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(ANTI) FUZZY IDEALS OF SHEFFER STROKE BCK-ALGEBRAS

T. ONER, T. KALKAN AND A. BORUMAND SAEID*

Abstract. This study aims to introduce the concept of (anti) fuzzy ideals of a Sheffer stroke BCK-algebra. After describing an anti fuzzy subalgebra and an anti fuzzy (sub-implicative) ideal of a Sheffer stroke BCK-algebra, the relationships of these notions are demonstrated. Also, a t-level cut and complement of a fuzzy subset are defined, and some of the properties are investigated. An implicative Sheffer stroke BCK-algebra is defined, and it is proved that a fuzzy subset of an implicative Sheffer stroke BCK-algebra is an anti fuzzy ideal if and only if it is an anti fuzzy sub-implicative ideal of this algebraic structure. A fuzzy congruence and quotient of Sheffer stroke BCK-algebra are studied in detail, and it is shown that there is a bijection between the set of fuzzy ideals and the set of fuzzy congruences on this algebraic structure.

1. Introduction

The notion of BCK-algebra was initiated by Imai and Iséki [7] in 1966. It is derived from two different ways: one of them is based on set theory; another is from classical and nonclassical propositional calculi. The BCK-operator $*$ is an analog of the set-theoretical difference. There is a close relationship between the notions of the set difference in set theory and the implication functor in logical systems.

Fuzzy set theory, which was introduced by Zadeh [17], has been known as a generalization of the ordinary set theory. Thus, fuzzy sets

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*Corresponding author.

have been applied to other algebraic structures such as semigroups, groups, rings, modules, vector spaces, and topologies. For example, the anti fuzzy subgroups of groups were introduced in [4]. In what follows, fuzzy sets [16] and anti fuzzy ideals [6] were applied to BCK-algebras. For further reading, the references [1, 3, 2] are suggested.

On the other hand, Sheffer stroke (or Sheffer operation), which was introduced by Sheffer [15], can only be used by itself to build a logical formal system. Since this operation reduces the number of axioms of many algebraic structures, studying the systems with the Sheffer operation is cheaper and easily controllable. The well-known examples are Boolean algebras [8] and ortholattices [5]. In recent times, it should be pointed out that the Sheffer operation has been applied to algebraic structures such as interval Sheffer stroke basic algebras [12], Sheffer stroke Hilbert algebras [10], filters of strong Sheffer stroke nonassociative MV-algebras [11], Sheffer stroke BG-algebras [14], and (fuzzy) filters of Sheffer stroke BL-algebras [9] as the corresponding literature.

In the next section, essentials on Sheffer stroke BCK-algebras are given. In the third section, the plan of the main results and outcomes of the study are presented with illustrative examples. The results of the study are new and original, thus, contributing to the ongoing theory of pure mathematics regarding Sheffer stroke algebraic structures.

2. Preliminaries

In this section, we give the fundamental concepts about a Sheffer stroke and a Sheffer stroke BCK-algebra.

Definition 2.1. [5] Let $\mathcal{A} = (A, |)$ be a groupoid. The operation $|$ is said to be Sheffer stroke if it satisfies the following conditions:

(S1) $x|y = y|x$,
(S2) $(x|x)|(x|y) = x$,
(S3) $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$,
(S4) $(x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x$.

Lemma 2.2. [5] *Let $\mathcal{A} = (A, |)$ be a groupoid. The binary relation $\leq$, defined on A by*

$$x \leq y \quad \Leftrightarrow \quad x|y = x|x,$$

is an order on A.

Definition 2.3. [13] A Sheffer stroke BCK-algebra is an algebra $(A, |, 0)$ of type $(2, 0)$ satisfying the following conditions:
(sBCK-1):

$$((((x|(y|y))|(x|(y|y)))|(x|(z|z)))|(((x|(y|y))|(x|(y|y)))|(x|(z|z))))|(z|(y|y)) = 0|0$$

for all $x, y, z \in A$.

(sBCK-2): $(x|(y|y))|(x|(y|y)) = 0$ and $(y|(x|x))|y|(x|x)) = 0$ imply $x = y$ for all $x, y \in A$.

A partial order $\leq$ on A can be defined by $x \leq y$ if and only if $(x|(y|y))|(x|(y|y)) = 0$.

Example 2.4. [13] Consider $(A, |, 0)$ with the Hasse diagram (see Figure 1) where $A = \{0, x, y, z, 1\}$.

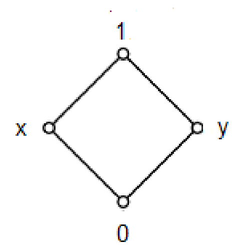

FIGURE 1. Hasse diagram of $(A, |, 0)$

The binary operation | has the Cayley table; see Table 1.

TABLE 1. Cayley table of Example 2.4

$\vert$	0	x	y	1
0	1	1	1	1
x	1	y	1	y
y	1	1	x	x
1	1	y	x	0

Then $(A, |, 0)$ is a Sheffer stroke BCK-algebra.

Lemma 2.5. [13] *Let A be a Sheffer stroke BCK-algebra. Then the following features hold for all $x, y, z \in A$:*

(1) $(x|(x|x))|(x|x) = x$,
(2) $(x|(x|x))|(x|(x|x)) = 0$,
(3) $x|(((x|(y|y))|(y|y))|((x|(y|y))|(y|y))) = 0|0$,
(4) $(0|0)|(x|x) = x$,
(5) $x|0 = 0|0$,
(6) $(x|(0|0))|(x|(0|0)) = x$,
(7) $(0|(x|x))|(0|(x|x)) = 0$,
(8) $x|((y|(z|z))|(y|(z|z))) = y|((x|(z|z))|(x|(z|z)))$,

(9) $(x|((y|(z|z))|(y|(z|z))))|((y|(x|(z|z))|(x|(z|z)))|(y|(x|(z|z))$
$|(x|(z|z)))) = 0|0$,

(10) $((x|(x|(y|y)))|(x|(x|(y|y))))|(y|y) = 0|0$.

Let A be a Sheffer stroke BCK-algebra. Then $1 = 0|0$ is the greatest element, and $0 = 1|1$ is the least element of A.

Proposition 2.6. [13] *Let $(A, |, 0)$ be a Sheffer stroke BCK-algebra. Then the following features hold for all $x, y, z \in A$:*

(i) $x \leq z$ *implies* $(y|(z|z))|(y|(z|z)) \leq (y|(x|x))|(y|(x|x))$,
(ii) $((x|(y|y))|(x|(y|y)))|(z|z) = ((x|(z|z))|(x|(z|z)))|(y|y)$,
(iii) $((x|(y|y))|(x|(y|y))) \leq z \Leftrightarrow ((x|(z|z))|(x|(z|z))) \leq y$,
(iv) $(x|(y|y))|(x|(y|y)) \leq x$,
(v) $x \leq y|(x|x)$,
(vi) $x \leq (x|(y|y))|(y|y)$,
(vii) *If* $x \leq y$, *then* $z|(x|x) \leq z|(y|y)$.

Let A be a Sheffer stroke BCK-algebra, unless otherwise indicated.

Definition 2.7. A nonempty subset I of A is called an ideal of A, if it satisfies

(I1) $0 \in I$,
(I2) $(x|(y|y))|(x|(y|y)) \in I$ and $y \in I$ imply $x \in I$.

Definition 2.8. A nonempty subset I of A is called an implicative ideal of A, if it satisfies
(i) $0 \in I$,
(ii)
$(((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))|(((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z)) \in I$
and $z \in I$ imply $x \in I$.

Definition 2.9. A nonempty subset I of A is called a sub-implicative ideal of A if it satisfies
(i) $0 \in I$,
(ii)
$(((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))|(((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z)) \in I$
and $z \in I$ imply $(y|(y|(x|x)))|(y|(y|(x|x))) \in I$.

Definition 2.10. Let S be a nonempty set. A fuzzy subset μ of S is a function $\mu : S \to [0, 1]$.

3. (Anti) Fuzzy Ideals of Sheffer stroke BCK-Algebras

Definition 3.1. A fuzzy subset μ of A is called a fuzzy subalgebra of A if $\mu(x|(y|y))|(x|(y|y)) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in A$.

Definition 3.2. Let A be a Sheffer stroke BCK-algebra.

(a) A fuzzy subset μ of A is called a fuzzy ideal of A if

- $\mu(0) \geq \mu(x)$ for all $x \in A$,
- $\mu(x) \geq \min\{\mu(x|(y|y))|(x|(y|y)), \mu(y)\}$ for all $x, y \in A$.

(b) A fuzzy subset μ of A is called a fuzzy sub-implicative ideal of A if

- $\mu(0) \geq \mu(x)$ for all $x \in A$,
-

$$\mu((y|(y|(x|x)))|(y|(y|(x|x)))) \geq \min\{\mu((((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))|(((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))), \mu(z)\}$$

for all $x, y, z \in A$.

(c) A fuzzy subset μ of A is called an anti fuzzy subalgebra of A if

- $\mu((x|(y|y))|(x|(y|y))) \leq \max\{\mu(x), \mu(y)\}$ for all $x, y \in A$.

Example 3.3. Consider the Sheffer stroke BCK-algebra $(A, |, 0)$ in Example 2.4. Let $t_0, t_1 \in [0, 1]$ such that $t_0 \leq t_1$. Define $A \to [0, 1]$ by $\mu(0) = t_0$, $\mu(x) = \mu(y) = \mu(1) = t_1$. Then μ is an anti fuzzy subalgebra of A.

Proposition 3.4. *If μ is an anti fuzzy subalgebra of A, then $\mu(0) \leq \mu(x)$ for all $x \in A$.*

Proof. From Lemma 2.5 (2), we have $(x|(x|x))|(x|(x|x)) = 0$. Hence

$$\mu(0) = \mu((x|(x|x))|(x|(x|x))) \leq \max\{\mu(x), \mu(x)\} = \mu(x).$$

□

Definition 3.5. A fuzzy subset μ of A is called an anti fuzzy ideal of A if it satisfies

(i) $\mu(0) \leq \mu(x)$ for all $x \in A$.

(ii) $\mu(x) \leq \max\{\mu(x|(y|y))|(x|(y|y)), \mu(y)\}$ for all $x, y \in A$.

Example 3.6. Consider the fuzzy subset μ of A in Example 3.3. Then it is obvious that μ is an anti fuzzy ideal of A.

