1}^{(ii)} = h_{j,j+2}^{(ii)} = \cdots = h_{jn}^{(ii)} = 0, \\ g_{j,j+1}^{(ii)} = g_{j,j+2}^{(ii)} = \cdots = g_{jn}^{(ii)} = 0. \end{aligned} \tag{2.15}$$

Now, using (2.11), (2.14) and (2.15), we have

$$f(e_{ij}e_{ii}) = 0 = g(e_{ij})e_{ii} + e_{ij}h(e_{ii}) = h(e_{ij})e_{ii} + e_{ij}g(e_{ii}). \tag{2.16}$$

**Case 2.** Claim:

$$f(e_{ii}e_{jk}) = g(e_{ii})e_{jk} + e_{ii}h(e_{jk}) = h(e_{ii})e_{jk} + e_{ii}g(e_{jk}),$$

where $i < j$. Using the relation $g(e_{ii})\circ e_{jk} + e_{ii}\circ h(e_{jk}) = f(e_{ii}\circ e_{jk}) = 0$,

$$\begin{aligned} &g_{ij}^{(ii)} + h_{ik}^{(jk)} = 0 \\ &h_{ii}^{(jk)} = h_{i,i+1}^{(jk)} = \cdots = h_{i,k-1}^{(jk)} = h_{i,k+1}^{(jk)} = \cdots = h_{in}^{(jk)} = 0. \end{aligned} \tag{2.17}$$

Hence, we have $f(e_{ii}e_{jk}) = 0 = g(e_{ii})e_{jk} + e_{ii}h(e_{jk})$ using (2.11) and (2.17). Similarly, using $h(e_{ii}) \circ e_{jk} + e_{ii} \circ g(e_{jk}) = f(e_{ii} \circ e_{jk})$, we have $h(e_{ii})e_{jk} + e_{ii}g(e_{jk}) = 0$.

**Case 3.** Claim:

$$f(e_{ii}e_{jk}) = g(e_{ii})e_{jk} + e_{ii}h(e_{jk}) = h(e_{ii})e_{jk} + e_{ii}g(e_{jk}), \tag{2.18}$$

where $i > j$.

**Subcase 1.** Let $k = i$. Then (2.18) is equivalent to

$$f(e_{jj}e_{ij}) = g(e_{jj})e_{ij} + e_{jj}h(e_{ij}) = h(e_{jj})e_{ij} + e_{jj}g(e_{ij}),$$

for $i < j$.

From (2.10) and (2.13), we have

$$\begin{aligned} g_{1i}^{(jj)} &= g_{2i}^{(jj)} = \cdots = g_{i-1,i}^{(jj)} = g_{ii}^{(jj)} = 0, \\ h_{jj}^{(ij)} &= h_{j,j+1}^{(ij)} = \cdots = h_{jn}^{(ij)} = 0. \end{aligned} \tag{2.19}$$

Using (2.19), we have $f(e_{jj}e_{ij}) = 0 = g(e_{jj})e_{ij} + e_{jj}h(e_{ij})$. In a similar way, $h(e_{jj})e_{ij} + e_{jj}g(e_{ij}) = 0$.

**Subcase 2.** Let $k \neq i$. From $g(e_{ii}) \circ e_{jk} + e_{ii} \circ h(e_{jk}) = f(e_{ii} \circ e_{jk}) = 0$ and (2.10), we have

$$\begin{aligned} h_{ii}^{(jk)} &= h_{i,i+1}^{(jk)} = \cdots = h_{in}^{(jk)} = 0, \\ g_{1j}^{(ii)} &= g_{2j}^{(ii)} = \cdots = g_{jj}^{(ii)} = 0. \end{aligned} \tag{2.20}$$

By (2.20), we get $f(e_{ii}e_{jk}) = 0 = g(e_{ii})e_{jk} + e_{ii}h(e_{jk})$. Similarly, $h(e_{ii})e_{jk} + e_{ii}g(e_{jk}) = 0$. □

**Lemma 2.12.** *$f(e_{ij}e_{kl}) = g(e_{ij})e_{kl} + e_{ij}h(e_{kl}) = h(e_{ij})e_{kl} + e_{ij}g(e_{kl})$, for $i < j,\ k < l$.*

*Proof.* We prove

$$f(e_{ij}e_{kl}) = g(e_{ij})e_{kl} + e_{ij}h(e_{kl}) = h(e_{ij})e_{kl} + e_{ij}g(e_{kl}), \tag{2.21}$$

where $i < j,\ k < l$.

**Case 1.** Let $j = k$. Then (2.21) equivalent to

$$f(e_{kj}e_{ik}) = g(e_{kj})e_{ik} + e_{kj}h(e_{ik}) = h(e_{kj})e_{ik} + e_{kj}g(e_{ik}),$$

where $i < k < j$.

