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ON THE FINITENESS OF FORMAL LOCAL COHOMOLOGY MODULES

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ABSTRACT. Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. In this paper, we prove some results concerning artinianness and finiteness of formal local cohomology modules. In particular, we investigate some properties of top formal local cohomology module $\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M)$ and we determine $\mathrm{Cos}_R(\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M))$ and $\mathrm{Ann}_R(\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M))$.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with identity, $\mathfrak{a}$ is an ideal of R and M is an R-module. Recall that the i-th local cohomology module of M with respect to $\mathfrak{a}$ is denoted by $\mathrm{H}_{\mathfrak{a}}^{i}(M)$. For basic facts about commutative algebra see [7]; for local cohomology refer to [4]. Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. For each $i \geq 0$; $\mathfrak{F}_{\mathfrak{a}}^{i}(M) := \varprojlim_n \mathrm{H}_{m}^{i}(M/\mathfrak{a}^n M)$ is called the i-th formal local cohomology of M with respect to $\mathfrak{a}$.

The basic properties of formal local cohomology modules are found in [1], [3], [5] and [12].

Recall that an R-module M is called minimax, if there is a finite submodule N of M, such that M/N is Artinian, The class of minimax modules was introduced by Zöschinger [14]. The class of minimax modules includes all finite and all Artinian modules. Moreover it is

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closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of R-modules.

In this paper, we investigate some artinianness and finiteness properties of formal local cohomology modules. At first, we obtain some results about the top formal local cohomology module. It is well known that $\dim M/\mathfrak{a}M$ is the largest integers i such that $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is non-zero (see [12, Theorem 4.5]). Here we determine $\mathrm{Cos}_R(\mathfrak{F}^{\dim M/\mathfrak{a}M}_{\mathfrak{a}}(M))$ and $\mathrm{Ann}_R(\mathfrak{F}^{\dim M/\mathfrak{a}M}_{\mathfrak{a}}(M))$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. Let $l := \dim(M/\mathfrak{a}M)$. We will prove that

i) $\mathrm{Cos}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Supp}_R((M/\mathfrak{a}M)/N)$ where N is the largest submodule of $M/\mathfrak{a}M$ such that $\dim N < l$.

ii) $\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \cap_{k\in\mathbb{N}} \mathrm{Ann}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M))$,

iii) If $\mathfrak{F}^l_{\mathfrak{a}}(M)$ is artinian, then there exists an integer $k \in \mathbb{N}$ such that $\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Ann}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M))$.

Also, we investigate the relation between artinianness and finiteness of $\mathfrak{F}^i_{\mathfrak{a}}(M)$ and $\mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M)$ for some integer i. Among other things, we show that if $\mathfrak{a} \subseteq \mathfrak{b}$ then

$$\inf\{i \in \mathbb{N}_0 : \mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M) \text{ is not representable}\}$$

and $\inf\{i \in \mathbb{N}_0 : \mathfrak{F}^i_{\mathfrak{a}}(M) \text{ is not artinian}\}$ are equal and

$$\sup\{i \in \mathbb{N}_0 : \mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M) \text{ is not representable}\}$$

and $\sup\{i \in \mathbb{N}_0 : \mathfrak{F}^i_{\mathfrak{a}}(M) \text{ is not artinian}\}$ are equal. As our last result, we will show that if $\mathfrak{a} \subseteq \mathfrak{b}$ then in each of the cases a) $\dim R \leq 2$, b) $\mathfrak{a}$ is principal, and c) $\dim R/\mathfrak{a} \leq 1$, the Betti number $\beta^j(\mathfrak{m}, \mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M))$ is finite for all i and j.

2. Main results

A non-zero R-module M is called secondary if its multiplication map by any element a of R is either surjective or nilpotent. A secondary representation for an R-module M is an expression for M as a finite sum of secondary submodules. If such a representation exists, we will say that M is representable. A prime ideal $\mathfrak{p}$ of R is said to be an attached prime of M if $\mathfrak{p} = (N :_R M)$ for some submodule N of M. If M admits a reduced secondary representation, $M = S_1 + S_2 + \ldots + S_n$, then the set of attached primes $\mathrm{Att}_R(M)$ of M is equal to $\{\sqrt{0 :_R S_i} : i = 1, \ldots, n\}$ (see [6]).