Theorem 3.7. (i) *Every anti fuzzy ideal of A is order preserving.*

(ii) *Every anti fuzzy ideal of A is an anti fuzzy subalgebra of A.*

Proof. (i) Let μ be an anti fuzzy ideal of A and let $x, y \in A$ such that $x \leq y$. Then

$$\begin{aligned}\mu(x) &\leq \max\{\mu(x|(y|y))|(x|(y|y)), \mu(y)\}\\ &= \max\{\mu(0), \mu(y)\}\\ &= \mu(y).\end{aligned}$$

(ii) Since $(x|(y|y))|(x|(y|y)) \leq x$ by Proposition 2.6 (iv), it follows from (i) that $\mu(x|(y|y))|(x|(y|y)) \leq \mu(x)$. By Definition 3.5 (ii),

$$\begin{aligned}\mu(x|(y|y))|(x|(y|y)) &\leq \mu(x)\\ &\leq \max\{\mu(x|(y|y))|(x|(y|y)), \mu(y)\}\\ &\leq \max\{\mu(x), \mu(y)\}.\end{aligned}$$

Therefore, μ is an anti fuzzy subalgebra of A. □

An anti fuzzy subalgebra of A may not be an anti fuzzy ideal of A as shown in the following example.

Example 3.8. Consider the Sheffer stroke BCK-algebra $(A, |, 0)$ in Example 2.4. Let $t_0, t_1, t_2 \in [0, 1]$ such that $t_0 \leq t_1 \leq t_2$. Define $A \to [0, 1]$ by $\mu(0) = t_0$, $\mu(x) = \mu(y) = t_1$, and $\mu(1) = t_2$. Then μ is an anti fuzzy subalgebra of A, but not an anti fuzzy ideal of A since $\mu(1) = t_2 > t_1 = \max\{\mu(1|(x|x))|(1|(x|x)), \mu(x)\}$.

Proposition 3.9. *Let μ be an anti fuzzy ideal of A. If $(x|(y|y))|(x|(y|y)) \leq z$ holds in A, then $\mu(x) \leq \max\{\mu(y), \mu(z)\}$ for all $x, y, z \in A$.*

Proof. Assume that $(x|(y|y))|(x|(y|y)) \leq z$ holds in A. Then

$$\begin{aligned}\mu(x|(y|y))|(x|(y|y)) &\leq \max\{(((x|(y|y))|(x|(y|y)))|(z|z))|\\ &\qquad (((x|(y|y))|(x|(y|y)))|(z|z)), \mu(z)\}\\ &= \max\{\mu(0), \mu(z)\}\\ &= \mu(z)\end{aligned}$$

from Definition 3.5 (ii). Thus,

$$\mu(x) \leq \max\{\mu(x|(y|y))|(x|(y|y)), \mu(y)\} \leq \max\{\mu(z), \mu(y)\}.$$

□

Proposition 3.10. *A fuzzy subset μ of A is a fuzzy ideal of A if and only if its complement μ^c is an anti fuzzy ideal of A.*

Proof. Let μ be a fuzzy ideal of A and let $x, y \in A$. Then $\mu^c(0) = 1 - \mu(0) \leq 1 - \mu(x) = \mu^c(x)$ and

$$\begin{aligned}\mu^c(x) &= 1 - \mu(x)\\ &\leq 1 - \min\{\mu(x|(y|y))|(x|(y|y)), \mu(y)\}\\ &= 1 - \min\{1 - \mu^c(x|(y|y))|(x|(y|y)), 1 - \mu^c(y)\}\\ &= \max\{\mu^c(x|(y|y))|(x|(y|y)), \mu^c(y)\}.\end{aligned}$$

Thus, μ^c is an anti fuzzy ideal of A. Similarly, the converse can be proved. □

Theorem 3.11. *Let μ be an anti fuzzy ideal of A. Then the set $A_\mu := \{x \in A | \mu(x) = \mu(0)\}$ is an ideal of A.*

Proof. Clearly, $0 \in A_\mu$. Let $x, y \in A$ such that $(x|(y|y))|(x|(y|y)) \in A_\mu$ and $y \in A_\mu$. Then $\mu(x|(y|y))|(x|(y|y)) = \mu(0) = \mu(y)$. From Definition 3.5 (ii), we get

$$\mu(x) \leq \max\{\mu(x|(y|y))|(x|(y|y)), \mu(y)\} = \mu(0).$$

We obtain from Definition 3.5 (i) that $\mu(x) = \mu(0)$, and hence $x \in A_\mu$. □

Definition 3.12. Let μ be a fuzzy subset of A. Then, for $t \in [0, 1]$, the t-level cut of μ is the set $\mu_t := \{x \in A | \mu(x) \geq t\}$. The lower t-level cut of μ is the set $\mu^t := \{x \in A | \mu(x) \leq t\}$. Clearly, $\mu^1 = A$ and $\mu_t \cup \mu^t = A$ for $t \in [0, 1]$. If $t_1 \leq t_2$, then $\mu^{t_1} \subseteq \mu^{t_2}$.

Theorem 3.13. *Let μ be a fuzzy subset of A. Then μ is an anti fuzzy ideal of A if and only if, for each $t \in [0, 1]$ with $t \geq \mu(0)$, the lower cut μ^t is an ideal of A.*

Proof. Let μ be an anti fuzzy ideal of A and let $t \in [0, 1]$ with $t \geq \mu(0)$. Clearly, $0 \in \mu^t$. Let $x, y \in A$ such that $(x|(y|y))|(x|(y|y)) \in \mu^t$ and $y \in \mu^t$. Then $\mu(x) \leq \max\{\mu(x|(y|y))|(x|(y|y)), \mu(y)\} \leq t$, and so $x \in \mu^t$. Hence μ^t is an ideal of A.

Conversely, we show that $\mu(0) \leq \mu(x)$ for all $x \in A$. If not, then there exists $x_0 \in A$ such that $\mu(0) > \mu(x_0)$. Putting $t_0 := \frac{1}{2}\{\mu(0) + \mu(x_0)\}$, then $0 \leq \mu(x_0) < t_0 < \mu(0) \leq 1$. It follows that $x_0 \in \mu^{t_0}$, so that $\mu^{t_0} \neq \emptyset$. Indeed μ^{t_0} is an ideal of A. Then $0 \in \mu^{t_0}$ or $\mu(0) \leq t_0$, a contradiction. Hence $\mu(0) \leq \mu(x)$ for all $x \in A$. Now, we prove that

$$\mu(x) \leq \max\{\mu(x|(y|y))|(x|(y|y)), \mu(y)\}$$

for all $x, y \in A$. If not, then there exist $x_0, y_0 \in A$ such that

$$\mu(x_0) > \max\{\mu(x_0|(y_0|y_0))|(x_0|(y_0|y_0)), \mu(y_0\}.$$

If

$$s_0 := \tfrac{1}{2}\{\mu(x_0) + \max\{\mu(x_0|(y_0|y_0))|(x_0|(y_0|y_0)), \mu(y_0\},$$

then $s_0 < \mu(x_0)$ and

$$0 \leq \max\{\mu(x_0|(y_0|y_0))|(x_0|(y_0|y_0)), \mu(y_0)\} < s_0 \leq 1.$$

Thus, we have $s_0 > \mu(x_0|(y_0|y_0))|(x_0|(y_0|y_0))$ and $s_0 > \mu(y_0)$, which imply that $(x_0|(y_0|y_0))|(x_0|(y_0|y_0)) \in \mu^{s_0}$ and $y_0 \in \mu^{s_0}$. Since μ^{s_0} is an ideal of A, it follows that $x_0 \in \mu^{s_0}$ or $\mu(x_0) \leq s_0$. This is a contradiction. Therefore, μ is an anti fuzzy ideal of A. □

Definition 3.14. Let μ be a fuzzy subset of A. The fuzzification of μ^t, $t \in [0,1]$, is the fuzzy subset λ_{μ^t} of A defined by

$$\lambda_{\mu^t}(x) := \begin{cases} \mu(x) & if x \in \mu^t, \\ 0 & otherwise. \end{cases}$$

It can be easily verified that $\lambda_{\mu^t} \leq \mu$, that is, $\lambda_{\mu^t}(x) \leq \mu(x)$ for all $x \in A$, and $(\lambda_{\mu^t})^t = \mu^t$.

Example 3.15. Consider the Sheffer stroke BCK-algebra $(A, |, 0)$ in Example 2.4. Let $t_0, t_1, t_2 \in [0,1]$ such that $t_0 \leq t_1 \leq t_2$. Define $A \to [0,1]$ by $\mu(0) = t_0$, $\mu(x) = \mu(y) = t_1$ and $\mu(1) = t_2$. Then $\mu^{t_0} = \{0\}$, $\mu^{t_1} = \{0, x, y\}$, and $\mu^{t_2} = \{0, x, y, 1\}$. Hence,

$$\lambda_{\mu^{t_1}}(0) := \mu(0) = t_0,\ \lambda_{\mu^{t_1}}(x) := \mu(x) = t_1,\ \lambda_{\mu^{t_1}}(y) := \mu(y) = t_1,$$

and $\lambda_{\mu^{t_1}}(1) := 0$. It can be easily proved that

$$\lambda_{\mu^{t_2}}(0) := \mu(0) = t_0,$$

that $\lambda_{\mu^{t_2}}(x) := \mu(x) = t_1$, that $\lambda_{\mu^{t_2}}(y) := \mu(y) = t_1$, and that $\lambda_{\mu^{t_2}}(1) := \mu(1) = t_2$.

Theorem 3.16. *If μ is an anti fuzzy ideal of A, then λ_{μ^t} is also an anti fuzzy ideal of A, where $t \in [0,1]$ and $t \geq \mu(0)$.*

Proof. By Theorem 3.13, it is sufficient to show that $(\lambda_{\mu^t})^s$ is an ideal of A, where $s \in [0,1]$ and $s \geq \lambda_{\mu^t}(0)$. Clearly, $0 \in (\lambda_{\mu^t})^s$. Let $x, y \in A$ such that $((x|(y|y))|(x|(y|y))), y \in (\lambda_{\mu^t})^s$. We claim that $x \in (\lambda_{\mu^t})^s$ or that $(\lambda_{\mu^t})(x) \leq s$. If $((x|(y|y))|(x|(y|y))), y \in \mu^t$, then $x \in \mu^t$ because μ^t is an ideal of A. Hence,

$$\begin{aligned} \lambda_{\mu^t} &= \mu(x) \\ &\leq \max\{\mu(x|(y|y))|(x|(y|y)), \mu(y)\} \\ &= \max\{\lambda_{\mu^t}(x|(y|y))|(x|(y|y)), \lambda_{\mu^t}(y)\} \\ &\leq s. \end{aligned}$$

Therefore, $x \in (\lambda_{\mu^t})^s$. If $(x|(y|y))|(x|(y|y)) \notin \mu^t$ or $y \in \mu^t$, then $\lambda_{\mu^t}(x) \leq s$. Thus, $x \in (\lambda_{\mu^t})^s$. Therefore, $(\lambda_{\mu^t})^s$ is an ideal of A. □

Definition 3.17. A fuzzy subset μ of A is called an anti fuzzy implicative ideal of A if it satisfies:

(i) $\mu(0) \leq \mu(x)$ for all $x \in A$,

(ii)

$$\mu(x) \leq \max\{\mu(z), \mu((((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))| (((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z)))\}$$

for all $x, y, z \in A$.