For $i < k < j$, from (2.13) and the relation

$$f(e_{ij}) = f(e_{kj} \circ e_{ik}) = g(e_{kj}) \circ e_{ik} + e_{kj} \circ h(e_{ik}),$$

we have

$$\begin{aligned} g_{1i}^{(kj)} &= g_{2i}^{(kj)} = \cdots = g_{i-1,i}^{(kj)} = 0, \\ h_{j,j+1}^{(ik)} &= h_{j,j+2}^{(ik)} = \cdots = h_{jn}^{(ik)} = 0. \end{aligned} \tag{2.22}$$

As discussed in Lemma (2.11), using the relations

$$h(e_{ii})e_{kj} + e_{ii}g(e_{kj}) = f(e_{ii}e_{kj}) = 0$$

and $h(e_{ik})e_{jj} + e_{ik}g(e_{jj}) = f(e_{ik}e_{jj}) = 0$, we get

$$g_{ii}^{(kj)} = 0 = h_{jj}^{(ik)}. \tag{2.23}$$

Therefore, by using (2.22) and (2.23), we have

$$f(e_{kj}e_{ik}) = 0 = g(e_{kj})e_{ik} + e_{kj}h(e_{ik}).$$

Similarly, we can prove $h(e_{kj})e_{ik} + e_{kj}g(e_{ik}) = 0$.

**Case 2.** Let $j < k$. For $i < j < k < l$, from the relation

$$0 = f(e_{ij} \circ e_{kl}) = g(e_{ij}) \circ e_{kl} + e_{ij} \circ h(e_{kl}),$$

we have

$$\begin{aligned} &g_{1k}^{(ij)} = g_{2k}^{(ij)} = \cdots = g_{i-1,k}^{(ij)} = g_{i+1,k}^{(ij)} = \cdots = g_{k-1,k}^{(ij)} = 0,\\ &g_{ik}^{(ij)} + h_{jl}^{(kl)} = 0,\\ &h_{j,j+1}^{(kl)} = h_{j,j+2}^{(kl)} = \cdots = h_{j,l-1}^{(kl)} = h_{j,l+1}^{(kl)} = \cdots = h_{jn}^{(kl)} = 0. \end{aligned} \tag{2.24}$$

By Lemma (2.11), from $0 = f(e_{ij}e_{kk}) = g(e_{ij})e_{kk} + e_{ij}h(e_{kk})$ and $0 = f(e_{jj}e_{kl}) = g(e_{jj})e_{kl} + e_{jj}h(e_{kl})$, respectively, we have

$$g_{kk}^{(ij)} = 0 = h_{jj}^{(kl)}. \tag{2.25}$$

By using (2.24) and (2.25), $f(e_{ij}e_{kl}) = 0 = g(e_{ij})e_{kl} + e_{ij}h(e_{kl})$. In a similar way, we have $h(e_{ij})e_{kl} + e_{ij}g(e_{kl}) = 0$.

**Case 3.** Let $j > k$. Then, we have $i < j, j > k, k < l$. From Lemma (2.11), the relation $0 = f(e_{ij}e_{kk}) = g(e_{ij})e_{kk} + e_{ij}h(e_{kk})$ implies that

$$g_{1k}^{(ij)} = g_{2k}^{(ij)} = \cdots = g_{kk}^{(ij)} = 0. \tag{2.26}$$

Similarly, from $0 = f(e_{jj}e_{kl}) = g(e_{jj})e_{kl} + e_{jj}h(e_{kl})$,

$$h_{jj}^{(kl)} = h_{j,j+1}^{(kl)} = \cdots = h_{jn}^{(kl)} = 0. \tag{2.27}$$

Hence, by using (2.26) and (2.27), $f(e_{ij}e_{kl}) = 0 = g(e_{ij})e_{kl}+e_{ij}h(e_{kl})$. Similarly, it can be proved that $h(e_{ij})e_{kl} + e_{ij}g(e_{kl}) = 0$. □

**Proof of Theorem 2.7:** Let $f$ be a Jordan $\{g,h\}$-derivation on $\mathcal{T}_n(C)$. Since $f(e_{ii}^2) = f(e_{ii}) = g(e_{ii})e_{ii} + e_{ii}h(e_{ii})$, we get

$$f(e_{ii}^2) = h(e_{ii})e_{ii} + e_{ii}g(e_{ii}),$$

for all $i = 1, 2, \ldots, n$ (by Lemma 2.8). By Lemmas 2.10-2.12, we get $f(xy) = g(x)y + xh(y) = h(x)y + xg(y)$ for all $x, y \in \mathcal{T}_n(C)$.

**Example 2.13.** This example is of a Jordan $\{g,h\}$-derivation which is not a $\{g,h\}$-derivation over $\mathcal{T}_2(C)$. In this case, the condition over the maps in Theorem 2.7 is not satisfied. Consider $g : \mathcal{T}_2(C) \to \mathcal{T}_2(C)$ defined by $g(x) = a \circ x$, where $a = e_{11} + e_{12} + e_{22}$. Then 0 is a Jordan $\{g, -g\}$ derivation, but $0(e_{11}^2) = 0 \neq -e_{12} = g(e_{11})e_{11} + e_{11}((-g)(e_{11}))$, i.e. 0 is not a $\{g, -g\}$ derivation.