Let $S \subseteq R$ be a multiplicative set. The R_S-module $\mathrm{Hom}_R(R_S, M)$ is called the colocalization of M with respect to S and denoted by ${}_SM$. When M is an artinian R-module, it is known that ${}_SM$ is almost never an artinian R_S -module (see [9]), while by [9, Theorem 3.2] ${}_SM$ is a representable R_S -module. Thus the functor colocalization is not closed on the category of artinian modules.

The cosupport of M is defined by

$$\operatorname{Cos}_R M = \{\mathfrak{p} \in \operatorname{Spec} R : {}_{\mathfrak{p}}M \neq 0\}$$

(see [9]). A module is called cocyclic if it is a submodule of $E(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m}$ of R. Yassemi [13] defined the co-support of an R-module M, denoted by $\operatorname{Cosupp}_R(M)$, to be the set of primes $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of M with $\operatorname{Ann}(L) \subseteq \mathfrak{p}$. We always have $\operatorname{Cos}_R(M) \subseteq \operatorname{Cosupp}_R(M)$. Also, it is well known that in the case where M is an artinian R-module the equality $\operatorname{Cos}_R(M) = \operatorname{Cosupp}_R(M)$ holds (see [9, Lemma 7.3] and [13, 2.3]).

A prime ideal $\mathfrak{p}$ is called coassociated to a non-zero R-module M if there is a cocyclic homomorphic image T of M with $\mathfrak{p} = \operatorname{Ann}_R T$ [13]. The set of coassociated primes of M is denoted by $\operatorname{Coass}_R(M)$. In [13] we can see that $\operatorname{Coass}_R(M) \subseteq \operatorname{Cosupp}_R(M)$ and every minimal element of the set $\operatorname{Cosupp}_R(M)$ belongs to $\operatorname{Coass}_R(M)$.

Theorem 2.1. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. Let $l := \dim(M/\mathfrak{a}M)$. Then*

$$\operatorname{Cos}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \operatorname{Cos}_R(\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M)) = \operatorname{V}(\operatorname{Ann}_R(\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M))).$$

Proof. At first, we show that for any integer k we have

$$\operatorname{Cos}_R(\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)) = \operatorname{Cos}_R(\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M)).$$

Since $\operatorname{Supp}_R(M/\mathfrak{a}^k M) = \operatorname{Supp}_R(M/\mathfrak{a}M)$, we have

$$\operatorname{Assh}_R(M/\mathfrak{a}^k M) = \operatorname{Assh}_R(M/\mathfrak{a}M)$$

and so by [4, Theorem 7.3.2] $\operatorname{Att}_R(\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)) = \operatorname{Att}_R(\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M))$. Thus by [4, Proposition 7.2.11]

$$\sqrt{(0 : \operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M))} = \sqrt{(0 : \operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M))}.$$

Hence $\operatorname{V}((0 : \operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M))) = \operatorname{V}((0 : \operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M)))$. On the other hand, since $\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)$ and $\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M)$ are artinian [9, Lemma 7.3] implies that

$$\operatorname{Cos}_R(\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)) = \operatorname{V}((0 : \operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)))$$

and also $\operatorname{Cos}_R(\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M)) = \operatorname{V}((0 : \operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M)))$. Thus

$$\operatorname{Cos}_R(\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)) = \operatorname{Cos}_R(\operatorname{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M))$$

for any integer k.