Example 3.18. Consider the Sheffer stroke BCK-algebra $(A, |, 0)$ in Example 2.4. Let $t_0, t_1 \in [0, 1]$ such that $t_0 \leq t_1$. Define $A \to [0, 1]$ by $\mu(0) = t_0$ and $\mu(x) = \mu(y) = \mu(1) = t_1$. Then μ is an anti fuzzy implicative ideal of A.

Definition 3.19. A fuzzy subset μ of A is called an anti fuzzy sub-implicative ideal of A if it satisfies

(i) $\mu(0) \leq \mu(x)$ for all $x \in A$,

(ii)

$$\mu((y|(y|(x|x)))|(y|(y|(x|x)))) \leq \max\{\mu((((x|(x|(y|y)))|(x|(x|(y|y))))| (z|z))|(((x|(x|(y|y)))| (x|(x|(y|y))))|(z|z))), \mu(z)\}$$

for all $x, y, z \in A$.

Example 3.20. Consider the Sheffer stroke BCK-algebra $(A, |, 0)$ in Example 2.4. Let $t_0, t_1 \in [0, 1]$ such that $t_0 \leq t_1$. Define $A \to [0, 1]$ by $\mu(0) = t_0$ and $\mu(x) = \mu(y) = \mu(1) = t_1$. Then μ is an anti fuzzy sub-implicative ideal of A.

Proposition 3.21. (i) *Every anti fuzzy sub-implicative ideal of A is order preserving.*

(ii) *Every anti fuzzy sub-implicative ideal of A is an anti fuzzy ideal of A.*

(iii) *Let μ be an anti fuzzy ideal of A. If*

$$\mu((y|(y|(x|x)))|(y|(y|(x|x)))) \leq \mu((x|(x|(y|y)))|(x|(x|(y|y)))) \tag{3.1}$$

holds for all $x, y \in A$, then μ is an anti fuzzy sub-implicative ideal of A.

Proof. (i) Let μ be an anti fuzzy sub-implicative ideal of A and let $x, y, z \in A$ such that $x \leq z$. Let $[y := x]$ in Definition 3.19 (ii), we get

$$\begin{aligned}\mu(x) &\leq \max\{\mu((((x|(x|(x|x)))|(x|(x|(x|x))))|(z|z))| (((x|(x|(x|x)))|(x|(x|(x|x))))|(z|z))), \mu(z)\} \\ &= \max\{\mu(x|(z|z))|(x|(z|z)), \mu(z)\} \\ &= \max\{\mu(0), \mu(z)\} \\ &= \mu(z).\end{aligned}$$

(ii) Let μ be an anti fuzzy sub-implicative ideal of A. Then $\mu(0) \leq \mu(x)$ from Definition 3.19 (i). Putting $[y := x]$ in Definition 3.19 (ii), using Definition 2.1 (S2), Lemma 2.5 (2) and (6), and Definition 3.5, we get

$$\begin{aligned}\mu(x) &\leq \max\{\mu((((x|(x|(x|x)))|(x|(x|(x|x))))|(z|z))|\\ &\qquad (((x|(x|(x|x)))|(x|(x|(x|x))))|(z|z))), \mu(z)\}\\ &= \max\{\mu((x|(z|z))|(x|(z|z))), \mu(z)\}.\end{aligned}$$

Therefore, μ is an anti fuzzy ideal of A.

(iii) Let μ be an anti fuzzy ideal of A satisfying the inequality (3.1). By using Definition 3.5 (ii), we have

$$\begin{aligned}\mu((y|(y|(x|x)))|(y|(y|(x|x)))) &\leq \mu((x|(x|(y|y)))|(x|(x|(y|y))))\\ &\leq \max\{\mu(((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))|\\ &\qquad (((x|(x|(y|y)))|(x|(x|(y|y))))|\\ &\qquad (z|z)), \mu(z)\}.\end{aligned}$$

Therefore, μ is an anti fuzzy sub-implicative ideal of A. □

Theorem 3.22. *Every anti fuzzy sub-implicative ideal of A is an anti fuzzy subalgebra of A.*

Proof. Let μ is an anti fuzzy sub-implicative ideal of A. Putting $[y := x]$ in Definition 3.19 (ii), we get

$$\begin{aligned}\mu((x|(x|(x|x)))|(x|(x|(x|x)))) &= \mu(x)\\ &\leq \max\{\mu((((x|(x|(x|x)))|(x|(x|(x|x))))\\ &\qquad |(z|z))|(((x|(x|(x|x)))\\ &\qquad |(x|(x|(x|x))))|(z|z))), \mu(z)\}\\ &= \max\{\mu(x|(z|z))|(x|(z|z)), \mu(z)\},\end{aligned}$$

which implies that

$$\begin{aligned}\mu(x|(z|z))|(x|(z|z)) &\leq \max\{\mu(((x|(z|z))|(x|(z|z)))|\\ &\qquad (z|z))|(((x|(z|z))|(x|(z|z)))|(z|z)), \mu(z)\}\end{aligned}$$

Since

$$\begin{aligned}(((x|(z|z))|(x|(z|z)))|(z|z))|(((x|(z|z))|(x|(z|z)))|(z|z)) &\leq (x|(z|z))|(x|(z|z))\\ &\leq x\end{aligned}$$

from Proposition 2.6 (iv), it follows from Proposition 3.21 (i) that

$$\mu(((x|(z|z))|(x|(z|z)))|(z|z))|(((x|(z|z))|(x|(z|z)))|(z|z)) \leq \mu(x).$$

Hence, $\mu(x|(z|z))|(x|(z|z)) \leq \max\{\mu(x), \mu(z)\}$. Therefore, μ is an anti fuzzy subalgebra of A. □

An anti fuzzy subalgebra of A may not be an anti fuzzy sub-implicative ideal of A as shown in the following example.

Example 3.23. Consider the Sheffer stroke BCK-algebra $(A, |, 0)$ in Example 2.4. Let $t_0, t_1, t_2 \in [0, 1]$ such that $t_0 \leq t_1 \leq t_2$. Define $A \to [0, 1]$ by $\mu(0) = t_0$, $\mu(x) = \mu(y) = t_1$, and $\mu(1) = t_2$. Therefore, μ is an anti fuzzy subalgebra of A, but not an anti fuzzy sub-implicative ideal of A since

$$\begin{aligned} \mu(1|(1|(1|1)))|(1|(1|(1|1))) &= \mu(1) \\ &= t_2 > t_1 \\ &= \max\{\mu(((1|(1|(1|1)))|(1|(1|(1|1)))) \\ &\qquad |(x|x))|(((1|(1|(1|1)))| \\ &\qquad (1|(1|(1|1))))|(x|x)), \mu(x)\}. \end{aligned}$$

Definition 3.24. A Sheffer stroke BCK-algebra is said to be implicative if it satisfies the condition

$$x|(x|(y|y)) = y|(y|(x|x)) \qquad \text{for all } x, y \in A.$$

Theorem 3.25. *Let A be an implicative Sheffer stroke BCK-algebra. Then every anti fuzzy ideal of A is an anti fuzzy sub-implicative ideal of A.*

Proof. Let μ be an anti fuzzy ideal of A. (i) $\mu(0) \leq \mu(x)$,
(ii) Since

$$\begin{aligned} \mu((y|(y|(x|x)))|(y|(y|(x|x)))) &\leq \max\{\mu((((y|(y|(x|x)))| \\ &\qquad (y|(y|(x|x))))|(z|z))| \\ &\qquad (((y|(y|(x|x)))|(y|(y|(x|x)))) \\ &\qquad |(z|z))), \mu(z)\} \\ &= \max\{\mu((((x|(x|(y|y)))|(x|(x|(y|y)))) \\ &\qquad |(z|z))|(((x|(x|(y|y))) \\ &\qquad |(x|(x|(y|y))))|(z|z)))), \mu(z)\}, \end{aligned}$$

μ is an anti fuzzy sub-implicative ideal of A. □

By applying Proposition 3.21 (ii) and Theorem 3.25, we have the following Theorem.

Theorem 3.26. *Let A be an implicative Sheffer stroke BCK-algebra. Then a fuzzy set μ is an anti fuzzy ideal of A if and only if it is an anti fuzzy sub-implicative ideal of A.*

Theorem 3.27. *For any anti fuzzy sub-implicative ideal μ of a Sheffer stroke BCK-algebra A, the set*

$$A_\mu = \{x \in A | \mu(x) = \mu(0)\}$$

is a sub-implicative ideal of A.

Proof. Clearly, $0 \in A_\mu$. Let $x, y, z \in A$ such that

$$(((x|(x|(y|y)))|(x|(x|(y|y))))|(z|z))|(((x|(x|(y|y)))| (x|(x|(y|y))))|(z|z)) \in A_\mu$$

and $z \in A_\mu$. By Definition 3.19 (ii), we have

$$\begin{aligned} \mu((y|(y|(x|x)))|(y|(y|(x|x)))) &\leq \max\{\mu(((x|(x|(y|y)))|(x|(x|(y|y))))| \\ &\qquad (z|z))|(((x|(x|(y|y))) \\ &\qquad |(x|(x|(y|y))))|(z|z)), \mu(z)\} \\ &= \mu(0), \end{aligned}$$

which implies from Definition 3.19 (i) that

$$\mu((y|(y|(x|x)))|(y|(y|(x|x)))) = \mu(0).$$

Then $((y|(y|(x|x)))|(y|(y|(x|x)))) \in A_\mu$. Therefore, A_μ is a sub-implicative ideal of A. □

Proposition 3.28. *A fuzzy subset μ of A is a fuzzy sub-implicative ideal of A if and only if its complement μ^c is an anti fuzzy sub-implicative ideal of A.*

Proof. Let μ be a fuzzy sub-implicative ideal of A and let $x, y, z \in A$. Then $\mu^c(0) = 1 - \mu(0) \leq 1 - \mu(x) = \mu^c(x)$ and

$$\begin{aligned} \mu^c((y|(y|(x|x)))|(y|(y|(x|x)))) &= 1 - \mu((y|(y|(x|x)))|(y|(y|(x|x)))) \\ &\leq 1 - \min\{\mu(((x|(x|(y|y)))| \\ &\qquad (x|(x|(y|y))))|(z|z))| \\ &\qquad (((x|(x|(y|y)))|(x|(x|(y|y)))) \\ &\qquad |(z|z)), \mu(z)\} \\ &= 1 - \min\{1 - \mu^c(((x|(x|(y|y)))| \\ &\qquad (x|(x|(y|y))))|(z|z))| \\ &\qquad (((x|(x|(y|y)))|(x|(x|(y|y)))) \\ &\qquad |(z|z)), 1 - \mu^c(z)\} \\ &= \max\{\mu^c(((x|(x|(y|y)))|(x|(x|(y|y)))) \\ &\qquad |(z|z))|(((x|(x|(y|y)))| \\ &\qquad (x|(x|(y|y))))|(z|z)), \mu^c(z)\}. \end{aligned}$$