**Example 2.14.** Now, we have an example of a non-zero Jordan $\{g,h\}$-derivation which is not a $\{g,h\}$-derivation over $\mathcal{T}_2(C)$. But, here, the condition over the underlying ring in Theorem 2.7 is not satisfied. Consider the ring $\mathbb{Z}_2$ which is not 2-torsion free. Define $f,\ g,\ h : \mathcal{T}_2(\mathbb{Z}_2) \to \mathcal{T}_2(\mathbb{Z}_2)$ by

$$f\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix},\ g\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{11}+a_{12} \end{pmatrix}$$

and

$$h\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} a_{12} & 0 \\ 0 & a_{11}+a_{22} \end{pmatrix},$$

respectively. Then $f$ is a Jordan $\{g,h\}$ derivation and

$$f(e_{11}^2) = g(e_{11})e_{11} + e_{11}h(e_{11}).$$

But $f$ is not $\{g,h\}$ derivation as

$$f(e_{11}e_{22}) = 0 \neq e_{22} = g(e_{11})e_{22} + e_{11}h(e_{22}).$$

**Example 2.15.** This example shows that $\mathcal{T}_n(C)$ is not always a semiprime algebra. The ring $\mathbb{Z}_9$ is a 2-torsion free commutative ring with unity $1 \neq 0$. Then $\mathcal{T}_2(\mathbb{Z}_9)$ is not a semiprime algebra as $3e_{11}\mathcal{T}_2(\mathbb{Z}_9)3e_{11} = 0$, but $3e_{11} \neq 0$ in $\mathcal{T}_2(\mathbb{Z}_9)$.

## 3. Jordan $\{g,h\}$-derivation on $\mathcal{M}_n(C)$

Let $\mathcal{M}_n(C)$ be the algebra of $n \times n$ matrices over $C$. Here, we study $\{g,h\}$-derivation and Jordan $\{g,h\}$-derivation over $\mathcal{M}_n(C)$.

**Example 3.1.** This is an example of a $\{g,h\}$-derivation on $\mathcal{M}_n(C)$. Let $g : \mathcal{M}_2(C) \to \mathcal{M}_2(C)$ defined by $g(A) = 2A$, where $A \in \mathcal{M}_2(C)$. It is easy to see that 0 is a $\{g, -g\}$-derivation. The following theorem describes about the converse of this example.

**Theorem 3.2.** *If 0 is a $\{g,h\}$-derivation over $\mathcal{M}_n(C)$, $n \geq 2$, then $h = -g$ and $g(A) = \gamma A$ for some $\gamma \in C$. In particular, $g$ is a two-sided centralizer.*

*Proof.* Let

$$g(e_{ij}) = \sum_{k=1}^{n}\sum_{l=1}^{n} g_{kl}^{(ij)} e_{kl}\ ,\ \text{where } g_{kl}^{(ij)} \in C \tag{3.1}$$

and

$$h(e_{ij}) = \sum_{k=1}^{n}\sum_{l=1}^{n} h_{kl}^{(ij)} e_{kl}\ ,\ \text{ where } h_{kl}^{(ij)} \in C. \tag{3.2}$$

By the relation $0 = g(e_{ii})e_{ii} + e_{ii}h(e_{ii}) = h(e_{ii})e_{ii} + e_{ii}g(e_{ii})$, we have

$$\begin{aligned} &g_{li}^{(ii)} = h_{li}^{(ii)} = 0, \\ &g_{il}^{(ii)} = h_{il}^{(ii)} = 0 \text{ for } l = \{1,\dots,i-1,i+1,\dots,n\}, \\ &g_{ii}^{(ii)} + h_{ii}^{(ii)} = 0, \text{ for all } i \in \{1,\dots,n\}. \end{aligned} \tag{3.3}$$

Let $i \neq j$. As the relation $g(e_{ii})e_{jj} + e_{ii}h(e_{jj}) = 0 = h(e_{ii})e_{jj} + e_{ii}g(e_{jj})$ holds, we have

$$\begin{aligned} &g_{lj}^{(ii)} = h_{lj}^{(ii)} = 0 \text{ for } l = \{1,\dots,n\}, \\ &\text{for all } i,\ j \in \{1,\dots,n\}. \end{aligned} \tag{3.4}$$