Now, let $\mathfrak{p} \in \operatorname{Cos}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M))$. Thus $\mathfrak{p} \in \operatorname{Cos}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M))$ and so ${}_{\mathfrak{p}}(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)) \neq 0$ for any integer k. On the other hand, $\dim(\mathfrak{a}^k M/\mathfrak{a}^{k+1}M) \leq l$ and so the short exact sequence

$$0 \to \mathfrak{a}^k M/\mathfrak{a}^{k+1}M \to M/\mathfrak{a}^{k+1}M \to M/\mathfrak{a}^k M \to 0$$

induces an epimorphism $\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M) \to \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)$, of non-zero artinian R-modules for all $k \in \mathbb{N}$. By [9, Proposition 2.4] we get an epimorphism ${}_{\mathfrak{p}}(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M)) \to {}_{\mathfrak{p}}(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M))$, of non-zero R-modules for all $k \in \mathbb{N}$. Thus, by using [10, Theorem 2.22] we can see that $\varprojlim_k({}_{\mathfrak{p}}(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M))) \neq 0$. Therefore

$${}_{\mathfrak{p}}\varprojlim_k(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)) = {}_{\mathfrak{p}}\mathfrak{F}^l_{\mathfrak{a}}(M) \neq 0$$

and we conclude that $\mathfrak{p} \in \operatorname{Cos}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))$.

Conversly, assume that $\mathfrak{p} \in \operatorname{Cos}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))$. Thus ${}_{\mathfrak{p}}\mathfrak{F}^l_{\mathfrak{a}}(M) \neq 0$ and so

$${}_{\mathfrak{p}}(\varprojlim_n(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^n M))) \simeq \varprojlim_n({}_{\mathfrak{p}}(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^n M))) \neq 0.$$

Hence there exists an integer k such that ${}_{\mathfrak{p}}(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)) \neq 0$ and therefore $\mathfrak{p} \in \operatorname{Cos}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)) = \operatorname{Cos}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M))$ and the proof is complete. $\square$

Theorem 2.2. *Let $(R, \mathfrak{m})$ be a local ring and M a non-zero finitely generated R-module of dimension d. Then*

$$\operatorname{Cos}_R(\mathrm{H}^d_{\mathfrak{m}}(M)) = \operatorname{Cosupp}_R(\mathrm{H}^d_{\mathfrak{m}}(M)) = \operatorname{Supp}_R M/N$$

where N is the largest submodule of M such that $\dim N < d$.

Proof. Since $\mathrm{H}^d_{\mathfrak{m}}(M)$ is artinian, $\operatorname{Cos}_R(\mathrm{H}^d_{\mathfrak{m}}(M)) = \operatorname{Cosupp}_R(\mathrm{H}^d_{\mathfrak{m}}(M))$. Set $G := M/N$. We know that $\operatorname{Ann}_R G \subseteq \operatorname{Ann}_R(\mathrm{H}^d_{\mathfrak{m}}(G))$ and so $\mathrm{V}(\operatorname{Ann}_R(\mathrm{H}^d_{\mathfrak{m}}(G))) \subseteq \mathrm{V}(\operatorname{Ann}_R G)$. But $\mathrm{H}^d_{\mathfrak{m}}(G)$ is artinian and so by [13, Proposition 2.3] we have

$$\mathrm{V}(\operatorname{Ann}_R(\mathrm{H}^d_{\mathfrak{m}}(G))) = \operatorname{Cosupp}_R(\mathrm{H}^d_{\mathfrak{m}}(G)).$$

Thus it follows that $\operatorname{Cosupp}_R(\mathrm{H}^d_{\mathfrak{m}}(G)) \subseteq \operatorname{Supp}_R G$. Since by [4, Lemma 7.3.1] $\mathrm{H}^d_{\mathfrak{m}}(G) \simeq \mathrm{H}^d_{\mathfrak{m}}(M)$ we get $\operatorname{Cosupp}_R(\mathrm{H}^d_{\mathfrak{m}}(M)) \subseteq \operatorname{Supp}_R G$. On the other hand, by [4, Lemma 7.3.1] $\operatorname{Ass}_R G = \operatorname{Att}_R(\mathrm{H}^d_{\mathfrak{m}}(M))$. Thus

$$\operatorname{Ass}_R G \subseteq \operatorname{Cosupp}_R(\mathrm{H}^d_{\mathfrak{m}}(M)) = \mathrm{V}(\operatorname{Ann}_R(\mathrm{H}^d_{\mathfrak{m}}(M)))$$

and so

$$\operatorname{Supp}_R G \subseteq \mathrm{V}(\operatorname{Ann}_R(\mathrm{H}^d_{\mathfrak{m}}(M))) = \operatorname{Cosupp}_R(\mathrm{H}^d_{\mathfrak{m}}(M)).$$