Thus, μ^c is an anti fuzzy sub-implicative ideal of A. Similarly, the converse can be proved. □

4. Homomorphism of Sheffer Stroke BCK-Algebra

Definition 4.1. Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Sheffer stroke BCK-algebras. A mapping $f : A \to B$ is called a homomorphism if

$$f(x|_A y) = f(x)|_B f(y) \qquad \text{for all } x, y \in A.$$

Theorem 4.2. *Let f be a homomorphism of Sheffer stroke BCK-algebra A into a Sheffer stroke BCK-algebra B.*

(i) *If 0 is the identity in A, then $f(0)$ is the identity in B.*
(ii) *If X is a sub-implicative ideal of A, then $f(X)$ is a sub-implicative ideal of B.*
(iii) *If Y is a sub-implicative ideal of B, then $f^{-1}(Y)$ is a sub-implicative ideal of A.*
(iv) *If A is an implicative Sheffer stroke BCK-algebra, then* $\ker f$ *is a sub-implicative ideal of A.*

Proof. (i) By using Definitions 2.3 and 4.1, we get

$$\begin{aligned} f(0) &= f((0|_A(0|_A 0))|_A(0|_A(0|_A 0))) \\ &= ((f(0)|_B(f(0)|_B f(0)))|_B(f(0)|_B(f(0)|_B f(0)))) \\ &= 0_B. \end{aligned}$$

(ii) Let X be a sub-implicative ideal of A. Clearly, $0_B \in f(X)$. Let

$$\begin{aligned} &((((f(x)|_B(f(x)|_B(f(y)|_B(f(y)))))|_B \\ &(f(x)|_B(f(x)|_B(f(y)|_B(f(y))))))|_B(f(z)|_B f(z)))|_B \\ &(((f(x)|_B(f(x)|_B(f(y)|_B(f(y)))))|_B(f(x)|_B \\ &(f(x)|_B(f(y)|_B(f(y))))))|_B(f(z)|_B f(z)))) \in f(X) \end{aligned}$$

and let $f(z) \in f(X)$. Since f is a homomorphism, we obtain

$$\begin{aligned} &((((x|_A(x|_A(y|_A y)))|_A(x|_A(x|_A(y|_A y))))|_A(z|_A z))|_A \\ &(((x|_A(x|_A(y|_A y)))|_A(x|_A(x|_A(y|_A y))))|_A(z|_A z))) \in X. \end{aligned}$$

Since X is a sub-implicative ideal, we get

$$((y|_A(y|_A(x|_A x)))|_A(y|_A(y|_A(x|_A x)))) \in X.$$

Hence,

$$\begin{aligned} &f(((y|_A(y|_A(x|_A x)))|_A(y|_A(y|_A(x|_A x))))) \\ &= (((f(y)|_B(f(y)|_B(f(x)|_B f(x))))|_B \\ &\quad (f(y)|_B(f(y)|_B(f(x)|_B(f(x)))))) \\ &\in f(X). \end{aligned}$$

Therefore, $f(X)$ is a sub-implicative ideal of B.

(iii) Let Y be a sub-implicative ideal of B. Since $f(0) = 0_B$, we obtain that $0 \in f^{-1}(Y)$. Assume that

$$((((x|_A(x|_A(y|_Ay)))|_A(x|_A(x|_A(y|_Ay))))|_A(z|_Az))|_A \\ (((x|_A(x|_A(y|_Ay)))|_A(x|_A(x|_A(y|_Ay))))|_A(z|_Az))) \in f^{-1}(Y)$$

and $z \in f^{-1}(Y)$ for any $x, y, z \in A$. Then

$$f((((x|_A(x|_A(y|_Ay)))|_A(x|_A(x|_A(y|_Ay))))|_A(z|_Az))|_A \\ (((x|_A(x|_A(y|_Ay)))|_A(x|_A(x|_A(y|_Ay))))|_A(z|_Az))) \in Y$$

and $f(z) \in Y$. Since f is homomorphism, it follows that

$$((((f(x)|_B(f(x)|_B(f(y)|_B(f(y)))))|_B(f(x)|_B(f(x)|_B(f(y)|_B \\ (f(y))))))|_B(f(z)|_Bf(z)))|_B(((f(x)|_B(f(x)|_B(f(y)|_B(f(y)))))|_B \\ (f(x)|_B(f(x)|_B(f(y)|_B(f(y))))))|_B(f(z)|_Bf(z)))) \in Y$$

and that $f(z) \in Y$. Since Y is a sub-implicative ideal, we have that

$$((f(y)|_B(f(y)|_B(f(x)|_B(f(x)))))|_B(f(y)|_B(f(y)|_B(f(x)|_Bf(x))))) \\ = f(((y|_A(y|_A(x|_Ax)))|_A(y|_A(y|_A(x|_Ax))))) \\ \in Y.$$

Hence,

$$((y|_A(y|_A(x|_Ax)))|_A(y|_A(y|_A(x|_Ax)))) \in f^{-1}(Y).$$

Therefore, $f^{-1}(Y)$ is a sub-implicative ideal of A.

(iv) Let $x, y, z \in A$ such that

$$((((x|_A(x|_A(y|_Ay)))|_A(x|_A(x|_A(y|_Ay))))|_A(z|_Az))|_A \\ (((x|_A(x|_A(y|_Ay)))|_A(x|_A(x|_A(y|_Ay))))|_A(z|_Az))) \in \ker f$$

and that $z \in \ker f$. Since f is a homomorphism, we have

$$((((f(x)|_B(f(x)|_B(f(y)|_B(f(y)))))|_B(f(x)|_B(f(x)|_B(f(y)|_B \\ (f(y))))))|_B(f(z)|_Bf(z)))|_B(((f(x)|_B(f(x)|_B(f(y)|_B(f(y)))))|_B \\ (f(x)|_B(f(x)|_B(f(y)|_B(f(y))))))|_B(f(z)|_Bf(z)))) = 0_B.$$

Then

$$\begin{aligned}
&((((f(x)|_B(f(x)|_B(f(y)|_B(f(y)))))|_B(f(x)|_B(f(x)|_B \\
&(f(y)|_B(f(y))))))|_B(0_B|_B0_B))|_B(((f(x)|_B(f(x)|_B(f(y)|_B \\
&(f(y)))))|_B(f(x)|_B(f(x)|_B(f(y)|_B(f(y))))))|_B(0_B|_B0_B))) \\
&= ((f(x)|_B(f(x)|_B(f(y)|_B(f(y)))))|_B \\
&\quad (f(x)|_B(f(x)|_B(f(y)|_B(f(y)))))) \\
&= f(((x|_A(x|_A(y|_Ay)))|_A(x|_A(x|_A(y|_Ay))))) \\
&= 0_B.
\end{aligned}$$

Since A is implicative, it is obtained that

$$f(((y|_A(y|_A(x|_Ax)))|_A(y|_A(y|_A(x|_Ax))))) = 0_B,$$

that is, $((y|_A(y|_A(x|_Ax)))|_A(y|_A(y|_A(x|_Ax)))) \in \ker f$. Therefore, $\ker f$ is a sub-implicative ideal of A. □

Definition 4.3. Let $f : A \to B$ be a mapping of Sheffer stroke BCK-algebras and let μ be a fuzzy subset of B. The map μ^f is the inverse image of μ under f if $\mu^f(x) = \mu(f(x))$ for all $x \in A$.

Theorem 4.4. *Let $f : A \to B$ be a homomorphism of Sheffer stroke BCK-algebras. If μ is an anti fuzzy ideal of B, then μ^f is an anti fuzzy ideal of A.*

Proof. Since μ is an anti fuzzy ideal of B, then $\mu(0^{'}) \leq \mu(f(x))$ for any $x \in A$. So, $\mu^f(0) = \mu(f(0)) = \mu(0^{'}) \leq \mu(f(x)) = \mu^f(x)$ and

$$\begin{aligned}
\mu^f(x) &= \mu(f(x)) \\
&\leq \max\{\mu((f(x)|(f(y)|f(y)))|(f(x)|(f(y)|f(y)))), \mu(f(y))\} \\
&= \max\{\mu(f((x|(y|y))|(x|(y|y)))), \mu(f(y))\} \\
&= \max\{\mu^f(((x|(y|y))|(x|(y|y)))), \mu^f(y)\}.
\end{aligned}$$

Thus, μ^f is an anti fuzzy ideal of A. □

Theorem 4.5. *Let $f : A \to B$ be an epimorphism of Sheffer stroke BCK-algebras. If μ^f is an anti fuzzy ideal of A, then μ is an anti fuzzy ideal of B.*

Proof. Let $y \in B$. Then there exists $x \in A$ such that $f(x) = y$. Thus,

$$\mu(y) = \mu(f(x)) = \mu^f(x) \geq \mu^f(0) = \mu(0^{'}).$$

Assume that $x^{'}, y^{'}, z^{'} \in B$. So, there exist $x, y, z \in A$ such that $f(x) = x^{'}$, $f(y) = y^{'}$, and $f(z) = z^{'}$. We obtain

$$\begin{aligned}
\mu(x^{'}) &= \mu(f(x)) \\
&= \mu^f(x) \\
&\leq \max\{\mu^f((x|(y|y))|(x|(y|y))), \mu^f(y)\} \\
&= \max\{\mu(f((x|(y|y))|(x|(y|y)))), \mu(f(y))\} \\
&= \max\{\mu((f(x)|(f(y)|f(y)))|(f(x)|(f(y)|f(y))), \mu(f(y))\} \\
&= \max\{\mu((x^{'}|(y^{'}|y^{'}))|(x^{'}|(y^{'}|y^{'}))), \mu(y^{'})\}.
\end{aligned}$$

Hence, μ is an anti fuzzy ideal of B. □

Definition 4.6. A fuzzy relation θ from $A \times A$ to $[0, 1]$ is called a fuzzy congruence in A if it satisfies the following conditions:

(C1) $\theta(0, 0) = \theta(x, x)$ for all $x \in A$,
(C2) $\theta(x, y) = \theta(y, x)$ for all $x, y \in A$,
(C3) $\theta(x, z) \geq \theta(x, y) \wedge \theta(y, z)$ for all $x, y, z \in A$,
(C4) $\theta(((x|(z|z))|(x|(z|z))), ((y|(z|z))|(y|(z|z)))) \geq \theta(x, y)$ and

$$\theta(((z|(x|x))|(z|(x|x))), ((z|(y|y))|(z|(y|y)))) \geq \theta(x, y)$$

for all $x, y, z \in A$.

