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# SOME RESULTS ON THE ARTINIAN COFINITE MODULES

G. PIRMOHAMMADI

Abstract. Let $I$ be an ideal of a commutative Noetherian ring $R$ and $M$ be a non-zero Artinian $R$-module with support contained in $V(I)$. In this paper it is shown that $M$ is $I$-cofinite if and only if $\mathrm{Rad}(I\widehat{R}^J + \mathrm{Ann}_{\widehat{R}^J} M) = J\widehat{R}^J$, where $J := \cap_{\mathfrak{m}\in \mathrm{Supp}\, M}\, \mathfrak{m}$ and $\widehat{R}^J$ denotes the $J$-adic comletion of $R$.

## 1. Introduction

Throughout this paper, let $R$ denote a commutative Noetherian ring (with identity) and $I$ be an ideal of $R$. Recall that an $R$-module $M$ is called $I$-*cofinite* if $\mathrm{Supp}(M) \subseteq V(I)$ and $\mathrm{Ext}^j_R(R/I, M)$ is finitely generated for all $i \geq 0$. The concept of $I$-cofinite for an $R$-module $M$ was first introduced by Hartshorne [6] and after that became one of the favorite issues for some algebraists. So far, many studies have been conducted in this area and some of them have led to some interesting results (see for example [3, 5, 7, 10]).

Melkersson in [10] proved that if $I$ is an ideal of a Noetherian complete local ring $(R, \mathfrak{m})$ and $M$ is a non-zero Artinian $R$-module, then $M$ is $I$-cofinite if and only if $\mathrm{Rad}(I + \mathfrak{p}) = \mathfrak{m}$ for each attached prime ideal $\mathfrak{p}$ of $M$, see [10, Theorem 1.6]. Abazari and Bahmanpour in [1, Theorem 2.2] generalized this result for the class of Noetherian local rings $(R, \mathfrak{m})$ for which $R \subseteq \widehat{R}^{\mathfrak{m}}$ is an integral extension.

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In this paper we shall establish the following generalization of Melkersson's result:

**Theorem 1.1.** *Let $I$ be a proper ideal of a Noetherian ring $R$ and $M$ be a non-zero Artinian $R$-module with support contained in $V(I)$. Set $J := \cap_{\mathfrak{m}\in \operatorname{Supp} M} \mathfrak{m}$ and let $\widehat{R}^J$ denote the $J$-adic comletion of $R$. Then the following statements are equivalent:*

(1) *$M$ is $I$-cofinite.*
(2) $\operatorname{Rad}(I\widehat{R}^J + \operatorname{Ann}_{\widehat{R}^J} M) = J\widehat{R}^J$.
(3) $J\widehat{R}^J \subseteq \operatorname{Rad}(I\widehat{R}^J + \mathfrak{P})$ *for every* $\mathfrak{P} \in \operatorname{Att}_{\widehat{R}^J} M$.

Also, as a generalization of [1, Theorem 2.2], we shall prove the following theorem:

**Theorem 1.2.** *Let $R$ be a Noetherian semi-local ring with Jacobson radical $J$ such that $\widehat{R}^J$ (the $J$-adic comletion of $R$) is integral over $R$. Let $I$ be an ideal of $R$ contained in $J$ and $M$ be a non-zero Artinian $R$-module with support contained in $V(I)$. Then the following statements are equivalent:*

(1) *$M$ is $I$-cofinite.*
(2) $J \subseteq \operatorname{Rad}(I + \mathfrak{p})$ *for each* $\mathfrak{p} \in \operatorname{Att}_R M$.
(3) $J \subseteq \operatorname{Rad}(I + \operatorname{Ann}_R M)$.

In this paper for any ideal $\mathfrak{b}$ of $R$, *the radical of* $\mathfrak{b}$ denoted by $\operatorname{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b} \text{ for some } n \in \mathbb{N}\}$. Also, the set $\{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{b}\}$ is denoted by $V(\mathfrak{b})$. For each Artinian $R$-module $M$, we denote by $\operatorname{Att}_R M$ the set of all attached prime ideals of $M$. Finally, for each $R$-module $M$, we show by $\Gamma_I(M)$ the submodule of $M$ consisting of all elements annihilated by some power of $I$, i.e., $\bigcup_{k=1}^{\infty}(0 :_M I^k)$. For any unexplained notation and terminology we refer the reader to [4, 8].