The proof is complete. □

Theorem 2.3. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. Let $l := \dim(M/\mathfrak{a}M)$. Set $G' := (M/\mathfrak{a}M)/N'$ where N' is the largest submodule of $M/\mathfrak{a}M$ such that $\dim N' < l$. Then $\mathrm{Cos}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Supp}_R(G')$.*

Proof. By Theorem 2.1 $\mathrm{Cos}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Cos}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M))$. But by Theorem 2.2 we have $\mathrm{Cos}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M)) = \mathrm{Supp}_R(G')$ and the proof is complete. □

Lemma 2.4. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. Let $l := \dim M/\mathfrak{a}M$. Then for any $k \in \mathbb{N}$, there exists an exact sequence $\mathfrak{F}^l_{\mathfrak{a}}(M) \to \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M) \to 0$.*

Proof. For any integer $k \in \mathbb{N}$, the short exact sequence

$$0 \to \mathfrak{a}^k M \to M \to M/\mathfrak{a}^k M \to 0$$

induces the following exact sequence

$$\mathfrak{F}^l_{\mathfrak{a}}(M) \to \mathfrak{F}^l_{\mathfrak{a}}(M/\mathfrak{a}^k M) \to \mathfrak{F}^{l+1}_{\mathfrak{a}}(\mathfrak{a}^k M).$$

Since $\dim(\mathfrak{a}^k M/\mathfrak{a}^{k+1}M) \leq \dim(M/\mathfrak{a}M) = l$ we have

$$\mathfrak{F}^{l+1}_{\mathfrak{a}}(\mathfrak{a}^k M) = 0.$$

On the other hand, $M/\mathfrak{a}^k M$ is an $\mathfrak{a}$-torsion R-module and so by [3, Lemma 2.1] $\mathfrak{F}^l_{\mathfrak{a}}(M/\mathfrak{a}^k M) \simeq \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)$. Thus from the above sequence we get the result. □

In the next result, we determine the annihilators of top formal local cohomology module.

Theorem 2.5. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. Let $l := \dim M/\mathfrak{a}M$. Then*

$$\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \cap_{k\in\mathbb{N}} \mathrm{Ann}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)).$$

Proof. By Lemma 2.4 for any $k \in \mathbb{N}$, there exists an exact sequence

$$\mathfrak{F}^l_{\mathfrak{a}}(M) \to \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M) \to 0.$$

Thus $\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \mathrm{Ann}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M))$ for any $k \in \mathbb{N}$ and so we conclude that

$$\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \cap_{k\in\mathbb{N}} \mathrm{Ann}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)).$$

On the other hand, assume that $u \in \cap_{k\in\mathbb{N}} \mathrm{Ann}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M))$. Thus we have $u\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M) = 0$ for any integer $k \in \mathbb{N}$. Hence

$$u\mathfrak{F}^l_{\mathfrak{a}}(M) = u \varprojlim_k \mathrm{H}^l_m(M/\mathfrak{a}^k M) = 0$$

and so $u \in \operatorname{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))$. It follows that

$$\cap_{k\in\mathbb{N}} \operatorname{Ann}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)) \subseteq \operatorname{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)),$$

as required. □

Theorem 2.6. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and M a finitely generated R-module. Let $l := \dim M/\mathfrak{a}M$. If $\mathfrak{F}^l_{\mathfrak{a}}(M)$ is artinian, then there exists an integer $k \in \mathbb{N}$ such that*

$$\operatorname{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \operatorname{Ann}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)).$$