Example 4.7. Consider the Sheffer stroke BCK-algebra $(A, |, 0)$ in Example 2.4. Let $t_0, t_1, t_2 \in [0, 1]$ such that $t_0 \leq t_1 \leq t_2$. Consider fuzzy relation θ from $A \times A$ to $[0, 1]$ with

$$\theta(0, 0) = \theta(x, x) = \theta(y, y) = \theta(1, 1) = t_2,$$
$$\theta(x, 1) = \theta(y, 0) = \theta(1, x) = \theta(0, y) = t_1,$$

and

$$\begin{aligned}
\theta(0, 1) &= \theta(1, 0) = \theta(x, y) = \theta(y, x) = \theta(x, 0) \\
&= \theta(y, 1) = \theta(0, x) = \theta(1, y) = t_0.
\end{aligned}$$

Then θ is a fuzzy congruence in A.

Definition 4.8. Let θ be a fuzzy congruence in A and let $x \in A$. Define the fuzzy set θ^x in A by $\theta^x(y) = \theta(x, y)$ for all $y \in A$. Then the fuzzy set θ^x is called a fuzzy congruence class of x by θ in A. The set $A/\theta = \{\theta^x | x \in A\}$ is called a fuzzy quotient set by θ.

Example 4.9. Consider the Sheffer stroke BCK-algebra $A = \{0, x, y, 1\}$ with fuzzy congruence relation θ in Example 4.7. A fuzzy quotient set by θ is $A/\theta = \{\theta^0, \theta^x, \theta^y, \theta^1\}$.

Definition 4.10. A fuzzy set μ in A is a fuzzy ideal of A if it satisfies the following properties:

(F1) $\mu(0) \geq \mu(x)$ for all $x \in A$.
(F2) $\mu(x) \geq \mu(y) \wedge \mu((x|(y|y))|(x|(y|y)))$ for all $x, y \in A$.

Example 4.11. Consider the Sheffer stroke BCK-algebra $(A, |, 0)$ in Example 2.4. Let $t_0, t_1 \in [0,1]$ such that $t_0 \leq t_1$. Define $A \to [0,1]$ by $\mu(0) = t_1$ and $\mu(x) = \mu(y) = \mu(1) = t_0$. Then μ is a fuzzy ideal of A.

Lemma 4.12. *Let θ be a fuzzy congruence in A. Then θ^0 is a fuzzy ideal in A.*

Proof. (i) Since θ is a fuzzy congruence in A, $\theta(0,0) = \theta(x,x)$ and $\theta(x,x) \geq \theta(x,y) \wedge \theta(y,x)$. Then, we obtain $\theta(0,0) \geq \theta(x,y)$ from Definition 4.6 (C2). Therefore, $\theta^0(0) = \theta(0,0) \geq \theta(0,x) = \theta^0(x)$ for all $x \in A$.
(ii) Since θ is a fuzzy congruence in A, we have

$$\theta(0,y) \geq \theta(0,(y|(x|x))|(y|(x|x))) \wedge \theta((y|(x|x))|(y|(x|x)),y)$$

and

$$\begin{aligned}&\theta(((y|(x|x))|(y|(x|x))),y) = \theta(((y|(x|x))|(y|(x|x))),\\ &((y|(0|0))|(y|(0|0)))) \geq \theta(x,0)\end{aligned}$$

from Definition 4.6 (C4). Hence,

$$\theta(0,y) \geq \theta(0,((y|(x|x))|(y|(x|x)))) \wedge \theta(0,x).$$

Thus, $\theta^0(y) \geq \theta^0((y|(x|x))|(y|(x|x))) \wedge \theta^0(x)$ for all $x, y \in A$. Therefore, θ^0 is a fuzzy ideal in A. □

Proposition 4.13. *Let μ be a fuzzy ideal of A.*

(i) *If $x \leq y$, then $\mu(x) \geq \mu(y)$ for all $x, y \in A$.*
(ii) *It holds that*

$$\begin{aligned}\mu((x|(y|y))|(x|(y|y))) \geq\ &\mu((x|(z|z))|(x|(z|z)))\\ &\wedge\mu((z|(y|y))|(z|(y|y)))\end{aligned}$$

for all $x, y, z \in A$.

Proof. (i) Let μ be a fuzzy ideal of A and let $x, y \in A$ such that $x \leq y$. Then

$$\begin{aligned}\mu(x) &\geq \min\{\mu((x|(y|y))|(x|(y|y))), \mu(y)\}\\ &= \min\{\mu(0), \mu(y)\}\\ &= \mu(y)\end{aligned}$$

from Definition 4.10 (F1).
(ii) It is obtained from Definitions 4.10 (F2), 2.1 (S2), and 2.3 (sBCK-1). □

Lemma 4.14. *Let μ be a fuzzy ideal in A. Then*
$$\theta_\mu(x,y) = \mu((x|(y|y))|(x|(y|y))) \wedge \mu((y|(x|x))|(y|(x|x)))$$
is a fuzzy congruence in A.

Proof. Definition 4.6 (C1) and (C2) clearly hold.
For condition (C3) of Definition 4.6, let $x, y, z \in A$. By Proposition 4.13 (ii), we have

$$\begin{aligned}\theta_\mu(x,z) &= \mu((x|(z|z))|(x|(z|z))) \wedge \mu((z|(x|x))|(z|(x|x)))\\ &\geq (\mu((x|(y|y))|(x|(y|y))) \wedge \mu((y|(z|z))|(y|(z|z))))\\ &\quad \wedge(\mu((z|(y|y))|(z|(y|y))) \wedge \mu((y|(x|x))|(y|(x|x))))\\ &= (\mu((x|(y|y))|(x|(y|y))) \wedge \mu((y|(x|x))|(y|(x|x))))\\ &\quad \wedge(\mu((y|(z|z))|(y|(z|z))) \wedge \mu((z|(y|y))|(z|(y|y))))\\ &= \theta_\mu(x,y) \wedge \theta_\mu(y,z).\end{aligned}$$

For condition (C4) of Definition 4.6, Let $x, y \in A$. By Definitions 2.3 (sBCK-1) and 2.1 (S2),

$$\begin{aligned}&((((x|(y|y))|(x|(y|y)))|(x|(z|z)))|(((x|(y|y))|(x|(y|y)))|(x|(z|z))))\\ &\leq ((z|(y|y))|(z|(y|y)).\end{aligned}$$

We get from Proposition 2.6 (iii) that

$$\begin{aligned}&((((x|(y|y))|(x|(y|y)))|(z|(y|y)))|(((x|(y|y))|(x|(y|y)))|(z|(y|y))))\\ &\leq (x|(z|z))|(x|(z|z)).\end{aligned}$$

Substituting $[x = y]$, $[y = z]$, and $[z = x]$, we obtain

$$\begin{aligned}&((((y|(z|z))|(y|(z|z)))|(x|(z|z)))|(((y|(z|z))|(y|(z|z)))|(x|(z|z))))\\ &\qquad\qquad\qquad\qquad \leq (y|(x|x))|(y|(x|x)). \quad (4.1)\end{aligned}$$

Then we have from Proposition 4.13 (i) and Definition 2.1 (S2) that

$$\begin{aligned}\theta_\mu(((x|(z|z))|(x|(z|z))), ((y|(z|z))|(y|(z|z)))) &= \mu((((x|(z|z))|(x|(z|z)))\\ &\quad |(y|(z|z)))|(((x|(z|z))|\\ &\quad (x|(z|z)))|(y|(z|z))))\\ &\quad \wedge\mu((((y|(z|z))\\ &\quad |(y|(z|z)))|(x|(z|z)))\\ &\quad |(((y|(z|z))|(y|(z|z)))\\ &\quad |(x|(z|z))))\\ &\geq \mu((x|(y|y))|(x|(y|y)))\\ &\quad \wedge\mu((y|(x|x))|(y|(x|x)))\\ &= \theta_\mu(x,y).\end{aligned}$$

Similarly, $\theta_\mu(((z|(x|x))|(z|(x|x))),((z|(y|y))|(z|(y|y)))) \geq \theta_\mu(x,y)$. Then $\theta_\mu(x,y) = \mu((x|(y|y))|(x|(y|y))) \wedge \mu((y|(x|x))|(y|(x|x)))$ is a fuzzy congruence in A. □

Example 4.15. Consider the Sheffer stroke BCK-algebra $A = \{0, x, y, 1\}$ with fuzzy ideal μ in Example 4.11. If we take $[x = x], [y = y]$, and $[z = 1]$, then we show that the conditions of Definition 4.6 hold:

(C1) :

$$\begin{aligned}
\theta_\mu(0,0) &= \mu((0|(0|0))|(0|(0|0))) \wedge \mu((0|(0|0))|(0|(0|0))) \\
&= \mu(0) \wedge \mu(0) \\
&= t_1 \\
&= \mu((x|(x|x))|(x|(x|x))) \wedge \mu((x|(x|x))|(x|(x|x))) \\
&= \theta_\mu(x,x).
\end{aligned}$$

(C2) :

$$\begin{aligned}
\theta_\mu(x,y) &= \mu((x|(y|y))|(x|(y|y))) \wedge \mu((y|(x|x))|(y|(x|x))) \\
&= \mu(x) \wedge \mu(y) \\
&= t_0 \\
&= \mu(y) \wedge \mu(x) \\
&= \mu((y|(x|x))|(y|(x|x))) \wedge \mu((x|(y|y))|(x|(y|y))) \\
&= \theta_\mu(y,x).
\end{aligned}$$

(C3) :

$$\begin{aligned}
\theta_\mu(x,1) &= \mu((x|(1|1))|(x|(1|1))) \wedge \mu((1|(x|x))|(1|(x|x))) \\
&= \mu(0) \wedge \mu(y) \\
&= t_0 \\
&\geq t_0 \wedge t_0 \\
&= (\mu(x) \wedge \mu(y)) \wedge (\mu(0) \wedge \mu(x)) \\
&= (\mu((x|(y|y))|(x|(y|y))) \wedge \mu((y|(x|x))|(y|(x|x)))) \\
&\quad \wedge(\mu((y|(1|1))|(y|(1|1))) \wedge \mu((1|(y|y))|(1|(y|y)))) \\
&= \theta_\mu(x,y) \wedge \theta_\mu(y,1).
\end{aligned}$$

(C4) :

$$\begin{aligned}\theta_\mu(((x|(1|1))|(x|(1|1))),((y|(1|1))|(y|(1|1)))) &= \theta_\mu(0,0)\\ &= \mu((0|(0|0))|(0|(0|0)))\\ &\quad \wedge\mu((0|(0|0))|(0|(0|0)))\\ &= \mu(0)\\ &= t_1\\ &\geq t_1\wedge t_1\\ &= \mu(x)\wedge\mu(y)\\ &= \mu((x|(y|y))|(x|(y|y)))\\ &\quad \wedge\mu((y|(x|x))|(y|(x|x)))\\ &= \theta_\mu(x,y).\end{aligned}$$

Similarly,

$$\theta_\mu(((1|(x|x))|(1|(x|x))),((1|(y|y))|(1|(y|y)))) \geq \theta_\mu(x,y).$$

Theorem 4.16. *There is a bijection between the set of fuzzy ideals and the set of fuzzy congruences in A.*

Proof. By Lemmas 4.12 and 4.14, it is easily checked that $\mu = (\theta_\mu)^0$ and that $\theta = \theta_{\theta^0}$, for each fuzzy ideal μ and fuzzy congruence θ in A. Hence, there is a bijection between the set of fuzzy ideals and the set of fuzzy congruences in A. □

Let μ be a fuzzy ideal in A, let μ^x denote the fuzzy congruence class of x by θ_μ in A for all $x \in A$, and let A/μ be the fuzzy quotient set by θ_μ. We introduce congruence relations induced by fuzzy ideals.