Using (3.3) and (3.4), $g(e_{ii}) = g_{ii}^{(ii)} e_{ii}$ and $h(e_{ii}) = h_{ii}^{(ii)} e_{ii} = -g_{ii}^{(ii)} e_{ii}$. Let $i \neq j$ and $j \neq k$. Since

$$g(e_{ij})e_{kk} + e_{ij}h(e_{kk}) = 0 = h(e_{ij})e_{kk} + e_{ij}g(e_{kk}),$$

we have

$$\begin{aligned} &g_{lk}^{(ij)} = h_{lk}^{(ij)} = 0 \text{ for } l = \{1,\dots,n\}, \\ &\text{for all } i,\ j,\ k \in \{1,\dots,n\}. \end{aligned} \tag{3.5}$$

From $g(e_{ii})e_{ij} + e_{ii}h(e_{ij}) = 0 = h(e_{ii})e_{ij} + e_{ii}g(e_{ij})$,

$$\begin{aligned} &g_{il}^{(ij)} = h_{il}^{(ij)} = 0 \text{ for } l = \{1,2,\dots,j-1,j+1,\dots,n\}, \\ &g_{ii}^{(ii)} + h_{ij}^{(ij)} = 0 = h_{ii}^{(ii)} + g_{ij}^{(ij)}, \text{ for all } i,\ j \in \{1,\dots,n\}. \end{aligned} \tag{3.6}$$

As $g(e_{ij})e_{jj} + e_{ij}h(e_{jj}) = 0 = h(e_{ij})e_{jj} + e_{ij}g(e_{jj})$ holds,

$$\begin{aligned} &g_{lj}^{(ij)} = h_{lj}^{(ij)} = 0 \text{ for } l = \{1,2,\dots,i-1,i+1,\dots,n\}, \\ &g_{ij}^{(ij)} + h_{jj}^{(jj)} = 0 = h_{ij}^{(ij)} + g_{jj}^{(jj)}, \text{ for all } i,\ j \in \{1,\dots,n\}. \end{aligned} \tag{3.7}$$

From (3.5), (3.6) and (3.7), $g(e_{ij}) = g_{ij}^{(ij)} e_{ij}$, $h(e_{ij}) = h_{ij}^{(ij)} e_{ij}$ and

$$g_{ij}^{(ij)} = g_{ii}^{(ii)} = g_{jj}^{(jj)} = -h_{ij}^{(ij)} = -h_{ii}^{(ii)} = -h_{jj}^{(jj)}$$

for all $i,\ j \in \{1,2,\dots,n\}$. Let $\gamma = g_{ii}^{(ii)} \in C$. Hence, $g(A) = \gamma A$ and $h(A) = -\gamma A$, for all $A \in \mathcal{M}_n(C)$. Thus, $h = -g$ and $g$ is a two-sided centralizer. □

**Example 3.3.** This is an example of a $\{g,g\}$-derivation on $\mathcal{M}_2(C)$. Let $f,\ g : \mathcal{M}_2(C) \to \mathcal{M}_2(C)$ be defined by

$$f\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 2a_{11} & a_{12} \\ 3a_{21} & 2a_{22} \end{pmatrix}$$

and

$$g\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 2a_{21} & a_{22} \end{pmatrix},$$

respectively. Then $f$ is $\{g, g\}$-derivation on $\mathcal{M}_2(C)$.

**Example 3.4.** Now we have an example of a $\{g, h\}$-derivation on $\mathcal{M}_2(C)$ for different $g \neq h$. Let $f,\ g,\ h : \mathcal{M}_2(C) \to \mathcal{M}_2(C)$ be defined by

$$f\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 3a_{11} & 2a_{12} \\ 4a_{21} & 3a_{22} \end{pmatrix},$$

$$g\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 2a_{21} & a_{22} \end{pmatrix}$$

and

$$h\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 2a_{11} & a_{12} \\ 3a_{21} & 2a_{22} \end{pmatrix},$$

respectively. It is easy to see that $f$ is a $\{g, h\}$-derivation on $\mathcal{M}_2(C)$.

Our next theorem tells about the converse of Theorem 3.2.

**Theorem 3.5.** *If $f$ is a Jordan $\{g, -g\}$-derivation over $\mathcal{M}_n(C)$ where $C$ is 2-torsion free, then $f = 0$.*

*Proof.* We can prove this in a similar way as in Theorem 2.5. □

**Example 3.6.** This is an example which supports the necessity of the torsion condition in Theorem 3.5. Let $f,\ g : \mathcal{M}_2(\mathbb{Z}_4) \to \mathcal{M}_2(\mathbb{Z}_4)$ be defined by

$$f\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 2a_{21} & 2a_{11} + 2a_{22} \\ 0 & 2a_{21} \end{pmatrix}$$

and

$$g\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + 2a_{21} & 2a_{11} + a_{12} + 2a_{22} \\ a_{21} & 2a_{21} + a_{22} \end{pmatrix},$$

respectively. Then $f$ is a nonzero Jordan $\{g, -g\}$-derivation on $\mathcal{M}_2(\mathbb{Z}_4)$.

The following theorem proves how a Jordan $\{g, h\}$-derivation over $\mathcal{M}_n(C)$ be a $\{g, h\}$-derivation under certain conditions over the underlying ring.