## 2. The results

Let $J$ be an ideal of a Noetherian ring $R$. In the sequel, we denote by $\widehat{R}^J$ the $J$-adic completion of $R$ which is defined as,

$$\widehat{R}^J = \varprojlim_{n\geq 1} R/J^n.$$

Recall that, $\widehat{R}^J$ is a $R$-flat Noetherian ring and its ideal $J\widehat{R}^J$ is contained in the Jacobson radical of $\widehat{R}^J$ (see [2]). Also, there is a natural homomorphism of rings $\tau : R \longrightarrow \widehat{R}^J$ with $\ker \tau = \bigcap_{n=1}^{\infty} J^n$

(see [2, Theorem 10.17]). This homomorphism is a monomorphism when the ideal $J$ is contained in the Jacobson radical of $R$. Note that by this homomorphism each $\widehat{R}^J$-module has an $R$-module structure. It is well-known that any $J$-torsion $R$-module $M$ enjoys a natural $\widehat{R}^J$-module structure in such a way that the lattices of its $R$-submodules and $\widehat{R}^J$-submodules coincide. Besides, we have

$$\widehat{R}^J \otimes_R M \simeq M$$

both as $R$-modules and $\widehat{R}^J$-modules. In general $\tau$ need not to be an isomorphism. In the case where $J$ is contained in the Jacobson radical of $R$, the following result shows that $\tau$ is an isomorphism if and only if $\widehat{R}^J$ is a finitely generated $R$-module.

**Theorem 2.1.** *Let $R$ be a Noetherian ring and $J$ be an ideal of $R$ contained in the Jacobson radical of $R$. Then the following statements are equivalent:*

(1) *The natural morphism $\tau : R \longrightarrow \widehat{R}^J$ is an isomorphism.*
(2) *$\widehat{R}^J$ is a finitely generated $R$-module.*

*Proof.* (1)$\Longrightarrow$(2) The assertion is clear.
(2)$\Longrightarrow$(1) Set $I := \ker \tau$. By [2, Theorem 10.17] we have $I = \bigcap_{n=1}^{\infty} J^n$. By *Krull's Intersection Theorem* we know that $I = JI$. Thus by *Nakayama's Lemma* it is concluded that $I = \{0\}$. Therefore, we may assume that $\tau$ is the inclusion map.

Since $R/J \simeq \widehat{R}^J/J\widehat{R}^J$, we see that $R \cap J\widehat{R}^J = J$. Thus, there are the following natural isomorphisms of $R$-modules:

$$\widehat{R}^J/J\widehat{R}^J \simeq R/J \simeq R/(R \cap J\widehat{R}^J) \simeq (R + J\widehat{R}^J)/J\widehat{R}^J.$$

Hence, the inclusion map $\iota : (R + J\widehat{R}^J)/J\widehat{R}^J \longrightarrow \widehat{R}^J/J\widehat{R}^J$ is an isomorphism of $R$-modules. Therefore,

$$(R + J\widehat{R}^J)/J\widehat{R}^J = \widehat{R}^J/J\widehat{R}^J.$$

This means that $R + J\widehat{R}^J = \widehat{R}^J$. Since by assumption $\widehat{R}^J$ is a finitely generated $R$-module and $J$ is contained in the Jacobson radical of $R$, by *Nakayama's Lemma* it is concluded that $R = \widehat{R}^J$. So, the map $\tau$ is an isomorphism. □

The following theorem is a generalization of Melkersson's result in [10].

**Theorem 2.2.** *Let $I$ be a proper ideal of a Noetherian ring $R$ and $M$ be a non-zero Artinian $R$-module with support contained in $V(I)$. Set $J := \cap_{\mathfrak{m} \in \operatorname{Supp} M} \mathfrak{m}$. Then the following statements are equivalent:*

(1) *$M$ is $I$-cofinite.*
(2) $\operatorname{Rad}(I\widehat{R}^J + \operatorname{Ann}_{\widehat{R}^J} M) = J\widehat{R}^J$.
(3) $J\widehat{R}^J \subseteq \operatorname{Rad}(I\widehat{R}^J + \mathfrak{P})$ *for each* $\mathfrak{P} \in \operatorname{Att}_{\widehat{R}^J} M$.