Proposition 4.17. *Let μ be a fuzzy ideal in A. Then $\mu^x = \mu^y$ if and only if $\mu((x|(y|y))|(x|(y|y))) = ((y|(x|x))|(y|(x|x))) = \mu(0)$ for all $x, y \in A$.*

Proof. Let $\mu^x = \mu^y$ for all $x, y \in A$. Then

$$\begin{aligned}\mu^u(v) &= \theta^u_\mu(v)\\ &= \theta_\mu(u,v)\\ &= \mu((u|(v|v))|(u|(v|v)))\wedge\mu((v|(u|u))|(v|(u|u)))\end{aligned}$$

for any $u, v \in A$. Since $\mu^x = \mu^y$, $\mu^x(x) = \mu^y(x)$ for all $x \in A$. It follows that

$$\begin{aligned}\mu((x|(x|x))|(x|(x|x)))\wedge\mu((x|(x|x))|(x|(x|x))) &= \mu((y|(x|x))|(y|(x|x)))\\ &\quad \wedge\mu((x|(y|y))|(x|(y|y)))\end{aligned}$$

and by Definition 4.10 (F1),

$$\mu((x|(y|y))|(x|(y|y))) = \mu((y|(x|x))|(y|(x|x))) = \mu(0).$$

Conversely, let $\mu((x|(y|y))|(x|(y|y))) = \mu((y|(x|x))|(y|(x|x))) = \mu(0)$. We get from Proposition 4.13 (ii) that

$$\mu((x|(z|z))|(x|(z|z))) \geq \mu((x|(y|y))|(x|(y|y))) \wedge \mu((y|(z|z))|(y|(z|z)))$$

and

$$\mu((y|(z|z))|(y|(z|z))) \geq \mu((y|(x|x))|(y|(x|x))) \wedge \mu((x|(z|z))|(x|(z|z)))$$

for all $z \in A$. We have from the hypothesis that

$$\mu((x|(z|z))|(x|(z|z))) \geq \mu((y|(z|z))|(y|(z|z)))$$

and $\mu((y|(z|z))|(y|(z|z))) \geq \mu((x|(z|z))|(x|(z|z)))$. Then

$$\mu((x|(z|z))|(x|(z|z))) = \mu((y|(z|z))|(y|(z|z))).$$

Similarly, $\mu((z|(x|x))|(z|(x|x))) = \mu((z|(y|y))|(z|(y|y)))$. Thus,

$$\begin{aligned} \mu^x(z) &= \mu((x|(z|z))|(x|(z|z))) \wedge \mu((z|(x|x))|(z|(x|x))) \\ &= \mu((y|(z|z))|(y|(z|z))) \wedge \mu((z|(y|y))|(z|(y|y))) \\ &= \mu^y(z) \end{aligned}$$

for all $z \in A$. Hence, $\mu^x = \mu^y$. □

By Proposition 4.17, consider the binary relation $\sim_\mu$ on A by

$$x \sim_\mu y \Leftrightarrow \mu((y|(x|x))|(y|(x|x))) = \mu((x|(y|y))|(x|(y|y))) = \mu(0),$$

where μ is a fuzzy ideal in A.

Lemma 4.18. *Let μ be a fuzzy ideal in A. Then $\sim_\mu$ is an equivalent relation on A.*

Proof. It is clear that $\sim_\mu$ is reflexive and symmetric. Let $x \sim_\mu y$ and $y \sim_\mu z$ for any $x, y, z \in A$. Then

$$\begin{aligned} \mu((y|(x|x))|(y|(x|x))) &= \mu((x|(y|y))|(x|(y|y))) \\ &= \mu((z|(y|y))|(z|(y|y))) \\ &= \mu((y|(z|z))|(y|(z|z))) \\ &= \mu(0). \end{aligned}$$

We have from (4.1) that

$$\begin{aligned} &(((x|(z|z))|(x|(z|z)))|(y|(z|z)))|(((x|(z|z))|(x|(z|z)))|(y|(z|z))) \\ &\leq (x|(y|y))|(x|(y|y)) \end{aligned}$$

and

$$\begin{aligned} &(((z|(x|x))|(z|(x|x)))|(y|(x|x)))|(((z|(x|x))|(z|(x|x)))|(y|(x|x))) \\ &\leq (z|(y|y))|(z|(y|y)). \end{aligned}$$

We get from Definitions 4.10 and 2.1 (S2) and Proposition 4.13 (i) that

$$\begin{aligned}\mu((z|(x|x))|(z|(x|x))) &\geq \mu((y|(x|x))|(y|(x|x))) \wedge \mu((((z|(x|x))\\&\quad|(z|(x|x)))|(y|(x|x)))|(((z|(x|x))|\\&\quad(z|(x|x)))|(y|(x|x))))\\&\geq \mu((y|(x|x))|(y|(x|x)))\\&\quad\wedge\mu((z|(y|y))|(z|(y|y)))\\&= \mu(0)\end{aligned}$$

and

$$\begin{aligned}\mu((x|(z|z))|(x|(z|z))) &\geq \mu((y|(z|z))|(y|(z|z))) \wedge \mu((((x|(z|z))\\&\quad|(x|(z|z)))|(y|(z|z)))|(((x|(z|z))\\&\quad|(x|(z|z)))|(y|(z|z))))\\&\geq \mu((y|(z|z))|(y|(z|z)))\\&\quad\wedge\mu((x|(y|y))|(x|(y|y)))\\&= \mu(0).\end{aligned}$$

So, $\mu((z|(x|x))|(z|(x|x))) = \mu((x|(z|z))|(x|(z|z))) = \mu(0)$. Hence, $x \sim_\mu z$. Therefore, $\sim_\mu$ is an equivalent relation on A. □

Example 4.19. Consider the Sheffer stroke BCK-algebra $(A, |, 0)$ in Example 2.4. For a fuzzy ideal of A defined by $\mu(0) = \mu(x) = t_1$, $\mu(y) = \mu(1) = t_0$ such that $t_0 \leq t_1$, where $t_0, t_1 \in [0, 1]$. Then

$$\sim_\mu = \{(0,0),(x,x),(y,y),(1,1),(0,x),(x,0),(y,1),(1,y)\}$$

is an equivalence relation on A.

Lemma 4.20. *Let μ be a fuzzy ideal in A. Then $\sim_\mu$ is a congruence relation on A.*

Proof. By Lemma 4.18, it is sufficient to prove that $x \sim_\mu y$ implies

$$((z|(x|x))|(z|(x|x))) \sim_\mu ((z|(y|y))|(z|(y|y)))$$

for any $x, y, z \in A$. Let $x \sim_\mu y$. Then

$$\mu((y|(x|x))|(y|(x|x))) = \mu((x|(y|y))|(x|(y|y))) = \mu(0).$$

We get from Definition 2.3 (sBCK-1) that

$$\begin{aligned}&((((z|(y|y))|(z|(y|y)))|(z|(x|x)))|(((z|(y|y))|(z|(y|y)))|(z|(x|x))))\\&\leq ((x|(y|y))|(x|(y|y)))\end{aligned}$$

and

$$\begin{aligned}&((((z|(x|x))|(z|(x|x)))|(z|(y|y)))|(((z|(x|x))|(z|(x|x)))|(z|(y|y))))\\&\leq ((y|(x|x))|(y|(x|x))).\end{aligned}$$

It follows from Proposition 4.13 (i) that

$$\begin{aligned}\mu(0) &= \mu((x|(y|y))|(x|(y|y))) \\ &\leq \mu((((z|(y|y))|(z|(y|y)))|(z|(x|x))) \\ &\quad |(((z|(y|y))|(z|(y|y)))|(z|(x|x))))\end{aligned}$$

and

$$\begin{aligned}\mu(0) &= \mu((y|(x|x))|(y|(x|x))) \\ &\leq \mu((((z|(x|x))|(z|(x|x)))|(z|(y|y)))| \\ &\quad (((z|(x|x))|(z|(x|x)))|(z|(y|y)))).\end{aligned}$$

So,

$$\begin{aligned}&\mu((((z|(x|x))|(z|(x|x)))|(z|(y|y)))|(((z|(x|x))|(z|(x|x)))|(z|(y|y)))) \\ &= \mu((((z|(y|y))|(z|(y|y)))|(z|(x|x)))|(((z|(y|y))|(z|(y|y)))|(z|(x|x)))) \\ &= \mu(0).\end{aligned}$$

Thus, $((z|(x|x))|(z|(x|x))) \sim_\mu ((z|(y|y))|(z|(y|y)))$. □

Example 4.21. Consider the equivalence relation $\sim_\mu$ on A in Example 4.19. Then it is a congruence relation on A.

Theorem 4.22. *Let μ be a fuzzy ideal of A and let $\sim$ be a congruence relation on A defined by μ. Then $A/\mu = (A/\sim, |_\sim)$ is a Sheffer stroke BCK-algebra, where the quotient set is $A/\sim = \{[x]_\sim : x \in A\}$ and the Sheffer stroke $|_\sim$ is defined by $[x]_\sim|_\sim[y]_\sim = [x|y]_\sim$ for all $x, y \in A$.*

Proof. For all $\mu^x, \mu^y \in A/\mu$, we define

$$(\mu^x|(\mu^y|\mu^y))|(\mu^x|(\mu^y|\mu^y)) = \mu^{(x|(y|y))|(x|(y|y))}.$$

Let $\mu^x = \mu^a$ and let $\mu^y = \mu^b$. Then $x \sim_\mu a$ and $y \sim_\mu b$. By Lemma 4.20, we obtain

$$(x|(a|a))|(x|(a|a)) \sim_\mu (y|(b|b))|(y|(b|b))$$

and so

$$\mu^{(x|(y|y))|(x|(y|y))} = \mu^{(a|(b|b))|(a|(b|b))}.$$

It is easily proved that $A/\mu = (A/\mu, |, \mu^0)$ is a Sheffer stroke BCK-algebra. □

Example 4.23. Consider $(A, |, 1)$ with the Hasse diagram (Figure 2), where $A = \{0, x, y, z, t, u, v, 1\}$.

The binary operation $|$ has the Cayley table; see Table 2.