Proof. Let k be an integer. Take $J_k := \operatorname{Ann}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M))$. The short exact sequence

$$0 \to \mathfrak{a}^k M/\mathfrak{a}^{k+1}M \to M/\mathfrak{a}^{k+1}M \to M/\mathfrak{a}^k M \to 0$$

induces the following epimorphism

$$\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M) \to \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M) \to 0.$$

Thus $\operatorname{Ann}_R \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^{k+1}M) \subseteq \operatorname{Ann}_R \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)$ and so $J_{k+1} \subseteq J_k$. On the other hand,

$$\begin{aligned}
\cap_t(J_t\mathfrak{F}^l_{\mathfrak{a}}(M)) &= \varprojlim_t J_t \varprojlim_n \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^n M)\\
&\subseteq \varprojlim_t \varprojlim_n J_t \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^n M)\\
&= \varprojlim_n \varprojlim_t J_t \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^n M)\\
&= 0
\end{aligned}$$

as $J_t \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^n M) = 0$ for all $t \geq n$. As $\mathfrak{F}^l_{\mathfrak{a}}(M)$ is artinian, the descending chain

$$\cdots \subseteq J_3\mathfrak{F}^l_{\mathfrak{a}}(M) \subseteq J_2\mathfrak{F}^l_{\mathfrak{a}}(M) \subseteq J_1\mathfrak{F}^l_{\mathfrak{a}}(M)$$

of submodules of $\mathfrak{F}^l_{\mathfrak{a}}(M)$ is stable. Thus there exists an integer k such that

$$\cap_t J_t\mathfrak{F}^l_{\mathfrak{a}}(M) = J_k\mathfrak{F}^l_{\mathfrak{a}}(M).$$

Thus $J_k\mathfrak{F}^l_{\mathfrak{a}}(M) = 0$ and so $J_k \subseteq \operatorname{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))$. But, by Theorem 2.5 $\operatorname{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq J_k$. Therefore

$$\operatorname{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = J_k = \operatorname{Ann}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)).$$

□

Corollary 2.7. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and M a finitely generated R-module. Let $l := \dim(M/\mathfrak{a}M)$. If $\mathfrak{F}^l_{\mathfrak{a}}(M)$ is artinian, then there exists a submodule W of M such that*

$$\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Ann}_R(M/W).$$

Proof. By Thorem 2.6, there exists an integer k such that

$$\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Ann}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)).$$

But by [2, Corollary 2.4]

$$\mathrm{Ann}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)) = \mathrm{Ann}_R((M/\mathfrak{a}^k M)/(W/\mathfrak{a}^k M))$$

and $W/\mathfrak{a}^k M$ is the largest submodule of $M/\mathfrak{a}^k M$ such that

$$\dim W/\mathfrak{a}^k M < l.$$

Hence, $\mathrm{Ann}_R(\mathfrak{F}^l_{\mathfrak{a}}(M)) = \mathrm{Ann}_R(M/W)$. □

Theorem 2.8. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. Let $l := \dim(M/\mathfrak{a}M)$. Then*

$$\bigcup_{\mathfrak{p} \in \mathrm{Coass}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))} \mathfrak{p} = \bigcup_{\mathfrak{p} \in \mathrm{Assh}_R(M/\mathfrak{a}M)} \mathfrak{p}.$$

Proof. By Lemma 2.4, there exists an ephimorphism

$$\mathfrak{F}^l_{\mathfrak{a}}(M) \to \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M) \to 0.$$

Thus $\mathrm{Coass}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M)) \subseteq \mathrm{Coass}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))$. But, $\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M)$ is artinian and so by [13, Theorem 1.14] and [4, Proposition 7.3.2] we have

$$\mathrm{Coass}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M)) = \mathrm{Att}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}M)) = \mathrm{Assh}(M/\mathfrak{a}M).$$

Therefore $\mathrm{Assh}(M/\mathfrak{a}M) \subseteq \mathrm{Coass}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))$ and so we get

$$\bigcup_{\mathfrak{p} \in \mathrm{Assh}_R(M/\mathfrak{a}M)} \mathfrak{p} \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Coass}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))} \mathfrak{p}.$$