*Proof.* (1)$\Longleftrightarrow$(2) Since $M$ is Artinian $R$-module it is clear that $\operatorname{Supp} M$ is a finite subset of $\operatorname{Max}(R)$. Suppose that $\operatorname{Supp} M = \{\mathfrak{m}_1, ..., \mathfrak{m}_n\}$. For each $k \in \mathbb{N}$, the map

$$\varphi_k : R/J^k \longrightarrow \oplus_{i=1}^n R/\mathfrak{m}_i^k,$$

defined by $\varphi_k(r + J^k) = (r + \mathfrak{m}_1^k, ..., r + \mathfrak{m}_n^k)$, $(r + J^k \in R/J^k)$, is an isomorphism of $R$-modules. Because,

$$\begin{aligned} R/J^k &= R/(\mathfrak{m}_1 \mathfrak{m}_2 \cdots \mathfrak{m}_n)^k \\ &= R/\mathfrak{m}_1^k \mathfrak{m}_2^k \cdots \mathfrak{m}_n^k \\ &= R/\cap_{i=1}^n \mathfrak{m}_i^k \\ &\simeq \oplus_{i=1}^n R/\mathfrak{m}_i^k. \end{aligned}$$

By getting inverse limit from the besides we obtain the following isomorphism of the rings,

$$\widehat{R}^J \simeq \oplus_{i=1}^n \widehat{R}^{\mathfrak{m}_i}.$$

Now, assume that $\operatorname{Rad}(I\widehat{R}^J + \operatorname{Ann}_{\widehat{R}^J} M) = J\widehat{R}^J$. Then there is a positive integer $\ell$ such that $J^\ell \widehat{R}^J \subseteq I\widehat{R}^J + \operatorname{Ann}_{\widehat{R}^J} M$. Therefore,

$$\begin{aligned} (0 :_M I) &= (0 :_M I\widehat{R}^J + \operatorname{Ann}_{\widehat{R}^J} M) \\ &\subseteq (0 :_M J^\ell \widehat{R}^J) \\ &= (0 :_M J^\ell) \\ &\simeq \oplus_{i=1}^n (0 :_M \mathfrak{m}_i^\ell), \end{aligned}$$

and hence the $R$-module $(0 :_M I)$ is of finite length. So, $M$ is $I$-cofinite, by [9, Proposition 4.1].

In order to establish the reverse implication, assume that $M$ is $I$-cofinite. Set $M_i := \Gamma_{\mathfrak{m}_i}(M)$, for $i = 1, 2, ..., n$. Clearly,

$$M \simeq \oplus_{i=1}^n M_i.$$

Since for each $1 \leq i \leq n$ the $R$-module $M_i$ is $I$-cofinite and $\widehat{R}^{\mathfrak{m}_i}$ is a flat $R$-algebra, it follows that $M_i$ is also $I\widehat{R}^{\mathfrak{m}_i}$-cofinite. Consequently, by using [10, Theorem 1.6] it can be deduced that

$$\mathfrak{m}_i \widehat{R}^{\mathfrak{m}_i} = \operatorname{Rad}(I\widehat{R}^{\mathfrak{m}_i} + \operatorname{Ann}_{\widehat{R}^{\mathfrak{m}_i}} M_i) \text{ for } i = 1, 2, ..., n.$$

As $\widehat{R}^J \simeq \oplus_{i=1}^n \widehat{R}^{\mathfrak{m}_i}$, for each $1 \le i \le n$, there exists an ideal $L_i$ of $\widehat{R}^J$ such that $\widehat{R}^J/L_i \simeq \widehat{R}^{\mathfrak{m}_i}$ and

$$L_i \simeq \widehat{R}^{\mathfrak{m}_1} \oplus \cdots \oplus \widehat{R}^{\mathfrak{m}_{i-1}} \oplus \widehat{R}^{\mathfrak{m}_{i+1}} \oplus \cdots \oplus \widehat{R}^{\mathfrak{m}_n}.$$