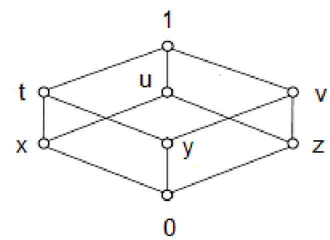

FIGURE 2. Hasse diagram of $(A, |, 1)$

TABLE 2. Cayley table of Example 4.23

$\|$	0	x	y	z	t	u	v	1
0	1	1	1	1	1	1	1	1
x	1	v	1	1	v	v	1	v
y	1	1	u	1	u	1	u	u
z	1	1	1	t	1	t	t	t
t	1	v	u	1	z	v	u	z
u	1	v	1	t	v	y	t	y
v	1	1	u	t	u	t	x	x
1	1	v	u	t	z	y	x	0

Then $(A, |, 0)$ is a Sheffer stroke BCK-algebra. Define a fuzzy ideal μ by $\mu(0) = \mu(z) = t_1$ and

$$\mu(x) = \mu(y) = \mu(t) = \mu(u) = \mu(v) = \mu(1) = t_0$$

such that $t_0 \leq t_1$, where $t_0, t_1 \in [0, 1]$. Let

$$\begin{aligned}\sim_\mu = \ &\{(0,0),(x,x),(y,y),(z,z),(t,t),(u,u),(v,v),(1,1),\\ &(0,z),(z,0),(t,1),(1,t),(v,y),(y,v),(x,u),(u,x)\}\end{aligned}$$

be a congruence relation on A defined by μ. Then $A/\mu = (A/\sim, |_\sim)$ is a Sheffer stroke BCK-algebra with the Hasse diagram (see Figure 3) and the quotient set $A/\sim = \{[0]_\sim, [x]_\sim, [y]_\sim, [1]_\sim\}$.

The binary operation $|_\sim$ on A/μ has the Cayley table; see Table 3.

TABLE 3. Cayley table of $|_\sim$ for Example 4.23

$\|_\sim$	$[0]_\sim$	$[x]_\sim$	$[y]_\sim$	$[1]_\sim$
$[0]_\sim$	$[1]_\sim$	$[1]_\sim$	$[1]_\sim$	$[1]_\sim$
$[x]_\sim$	$[1]_\sim$	$[y]_\sim$	$[1]_\sim$	$[y]_\sim$
$[y]_\sim$	$[1]_\sim$	$[1]_\sim$	$[x]_\sim$	$[x]_\sim$
$[1]_\sim$	$[1]_\sim$	$[y]_\sim$	$[x]_\sim$	$[0]_\sim$

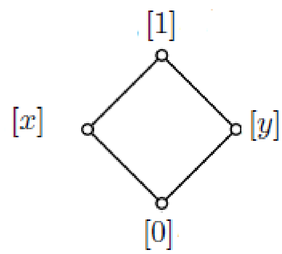

FIGURE 3. Hasse diagram of $A/\mu = (A/\sim, |_\sim)$

5. Anti cartesian product of anti fuzzy ideals

Definition 5.1. Let λ and μ be the fuzzy subsets of a set A. The Cartesian product $\lambda \times \mu : A \times A \to [0,1]$ is defined by

$$(\lambda \times \mu)(x,y) = \max\{\lambda(x), \mu(y)\}$$

for all $x, y \in A$.

Definition 5.2. Let μ be a fuzzy relation on a set A and β be a fuzzy subset of A. Then μ is an anti fuzzy relation on β if

$$\mu(x,y) \geq \max\{\beta(x), \beta(y)\}$$

for all $x, y \in A$.

Definition 5.3. If β is a fuzzy set of a set A, the the strongest anti fuzzy relation on A that is an anti fuzzy relation on β, is μ_β given by $\mu_\beta(x,y) = \max\{\beta(x), \beta(y)\}$ for all $x, y \in A$.

Example 5.4. Consider the Sheffer stroke BCK-algebra in Example 2.4. Let a fuzzy subset $\beta : A \to [0,1]$ be defined by

$$\beta(0) = t_3,\ \beta(x) = t_2,\ \beta(y) = t_1,$$

and $\beta(1) = t_0$, where $t_0, t_1, t_2, t_3 \in [0,1]$ such that $t_0 \leq t_1 \leq t_2 \leq t_3$. Then

$$\mu_\beta(x,y) = \begin{cases} t_3 & if\ x = 0\ or\ y = 0, \\ \beta(x) & if\ x = y, \\ \beta(x) & if\ x = 1\ or\ y = 1, \\ t_2 & otherwise, \end{cases}$$

is the strongest anti fuzzy relation on A.

Proposition 5.5. *Let β be a fuzzy set and let μ_β be the strongest anti-fuzzy relation on A. If μ_β is an anti fuzzy sub-implicative ideal of $A \times A$, then $\beta(x) \geq \beta(0)$ for all $x \in A$.*

Proof. We have

$$\max\{\beta(x), \beta(x)\} = \mu_\beta(x,x) \geq \mu_\beta(0,0) = \max\{\beta(0), \beta(0)\},$$

where $(0,0) \in A \times A$. Therefore, $\beta(x) \geq \beta(0)$ for all $x \in A$. □

Theorem 5.6. *Let β be a fuzzy subset of A and let μ_β be the strongest anti fuzzy relation on A. Then β is an anti fuzzy ideal of A if and only if μ_β is an anti fuzzy ideal of $A \times A$.*

Proof. It is known that $A \times A$ is a Sheffer stroke BCK-algebra. Let β be an anti fuzzy ideal of A. Then

$$\mu_\beta(0,0) = \max\{\beta(0), \beta(0)\} \leq \max\{\beta(x), \beta(y)\} = \mu_\beta(x,y)$$

for all $x, y \in A$. Since

$$\begin{aligned}
&\max\{\mu_\beta(((x_1,x_2)|_{A\times A}((y_1,y_2)|_{A\times A}(y_1,y_2)))|_{A\times A}\\
&\qquad ((x_1,x_2)|_{A\times A}((y_1,y_2)|_{A\times A}(y_1,y_2)))), \mu_\beta(y_1,y_2)\}\\
&= \max\{\mu_\beta(((x_1|(y_1|y_1))|(x_1|(y_1|y_1))),\\
&\qquad ((x_2|(y_2|y_2))|(x_2|(y_2|y_2)))), \mu_\beta(y_1,y_2)\}\\
&= \max\{\max\{\beta((x_1|(y_1|y_1))|(x_1|(y_1|y_1))),\\
&\qquad \beta((x_2|(y_2|y_2))|(x_2|(y_2|y_2)))\}, \max\{\beta(y_1), \beta(y_2)\}\}\\
&= \max\{\max\{\beta((x_1|(y_1|y_1))|(x_1|(y_1|y_1))), \beta(y_1)\},\\
&\quad \max\{((x_2|(y_2|y_2))|(x_2|(y_2|y_2))), \beta(y_2)\}\}\\
&\geq \max\{\beta(x_1), \beta(x_2)\}\}\\
&= \mu_\beta(x_1,x_2),
\end{aligned}$$

for all $x_1, x_2, y_1, y_2 \in A$, it follows that μ_β is an anti fuzzy ideal of $A \times A$.

Conversely, let μ_β be an anti fuzzy ideal of $A \times A$. Then

$$\max\{\beta(0), \beta(0)\} = \mu_\beta(0,0) \leq \mu_\beta(x,y) = \max\{\beta(x), \beta(y)\}.$$

Therefore, $\beta(0) \leq \beta(x)$. Since

$$\begin{aligned}
\max\{\beta(x|(y|y))|(x|(y|y)), \beta(y)\} &= \max\{\max\{\beta((x|(y|y))|(x|(y|y))), \beta((x|(y|y))|(x|(y|y)))\}, \max\{\beta(y), \beta(y)\}\} \\
&= \max\{\mu_\beta((x|(y|y))|(x|(y|y))), ((x|(y|y))|(x|(y|y))), \mu_\beta(y, y)\} \\
&= \max\{\mu_\beta(((x,x)|_{A\times A}((y,y)|_{A\times A}(y,y)))|_{A\times A}((x,x)|_{A\times A}((y,y)|_{A\times A}(y,y)))), \mu_\beta(y,y)\} \\
&\geq \mu_\beta(x,x) \\
&= \max\{\beta(x), \beta(x)\} \\
&= \beta(x),
\end{aligned}$$

for all $x, y \in A$, it is obtained that β is an anti fuzzy ideal of A.

□

Theorem 5.7. *If λ and μ are anti fuzzy ideals of A, then $\lambda \times \mu$ is an anti fuzzy ideal of $A \times A$.*

Proof. Let $x, x^{'} \in A$. Then

$$(\lambda \times \mu)(0,0) = \max\{\lambda(0), \mu(0)\} \leq \max\{\lambda(x), \mu(x)\} = (\lambda \times \mu)(x, x^{'}).$$

For any $(x, x^{'}), (y, y^{'}) \in A \times A$, we obtain

$$\begin{aligned}
(\lambda \times \mu)(x, x^{'}) &= \max\{\lambda(x), \mu(x^{'})\} \\
&\leq \max\{\max\{\lambda(x|(y|y))|(x|(y|y)), \lambda(y)\}, \\
&\quad \max\{\mu(x^{'}|(y^{'}|y^{'}))|(x^{'}|(y^{'}|y^{'})), \mu(y^{'})\}\} \\
&= \max\{\max\{\lambda(x|(y|y))|(x|(y|y)), \mu(x^{'}|(y^{'}|y^{'}))|(x^{'}|(y^{'}|y^{'}))\}, \max\{\lambda(y), \mu(y^{'})\}\} \\
&= \max\{(\lambda \times \mu)(((x,x^{'})|((y,y^{'})|(y,y^{'})))|((x,x^{'})|((y,y^{'})|(y,y^{'})))), (\lambda \times \mu)(y,y^{'})\}.
\end{aligned}$$

Hence, $\lambda \times \mu$ is an anti fuzzy ideal of $A \times A$. □

Theorem 5.8. *If λ and μ are anti fuzzy sub-implicative ideals of A, then $\lambda \times \mu$ is an anti fuzzy sub-implicative ideal of $A \times A$.*

Proof. Let $x, y \in A$. Then

$$(\lambda \times \mu)(0,0) = \max\{\lambda(0), \mu(0)\} \leq \max\{\lambda(x), \mu(y)\} = (\lambda \times \mu)(x, y).$$