**Theorem 3.7.** *Let $\mathcal{M}_n(C)$ be an algebra of $n \times n$, $n \geq 2$, matrices over a 2-torsion free ring $C$. Then every Jordan $\{g, h\}$-derivation on $\mathcal{M}_n(C)$ into itself is a $\{g, h\}$-derivation.*

Let $f:\mathcal{M}_n(C)\to\mathcal{M}_n(C)$ be a Jordan $\{g,h\}$-derivation.
First, we prove

$$f(e_{ij}e_{kl}) = g(e_{ij})e_{kl} + e_{ij}h(e_{kl}) = h(e_{ij})e_{kl} + e_{ij}g(e_{kl}), \tag{3.8}$$

which is equivalent to

$$\begin{aligned} f(e_{kl}e_{ij}) &= g(e_{kl})e_{ij} + e_{kl}h(e_{ij}) \\ &= h(e_{kl})e_{ij} + e_{kl}g(e_{ij}) \text{ (by Lemma 2.9).} \end{aligned} \tag{3.9}$$

Suppose $g(e_{ij})$ and $h(e_{ij})$ are of the form (3.1) and (3.2), respectively. Here, we derive few more identities by using (3.1) and (3.2). Let $i\neq j$.

$$\begin{aligned} 0 = f(e_{ii}\circ e_{jj}) &= g(e_{ii})\circ e_{jj} + e_{ii}\circ h(e_{jj}) \\ &= h(e_{ii})\circ e_{jj} + e_{ii}\circ g(e_{jj}). \end{aligned} \tag{3.10}$$

$$\begin{aligned} f(e_{ij}) &= f(e_{ii}\circ e_{ij}) \\ &= g(e_{ii})\circ e_{ij} + e_{ii}\circ h(e_{ij}) \\ &= h(e_{ii})\circ e_{ij} + e_{ii}\circ g(e_{ij}) \\ &= f(e_{ij}\circ e_{jj}) \\ &= g(e_{ij})\circ e_{jj} + e_{ij}\circ h(e_{jj}) \\ &= h(e_{ij})\circ e_{jj} + e_{ij}\circ g(e_{jj}). \end{aligned} \tag{3.11}$$

**Lemma 3.8.** *$f(e_{ii}e_{ii}) = g(e_{ii})e_{ii} + e_{ii}h(e_{ii}) = h(e_{ii})e_{ii} + e_{ii}g(e_{ii})$, for $i=1,2,\ldots,n$.*

*Proof.* From (3.11),

$$\begin{aligned} &g_{1i}^{(ii)} = h_{1i}^{(ii)},\ldots,g_{i-1,i}^{(ii)} = h_{i-1,i}^{(ii)}, g_{i+1,i}^{(ii)} = h_{i+1,i}^{(ii)},\ldots,g_{ni}^{(ii)} = h_{ni}^{(ii)}, \\ &g_{i1}^{(ii)} = h_{i1}^{(ii)},\ldots,g_{i,i-1}^{(ii)} = h_{i,i-1}^{(ii)}, g_{i,i+1}^{(ii)} = h_{i,i+1}^{(ii)},\ldots,g_{in}^{(ii)} = h_{in}^{(ii)}. \end{aligned} \tag{3.12}$$

By using (3.12), $f(e_{ii}^2) = g(e_{ii})e_{ii} + e_{ii}h(e_{ii})$. Similarly $f(e_{ii}^2) = h(e_{ii})e_{ii} + e_{ii}g(e_{ii})$. □

**Lemma 3.9.** *$f(e_{ii}e_{jj}) = g(e_{ii})e_{jj} + e_{ii}h(e_{jj}) = h(e_{ii})e_{jj} + e_{ii}g(e_{jj})$, for $i\neq j$.*

*Proof.* Using the relation (3.10),

$$\begin{aligned} &g_{1j}^{(ii)} = \cdots = g_{i-1,j}^{(ii)} = g_{i+1,j}^{(ii)} = \cdots = g_{nj}^{(ii)} = 0, \\ &g_{ij}^{(ii)} + h_{ij}^{(jj)} = 0, \\ &h_{i1}^{(jj)} = \cdots = h_{i,j-1}^{(jj)} = h_{i,j+1}^{(jj)} = \cdots = h_{in}^{(jj)} = 0. \end{aligned} \tag{3.13}$$

Hence, we have $f(e_{ii}e_{jj}) = 0 = g(e_{ii})e_{jj} + e_{ii}h(e_{jj})$ by (3.13). Similarly, we can show that $f(e_{ii}e_{jj}) = 0 = h(e_{ii})e_{jj} + e_{ii}g(e_{jj})$. □

**Lemma 3.10.** *$f(e_{ii}e_{jk}) = g(e_{ii})e_{jk} + e_{ii}h(e_{jk}) = h(e_{ii})e_{jk} + e_{ii}g(e_{jk})$, for $j \neq k$.*

*Proof.* We have two cases.