For each $1 \le i \le n$, we have $\mathfrak{m}_i\, L_i = L_i$ since $\mathfrak{m}_i\, \widehat{R}^{\mathfrak{m}_j} = \widehat{R}^{\mathfrak{m}_j}$ for each $1 \le j \le n$ with $j \ne i$. In particular, if $i_1, i_2, ..., i_t$ are integers with

$$1 \le i_1 < i_2 < \cdots < i_t \le n,$$

then $L_{i_1} L_{i_2} \cdots L_{i_t} = \mathfrak{m}_{i_1}\, \mathfrak{m}_{i_2} \cdots \mathfrak{m}_{i_t}\, L_{i_1} L_{i_2} \cdots L_{i_t}$. Therefore,

$$(\mathfrak{m}_1\, \widehat{R}^J + L_1)(\mathfrak{m}_2\, \widehat{R}^J + L_2) \cdots (\mathfrak{m}_n\, \widehat{R}^J + L_n) = \mathfrak{m}_1\, \mathfrak{m}_2 \cdots \mathfrak{m}_n\, \widehat{R}^J = J\widehat{R}^J.$$

For each $1 \le i \le n$, the relation

$$\mathfrak{m}_i\, \widehat{R}^{\mathfrak{m}_i} = \operatorname{Rad}(I\widehat{R}^{\mathfrak{m}_i} + \operatorname{Ann}_{\widehat{R}^{\mathfrak{m}_i}} M_i),$$

implies that

$$(\mathfrak{m}_i\, \widehat{R}^J + L_i)/L_i = \operatorname{Rad}\big((I\widehat{R}^J + \operatorname{Ann}_{\widehat{R}^J} M_i)/L_i\big).$$

Thus, $(\mathfrak{m}_i\, \widehat{R}^J + L_i) = \operatorname{Rad}(I\widehat{R}^J + \operatorname{Ann}_{\widehat{R}^J} M_i)$ for each $1 \le i \le n$. Hence,

$$\operatorname{Rad}(I\widehat{R}^J + \cap_{i=1}^n \operatorname{Ann}_{\widehat{R}^J} M_i) = \cap_{i=1}^n (\mathfrak{m}_i\, \widehat{R}^J + L_i).$$

Clearly, $\cap_{i=1}^n \operatorname{Ann}_{\widehat{R}^J} M_i = \operatorname{Ann}_{\widehat{R}^J} M$. Also, for each $1 \le i \le n$ we have

$$\widehat{R}^J/(\mathfrak{m}_i\, \widehat{R}^J + L_i) \simeq \widehat{R}^{\mathfrak{m}_i}/\, \mathfrak{m}_i\, \widehat{R}^{\mathfrak{m}_i},$$

which shows that $(\mathfrak{m}_i\, \widehat{R}^J + L_i)$ is a maximal ideal of $\widehat{R}^J$. Therefore,

$$\bigcap_{i=1}^n (\mathfrak{m}_i\, \widehat{R}^J + L_i) = (\mathfrak{m}_1\, \widehat{R}^J + L_1)(\mathfrak{m}_2\, \widehat{R}^J + L_2) \cdots (\mathfrak{m}_n\, \widehat{R}^J + L_n) = J\widehat{R}^J.$$

So, $\operatorname{Rad}(I\widehat{R}^J + \operatorname{Ann}_{\widehat{R}^J} M) = J\widehat{R}^J$, as required.

(2)$\Longleftrightarrow$(3) The assertion easily follows from the assumption $J = \cap_{\mathfrak{m} \in \operatorname{Supp} M}\, \mathfrak{m}$. $\square$

Recall that a Noetherian ring is called *semi-local* if it has only finitely many maximal ideals. The following result is a generalization of [1, Theorem 2.2].

**Theorem 2.3.** *Let $R$ be a Noetherian semi-local ring with Jacobson radical $J$ such that $\widehat{R}^J$ is integral over $R$. Let $I$ be an ideal of $R$ contained in $J$ and $M$ be a non-zero Artinian $R$-module with support contained in $V(I)$. Then the following statements are equivalent:*

(1) *$M$ is $I$-cofinite.*
(2) *$J \subseteq \operatorname{Rad}(I + \mathfrak{p})$ for each $\mathfrak{p} \in \operatorname{Att}_R M$.*

(3) $J \subseteq \operatorname{Rad}(I + \operatorname{Ann}_R M)$.