For any $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in A \times A$, we have

$$
\begin{aligned}
&(\lambda \times \mu)((y_1, y_2)|((y_1, y_2)|((x_1, x_2)|(x_1, x_2))))\\
&|((y_1, y_2)|((y_1, y_2)|((x_1, x_2)|(x_1, x_2))))\\
&= (\lambda \times \mu)((y_1|(y_1|(x_1|x_1)))|(y_1|(y_1|(x_1|x_1)))),\\
&\quad ((y_2|(y_2|(x_2|x_2)))|(y_2|(y_2|(x_2|x_2))))\\
&= \max\{\lambda((y_1|(y_1|(x_1|x_1)))|(y_1|(y_1|(x_1|x_1)))),\\
&\qquad \mu(y_2|(y_2|(x_2|x_2))|(y_2|(y_2|(x_2|x_2))))\}\\
&\leq \max\{\max\{\lambda((((x_1|(x_1|(y_1|y_1)))|(x_1|(x_1|(y_1|y_1))))|(z_1\\
&\qquad |z_1))|(((x_1|(x_1|(y_1|y_1)))|(x_1|(x_1|(y_1|y_1))))|(z_1|z_1))),\\
&\qquad \mu((((x_2|(x_2|(y_2|y_2)))|(x_2|(x_2|(y_2|y_2))))|(z_2|z_2))|(((x_2|(x_2|\\
&\qquad (y_2|y_2)))|(x_2|(x_2|(y_2|y_2))))|(z_2|z_2)))\}, \max\{\lambda(z_1), \mu(z_2)\}\}\\
&= \max\{(\lambda \times \mu)(((x_1|(x_1|(y_1|y_1)))|(x_1|(x_1|(y_1|y_1))))|(z_1\\
&\qquad ((((x_2|(x_2|(y_2|y_2)))|(x_2|(x_2|(y_2|y_2))))|(z_2|z_2))|(((x_2|(x_2\\
&\qquad |(y_2|y_2)))|(x_2|(x_2|(y_2|y_2))))|(z_2|z_2))), (\lambda \times \mu)(z_1, z_2)\}\\
&= \max\{(\lambda \times \mu)((((x_1, x_2)|((x_1, x_2)|(y_1, y_2)|(y_1, y_2)))\\
&\qquad |((x_1, x_2)|((x_1, x_2)|(y_1, y_2)|(y_1, y_2))))|(z_1, z_2))|\\
&\qquad ((((x_1, x_2)|((x_1, x_2)|(y_1, y_2)|(y_1, y_2)))|((x_1, x_2)|((x_1,\\
&\qquad x_2)|(y_1, y_2)|(y_1, y_2))))|(z_1, z_2)), (\lambda \times \mu)(z_1, z_2)\}.
\end{aligned}
$$

Hence, $\lambda \times \mu$ is an anti fuzzy sub-implicative ideal of $A \times A$. □

Theorem 5.9. *Let λ and μ be the fuzzy subsets in A such that $\lambda \times \mu$ is an anti fuzzy ideal of $A \times A$. Then the following properties hold:*

(i) *Either $\lambda(x) \geq \lambda(0)$ or $\mu(x) \geq \mu(0)$ for all $x \in A$.*
(ii) *If $\lambda(x) \geq \lambda(0)$ for all $x \in A$, then either $\lambda(x) \geq \mu(0)$ or $\mu(x) \geq \mu(0)$.*
(iii) *If $\mu(x) \geq \mu(0)$ for all $x \in A$, then either $\lambda(x) \geq \lambda(0)$ or $\mu(x) \geq \lambda(0)$.*

Proof. (i) Suppose that $\lambda(x) \leq \lambda(0)$ and $\mu(y) \leq \mu(0)$ for some $x, y \in A$. Then

$$(\lambda \times \mu)(x, y) = \max\{\lambda(x), \mu(y)\} \leq \max\{\lambda(0), \mu(0)\} = (\lambda \times \mu)(0, 0),$$

which is a contradiction. Thus, $\lambda(x) \geq \lambda(0)$ or $\mu(x) \geq \mu(0)$ for all $x \in A$.

(ii) Assume that there exist $x, y \in A$ such that $\lambda(x) \leq \mu(0)$ and $\mu(x) \leq \mu(0)$. Then $(\lambda \times \mu)(0, 0) = \max\{\lambda(0), \mu(0)\} = \mu(0)$. It follows that

$$(\lambda \times \mu)(x, y) = \max\{\lambda(x), \mu(y)\} \leq \mu(0) = (\lambda \times \mu)(0, 0),$$

which is a contradiction.

(iii) The proof is similar to (ii). □

Theorem 5.10. *Let λ and μ be the fuzzy subsets in A such that $\lambda \times \mu$ is an anti fuzzy ideal of $A \times A$. Then either λ or μ is an anti fuzzy ideal of A.*

Proof. By Theorem 5.9 (i), we assume that $\mu(x) \geq \mu(0)$ for all $x \in A$. It follows from (iii) that either $\lambda(x) \geq \lambda(0)$ or $\mu(x) \geq \lambda(0)$. If $\mu(x) \geq \lambda(0)$ for all $x \in A$, then $(\lambda \times \mu)(0, x) = \max\{\lambda(0), \mu(x)\} = \mu(x)$. Let $(x, x^{'}), (y, y^{'}) \in A \times A$. Since $\lambda \times \mu$ is an anti fuzzy ideal of $A \times A$, we have

$$\begin{aligned}(\lambda \times \mu)(x, x^{'}) &\leq \max\{(\lambda \times \mu)(((x, x^{'})|((y, y^{'})|(y, y^{'})))| \\ &\qquad ((x, x^{'})|((y, y^{'})|(y, y^{'})))), (\lambda \times \mu)(y, y^{'})\} \\ &= \max\{(\lambda \times \mu)((x|(y|y))|(x|(y|y)), \\ &\qquad (x^{'}|(y^{'}|y^{'}))|(x^{'}|(y^{'}|y^{'}))), (\lambda \times \mu)(y, y^{'})\}.\end{aligned}$$

Putting $x = y = 0$, we get

$$\begin{aligned}\mu(x^{'}) &= (\lambda \times \mu)(0, x^{'}) \\ &\leq \max\{(\lambda \times \mu)(0, (x^{'}|(y^{'}|y^{'}))|(x^{'}|(y^{'}|y^{'}))), (\lambda \times \mu)(0, y^{'})\} \\ &= \max\{\max\{\lambda(0), \mu((x^{'}|(y^{'}|y^{'}))|(x^{'}|(y^{'}|y^{'})))\}, \\ &\qquad \max\{\lambda(0), \mu(y^{'})\}\} \\ &= \max\{\mu((x^{'}|(y^{'}|y^{'}))|(x^{'}|(y^{'}|y^{'}))), \mu(y^{'})\}.\end{aligned}$$

Hence, μ is an anti fuzzy ideal of A. The second part is similar. □

Example 5.11. Consider the Sheffer stroke BCK-algebra in Example 2.4. Let fuzzy subsets λ and μ be defined by defined by $\lambda(x) = t_3$ for all $x \in A$, and

$$\mu(x) = \begin{cases} t_2 & if\ x = 0, \\ t_1 & otherwise, \end{cases}$$

where $t_1, t_2, t_3 \in [0, 1]$ such that $t_1 \leq t_2 \leq t_3$. Then

$$(\lambda \times \mu)(x, y) = \max\{\lambda(x), \mu(y)\} = t_3 = \lambda(x)$$

for all $x, y \in A$. It is an anti fuzzy ideal of A. Indeed μ is not an anti fuzzy ideal of A. Since, $\mu(0) = t_2 \geq \mu(x)$ for all $x \in A - \{0\}$.

6. Conclusion

In this study, a Sheffer stroke BCK-algebra, an anti fuzzy subalgebra, an anti fuzzy (sub-implicative) ideal, and their properties were investigated. An anti fuzzy subalgebra and an anti fuzzy ideal on a Sheffer stroke BCK-algebra were defined, and related notions were given. It was proved that every anti fuzzy ideal of a Sheffer stroke BCK-algebra is an anti fuzzy subalgebra. Then it was shown that the complement of a fuzzy subset on a Sheffer stroke BCK-algebra is an anti fuzzy ideal if and only if it is a fuzzy ideal. It was observed that a lower t-level cut of a fuzzy subset on a Sheffer stroke BCK-algebra is an ideal if and only if this fuzzy subset is an anti fuzzy ideal. By defining an anti fuzzy (sub-)implicative ideal, it was presented that every anti fuzzy sub-implicative ideal of a Sheffer stroke BCK-algebra is an anti fuzzy ideal. Also, a homomorphism on Sheffer stroke BCK-algebra was described, and an inverse image of a fuzzy subset on a Sheffer stroke BCK-algebra was identified. It was proved that the inverse image of a fuzzy subset on a Sheffer stroke BCK-algebra is an anti fuzzy ideal if this fuzzy subset is an anti fuzzy ideal. Besides, it was demonstrated that there is a bijection between the set of fuzzy ideals and the set of fuzzy congruences on a Sheffer stroke BCK-algebra. In the last part of the study, it was propounded that the Cartesian product of two fuzzy subsets on a Sheffer stroke BCK-algebra is an anti fuzzy ideal if these fuzzy subsets are anti fuzzy ideals. To prove the inverse, it was enough that at least one of these fuzzy subsets is an anti fuzzy ideal.

In the future works, anti fuzzy ideals for different Sheffer stroke algebraic structures and so related notions will be revealed.

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Tahsin Oner
Department of Mathematics, Ege University, Bornova, Izmir, Turkey.
Email: tahsin.oner@ege.edu.tr

Tugce Kalkan
Department of Mathematics, Ege University, Bornova, Izmir, Turkey.
Email: tugcekalkan92@gmail.com

Arsham Borumand Saeid
Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran.
Email:arsham@uk.ac.ir

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(ANTI) FUZZY IDEALS OF SHEFFER STROKE BCK-ALGEBRAS

T. ONER, T. KALKAN AND A. BORUMAND SAEID

ایده‌آل‌های (پاد) فازی در BCK- جبرهای شفر استروک

تحسین اونر[1]، توچه کالکان[2] و ارشام برومند سعید[3]

[1,2]گروه ریاضی، دانشگاه اگ، برنوا، ازمیر، ترکیه

[3]گروه ریاضی محض، دانشکده ریاضی و کامپیوتر، دانشگاه شهید باهنر کرمان، کرمان، ایران

هدف از این مقاله معرفی ایده‌آل‌های (پاد) فازی در BCK-جبرهای شفر استروک است. پس از توصیف یک زیر جبر پاد فازی و ایده‌آل (زیر -استلزامی) پاد فازی در BCK-جبرهای شفر استروک، روابط این مفاهیم نشان داده می‌شوند. همچنین، یک برش سطح t و مکمل یک زیرمجموعه فازی تعریف شده و برخی از ویژگی‌های آن بررسی می‌شوند. یک BCK-جبرهای شفر استروک استلزامی تعریف شده و ثابت شده است که یک زیرمجموعه فازی از BCK-جبرهای شفر استروک استلزامی یک ایده‌آل پاد فازی است اگر و فقط اگر یک ایده‌آل زیر استلزامی پاد فازی از این ساختار جبری باشد. یک همنهشتی فازی و BCK-جبرهای شفر استروک خارج قسمتی با جزئیات مورد مطالعه قرار گرفته است و نشان داده شده است که بین مجموعه‌ی ایده‌آل‌های فازی و مجموعه‌ی همنهشتی‌های فازی در این ساختار جبری یک تناظر یک به یک وجود دارد.

کلمات کلیدی: BCK-جبر شفر استروک، ایده‌آل (پاد) فازی.