**Case1.** Let $i = j$. We prove

$$f(e_{ik}e_{ii}) = g(e_{ik})e_{ii} + e_{ik}h(e_{ii}) = h(e_{ik})e_{ii} + e_{ik}g(e_{ii}),$$

for $i \neq k$, which is equivalent to

$$f(e_{ij}e_{ii}) = g(e_{ij})e_{ii} + e_{ij}h(e_{ii}) = h(e_{ij})e_{ii} + e_{ij}g(e_{ii}),$$

for $i \neq j$.

By the relation $0 = f(e_{ij} \circ e_{ij})$, we have

$$g_{1i}^{(ij)} + h_{1i}^{(ij)} = g_{2i}^{(ij)} + h_{2i}^{(ij)} = \cdots = g_{ni}^{(ij)} + h_{ni}^{(ij)} = 0, \tag{3.14}$$

$$g_{j1}^{(ij)} + h_{j1}^{(ij)} = g_{j2}^{(ij)} + h_{j2}^{(ij)} = \cdots = g_{jn}^{(ij)} + h_{jn}^{(ij)} = 0. \tag{3.15}$$

From (3.11),

$$g_{1i}^{(ij)} = h_{1i}^{(ij)}, \ldots, g_{i-1,i}^{(ij)} = h_{i-1,i}^{(ij)}, g_{i+1,i}^{(ij)} = h_{i+1,i}^{(ij)}, \ldots, g_{ni}^{(ij)} = h_{ni}^{(ij)}. \tag{3.16}$$

Using (3.14) and (3.16), we get

$$g_{1i}^{(ij)} = \cdots = g_{i-1,i}^{(ij)} = g_{i+1,i}^{(ij)} = \cdots = g_{ni}^{(ij)} = 0, \tag{3.17}$$

$$h_{1i}^{(ij)} = \cdots = h_{i-1,i}^{(ij)} = h_{i+1,i}^{(ij)} = \cdots = h_{ni}^{(ij)} = 0. \tag{3.18}$$

Again, by the relation (3.11),

$$h_{ji}^{(ii)} + 2g_{ii}^{(ij)} = g_{ji}^{(jj)}. \tag{3.19}$$

From (3.10),

$$h_{ji}^{(ii)} + g_{ji}^{(jj)} = 0. \tag{3.20}$$

By (3.19) and (3.20),

$$g_{ii}^{(ij)} + h_{ji}^{(ii)} = 0. \tag{3.21}$$

From the relation (3.10),

$$h_{j1}^{(ii)} = \cdots = h_{j,i-1}^{(ii)} = h_{j,i+1}^{(ii)} = \cdots = h_{jn}^{(ii)} = 0. \tag{3.22}$$

Using (3.17),(3.21) and (3.22), $g(e_{ij})e_{ii} + e_{ij}h(e_{ii}) = 0 = f(e_{ij}e_{ii})$. Similarly, we can prove that $f(e_{ij}e_{ii}) = h(e_{ij})e_{ii} + e_{ij}g(e_{ii})$.

**Case 2.** Let $j \neq k$. From the relation $0 = f(0) = f(e_{ii} \circ e_{jk})$,

$$h_{i1}^{(jk)} = \cdots = h_{i,k-1}^{(jk)} = h_{i,k+1}^{(jk)} = \cdots = h_{in}^{(jk)} = 0, \tag{3.23}$$

$$g_{ij}^{(ii)} + h_{ik}^{(jk)} = 0. \tag{3.24}$$

Using (3.13),(3.23) and (3.24), $g(e_{ii})e_{jk} + e_{ii}h(e_{jk}) = 0 = f(e_{ii}e_{jk})$. Similarly, $f(e_{ii}e_{jk}) = h(e_{ii})e_{jk} + e_{ii}g(e_{jk})$. □

**Lemma 3.11.** *$f(e_{ij}e_{kl}) = g(e_{ij})e_{kl} + e_{ij}h(e_{kl}) = h(e_{ij})e_{kl} + e_{ij}g(e_{kl})$, for $i \neq j$ and $k \neq l$.*

*Proof.* To complete the proof, we have to consider three cases.