*Proof.* (1)$\Longrightarrow$(2) By replacing $M$ with the $R$-module

$$M' := M \oplus (\oplus_{\mathfrak{m} \in \operatorname{Max}(R)} R/\mathfrak{m}),$$

we can make the additional assumption that $\operatorname{Supp} M = V(J)$. Now assume the opposite. Then, $J \not\subseteq \operatorname{Rad}(I + \mathfrak{q})$ for some $\mathfrak{q} \in \operatorname{Att}_R M$. By [4, Exercise 8.2.5] we have $\mathfrak{Q} \cap R = \mathfrak{q}$ for some $\mathfrak{Q} \in \operatorname{Att}_{\widehat{R}^J} M$. As $J \not\subseteq \operatorname{Rad}(I + \mathfrak{q})$, we can deduce that $J \not\subseteq \mathfrak{q}'$ for some $\mathfrak{q}' \in V(I + \mathfrak{q})$. Therefore, $\dim R/\mathfrak{q}' \geq 1$. Set $t := \operatorname{height}(\mathfrak{q}'/\mathfrak{q})$. Then we can find a chain of prime ideals of $R$ as,

$$\mathfrak{q} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_t = \mathfrak{q}'.$$

By [8, Theorem 4.9] there exists a chain of prime ideals of $\widehat{R}^J$ such as,

$$\mathfrak{Q} = \mathfrak{Q}_0 \subset \mathfrak{Q}_1 \subset \cdots \subset \mathfrak{Q}_t,$$

for which $\mathfrak{Q}_i \cap R = \mathfrak{q}_i$ for each $0 \leq i \leq t$.

Since $R/\mathfrak{q}' \subset \widehat{R}/\mathfrak{Q}_t$ is an integral extension, by using [8, Exercise 9.2] it can be seen that $\dim \widehat{R}^J/\mathfrak{Q}_t = \dim R/\mathfrak{q}' \geq 1$ and $J\widehat{R}^J \not\subseteq \mathfrak{Q}_t$. Since $\widehat{R}^J$ is an $R$-flat algebra, it is clear that $M$ is $I\widehat{R}^J$-cofinite as an $\widehat{R}^J$-module. Hence, Theorem 2.2. together with the assumption $\mathfrak{Q} \in \operatorname{Att}_{\widehat{R}^J} M$ implies that $J\widehat{R}^J \subseteq \operatorname{Rad}(I\widehat{R}^J + \mathfrak{Q})$. Clearly,

$$I\widehat{R}^J + \mathfrak{Q} \subseteq \mathfrak{Q}_t.$$

Thus, $J\widehat{R}^J \subseteq \mathfrak{Q}_t$, which is a contradiction.

(2)$\Longleftrightarrow$(3) The assertion is trivial.

(3)$\Longrightarrow$(1) The assertion follows by the proof of [1, Theorem 2.2]. □

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**Gholamreza Pirmohammadi**
Payame Noor University, P.O. Box 19395-3697, Tehran, Iran.
Email: gh_pirmohammadi@pnu.ac.ir

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G. PIRMOHAMMADI

نتایجی پیرامون مدول‌های هم‌متناهی آرتینی

غلامرضا پیرمحمدی

دانشگاه پیام نور، تهران، ایران

فرض کنید $R$ یک حلقه نوتری و $I$ ایده‌آلی از آن باشد. به علاوه، فرض کنید $M$ یک $R$–مدول آرتینی ناصفر با محمل مشمول در $V(I)$ باشد. در این مقاله نشان می دهیم $M$ یک $R$–مدول $I$–هم‌متناهی است اگر و فقط اگر $\mathrm{Rad}(I\widehat{R}^J + \mathrm{Ann}_{\widehat{R}^J} M) = J\widehat{R}^J$، که در آن $J := \cap_{\mathfrak{m}\in \mathrm{Supp}\, M}\mathfrak{m}$ و $\widehat{R}^J$ نشانگر تتمیم $R$ نسبت به توپولوژی $I$–ادیک است.

کلمات کلیدی: مدول آرتینی، مدول هم‌متناهی، حلقه نوتری، ایده‌آل اول چسبیده.