**Case 1.** Let $j \neq k$. By Lemma 3.10, we have

$$0 = f(e_{ij}e_{kk}) = g(e_{ij})e_{kk} + e_{ij}h(e_{kk}).$$

From this relation, we have

$$g^{(ij)}_{1k} = \cdots = g^{(ij)}_{i-1,k} = g^{(ij)}_{i+1,k} = \cdots = g^{(ij)}_{nk} = 0, \tag{3.25}$$

$$g^{(ij)}_{ik} + h^{(kk)}_{jk} = 0. \tag{3.26}$$

Similarly, from $0 = f(e_{jj}e_{kl})$,

$$h^{(kl)}_{j1} = \cdots = h^{(kl)}_{j,l-1} = h^{(kl)}_{j,l+1} = \cdots = h^{(kl)}_{jn} = 0, \tag{3.27}$$

$$g^{(jj)}_{jk} + h^{(kl)}_{jl} = 0. \tag{3.28}$$

By Lemma 3.9, $0 = f(e_{jj}e_{kk}) = g(e_{jj})e_{kk} + e_{jj}h(e_{kk})$. From that relation,

$$g^{(jj)}_{jk} + h^{(kk)}_{jk} = 0. \tag{3.29}$$

By (3.26),(3.28) and (3.29),

$$g^{(ij)}_{ik} + h^{(kl)}_{jl} = 0. \tag{3.30}$$

Hence, by (3.25),(3.27) and (3.30), $g(e_{ij})e_{kl}+e_{ij}h(e_{kl}) = 0 = f(e_{ij}e_{kl})$. Similarly, we have $f(e_{ij}e_{kl}) = h(e_{ij})e_{kl} + e_{ij}g(e_{kl})$.

**Case 2.** Let $j = k$ and $i \neq l$. We prove

$$f(e_{kl}e_{ik}) = g(e_{kl})e_{ik} + e_{kl}h(e_{ik}) = h(e_{kl})e_{ik} + e_{kl}g(e_{ik}).$$

Now, from $0 = f(e_{kl}e_{ii}) = g(e_{kl})e_{ii} + e_{kl}h(e_{ii})$,

$$g^{(kl)}_{1i} = \cdots = g^{(kl)}_{k-1,i} = g^{(kl)}_{k+1,i} = \cdots = g^{(kl)}_{ni} = 0, \tag{3.31}$$

$$g^{(kl)}_{ki} + g^{(ii)}_{li} = 0. \tag{3.32}$$

From $0 = f(e_{ll}e_{ik}) = g(e_{ll})e_{ik} + e_{ll}h(e_{ik})$,

$$h^{(ik)}_{l1} = \cdots = h^{(ik)}_{l,k-1} = h^{(ik)}_{l,k+1} = \cdots = h^{(ik)}_{ln} = 0, \tag{3.33}$$

$$g^{(ll)}_{li} + h^{(ik)}_{lk} = 0. \tag{3.34}$$

From $0 = f(e_{ll}e_{ii}) = g(e_{ll})e_{ii} + e_{ll}h(e_{ii})$,

$$g_{li}^{(ll)} + h_{li}^{(ii)} = 0. \tag{3.35}$$

By (3.32),(3.34) and (3.35),

$$g_{ki}^{(kl)} + h_{lk}^{(ik)} = 0. \tag{3.36}$$

Hence, by (3.31),(3.33) and (3.36), $g(e_{kl})e_{ik}+e_{kl}h(e_{ik}) = 0 = f(e_{kl}e_{ik})$. Similarly, we have $f(e_{kl}e_{ik}) = h(e_{kl})e_{ik} + e_{kl}g(e_{ik})$.

**Case 3.** Let $j = k$ and $i = l$. We prove

$$f(e_{ij}e_{ji}) = g(e_{ij})e_{ji} + e_{ij}h(e_{ji}) = h(e_{ij})e_{ji} + e_{ij}g(e_{ji}).$$

From $e_{ij} \circ e_{ji} = e_{ii} + e_{jj}$, and by using Lemma 3.8, we get

$$g(e_{ij})\circ e_{ji}+e_{ij}\circ h(e_{ji}) = g(e_{ii})e_{ii}+e_{ii}h(e_{ii})+g(e_{jj})e_{jj}+e_{jj}h(e_{jj}). \tag{3.37}$$

From (3.37),

$$\begin{aligned} &g_{1j}^{(ij)} = g_{1i}^{(ii)}, \ldots, g_{i-1,j}^{(ij)} = g_{i-1,i}^{(ii)}, g_{i+1,j}^{(ij)} = g_{i+1,i}^{(ii)}, \\ &\ldots, g_{j-1,j}^{(ij)} = g_{j-1,i}^{(ii)}, g_{j+1,j}^{(ij)} = g_{j+1,i}^{(ii)}, \ldots, g_{nj}^{(ij)} = g_{ni}^{(ii)}, \end{aligned} \tag{3.38}$$

$$g_{ij}^{(ij)} + h_{ji}^{(ji)} = g_{ii}^{(ii)} + h_{ii}^{(ii)}, \tag{3.39}$$

$$\begin{aligned} &h_{j1}^{(ji)} = h_{i1}^{(ii)}, \ldots, h_{j,i-1}^{(ji)} = h_{i,i-1}^{(ii)}, h_{j,i+1}^{(ji)} = h_{i,i+1}^{(ii)}, \\ &\ldots, h_{j,j-1}^{(ji)} = h_{i,j-1}^{(ii)}, h_{j,j+1}^{(ji)} = h_{i,j+1}^{(ii)}, \ldots, h_{jn}^{(ji)} = h_{in}^{(ii)}. \end{aligned} \tag{3.40}$$

From $g(e_{ij})e_{jj} + e_{ij}h(e_{jj}) = f(e_{ij}) = g(e_{ii})e_{ij} + e_{ii}h(e_{ij})$,

$$g_{jj}^{(ij)} = g_{ji}^{(ii)}. \tag{3.41}$$

From $g(e_{jj})e_{ji} + e_{jj}h(e_{ji}) = f(e_{ji}) = g(e_{ji})e_{ii} + e_{ji}h(e_{ii})$, we have

$$h_{jj}^{(ji)} = h_{ij}^{(ii)}. \tag{3.42}$$

Using (3.38)-(3.42), we get

$$g(e_{ij})e_{ji} + e_{ij}h(e_{ji}) = g(e_{ii})e_{ii} + e_{ii}h(e_{ii}) = f(e_{ii}) = f(e_{ij}e_{ji}).$$

Similarly, we can prove that $f(e_{ij}e_{ji}) = h(e_{ij})e_{ji} + e_{ij}g(e_{ji})$. □

**Proof of Theorem 3.7:** Let $f$ be a Jordan $\{g, h\}$-derivation on $\mathcal{M}_n(C)$. By Lemmas 3.8-3.11, we get

$$f(xy) = g(x)y + xh(y) = h(x)y + xg(y) \text{ for all } x, y \in \mathcal{M}_n(C).$$

**Example 3.12.** We give an example of a Jordan $\{g, h\}$-derivation which is not a $\{g, h\}$-derivation over $\mathcal{M}_2(C)$ where $C$ is not considered as 2-torsion free mentioned in Theorem 3.7. Let

$$f,\ g,\ h : \mathcal{M}_2(\mathbb{Z}_2) \to \mathcal{M}_2(\mathbb{Z}_2)$$

be defined by

$$f\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}, \; g\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$$

and

$$h\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{22} & 0 \\ 0 & a_{11} \end{pmatrix},$$

respectively. It can be shown that $f$ is a Jordan $\{g,h\}$ derivation over $\mathcal{M}_2(\mathbb{Z}_2)$. But $f$ is not a $\{g,h\}$ derivation on $\mathcal{M}_2(\mathbb{Z}_2)$ as

$$f(e_{11}e_{22}) = 0 \neq e_{11} = g(e_{11})e_{22} + e_{11}h(e_{22}).$$

**Example 3.13.** This example shows that $\mathcal{M}_n(C)$ is not always a semiprime algebra. Therefore, Theorem 3.7 is no more a consequence of any result from [4]. The algebra $\mathcal{M}_2(\mathbb{Z}_9)$ is not a semiprime, since $3e_{11}\mathcal{M}_2(\mathbb{Z}_9)3e_{11} = 0$, but $3e_{11} \neq 0$ in $\mathcal{M}_2(\mathbb{Z}_9)$.

We have a result motivated by Theorem 3.1 of [4]. It can be easily proved by following the proof of Theorem 3.1 in [4].

**Theorem 3.14.** *If every Jordan $\{g,h\}$-derivation over an algebra $A$ is a $\{g,h\}$-derivation, then the same is true for $A \otimes S$, where $A$ is an algebra over $C$, $S$ is a commutative algebra over $C$, and $A \otimes S$ represents the tensor product of two algebras $A$ and $S$.*

Finally, we have a result as a corollary of Theorem 3.7 and 3.14 as follows:

**Corollary 3.15.** *Every Jordan $\{g,h\}$-derivation over $\mathcal{M}_n(C) \otimes S$ is a $\{g,h\}$-derivation, where $S$ is a commutative algebra over a 2-torsion free ring $C$.*

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# CHARACTERIZATION OF JORDAN $\{g,h\}$-DERIVATIONS OVER MATRIX ALGEBRAS

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رده‌بندی $\{g,h\}$-مشتق‌های جردن روی جبرهای ماتریسی

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ما در این مقاله، $\{g,h\}$-مشتق روی جبر ماتریسی بالامثلثی $\mathcal{T}_n(C)$ را مشخص می‌کنیم و نشان می‌دهیم که تحت یک شرط مشخص، هر $\{g,h\}$-مشتق روی $\mathcal{T}_n(C)$، یک $\{g,h\}$-مشتق است، جایی که $C$ حلقه‌ای جابه‌جایی ۲-بی‌تاب با ۱ $\neq$ ۰ می‌باشد. همچنین ما به مطالعه‌ی $\{g,h\}$-مشتق و $\{g,h\}$-مشتق جردن روی جبر ماتریسی کامل $\mathcal{M}_n(C)$ می‌پردازیم.

کلمات کلیدی: مشتق، مشتق جردن، $\{g,h\}$-مشتق، $\{g,h\}$-مشتق جردن، جبر ماتریسی.