Now we show that

$$\textstyle\bigcup_{\mathfrak{p} \in \mathrm{Coass}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))} \mathfrak{p} \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Assh}_R(M/\mathfrak{a}M)} \mathfrak{p}.$$

Take $\mathfrak{p}_0 \in \mathrm{Coass}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))$. We claim that $\mathfrak{p}_0 \subseteq \cup_{\mathfrak{p} \in \mathrm{Assh}_R(M/\mathfrak{a}M)} \mathfrak{p}$. If not, there exists $u \in \mathfrak{p}_0$ such that $u \notin \cup_{\mathfrak{p} \in \mathrm{Assh}_R(M/\mathfrak{a}M)} \mathfrak{p}$. Since $u \in \mathfrak{p}_0$ and $\mathfrak{p}_0 \in \mathrm{Coass}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))$ by [13, Theorem 1.13] it follows that $u\mathfrak{F}^l_{\mathfrak{a}}(M) \neq \mathfrak{F}^l_{\mathfrak{a}}(M)$. On the other hand, $u \notin \cup_{\mathfrak{p} \in \mathrm{Assh}_R(M/\mathfrak{a}M)} \mathfrak{p}$ and it is easy to see that $\mathrm{Assh}_R(M/\mathfrak{a}M) = \mathrm{Assh}_R(M/\mathfrak{a}^k M)$ for any $k \in \mathbb{N}$. Thus $u \notin \cup_{\mathfrak{p} \in \mathrm{Att}_R(\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M))} \mathfrak{p}$ for any $k \in \mathbb{N}$. Thus, by [4, Proposition 7.2.11] we have $u\,\mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M) = \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)$ for any $k \in \mathbb{N}$ but the inverse limit on inverse system of artinian modules is exact and so

$u\varprojlim_k \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M) = \varprojlim_k \mathrm{H}^l_{\mathfrak{m}}(M/\mathfrak{a}^k M)$. Thus $u\mathfrak{F}^l_{\mathfrak{a}}(M) = \mathfrak{F}^l_{\mathfrak{a}}(M)$ which is a contradiction. Thus $\mathfrak{p}_0 \subseteq \cup_{\mathfrak{p}\in \mathrm{Assh}_R(M/\mathfrak{a}M)}\mathfrak{p}$. Therefore

$$\bigcup_{\mathfrak{p}\in \mathrm{Coass}_R(\mathfrak{F}^l_{\mathfrak{a}}(M))} \mathfrak{p} \subseteq \bigcup_{\mathfrak{p}\in \mathrm{Assh}_R(M/\mathfrak{a}M)} \mathfrak{p},$$

as required. □

Proposition 2.9. *Let R be a ring and $(Q_n)_{n\geq 1}$ be an inverse system of R-modules, with maps $\varphi_{mn} : Q_m \to Q_n$ for $m \geq n$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $u^k Q_k = 0$ for all $u \in \mathfrak{a}$ and all $k \in \mathbb{N}$. If $\mathfrak{b}\varprojlim_n Q_n$ is non-zero and representable, then $\mathfrak{a} \subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Att}_R(\mathfrak{b}\varprojlim_n Q_n)$.*

Proof. Let $\mathfrak{b}\varprojlim_n Q_n = S_1 + S_2 + ... + S_n$ be a minimal secondary representation of $\mathfrak{b}\varprojlim_n Q_n$ where S_j is $\mathfrak{p}_j$-secondary for $j = 1, 2, ..., n$. Assume that there exists an integer $j \in \{1, \dots, n\}$ such that $\mathfrak{a} \nsubseteq \mathfrak{p}_j$. Then there exists $u \in \mathfrak{a} \setminus \mathfrak{p}_j$. Since $S_j \neq 0$ there exists an element $0 \neq q = (q_k) \in S_j \subseteq \mathfrak{b}\varprojlim_n Q_n$. Let q_k be the first non-zero component of q. Since $u \notin \mathfrak{p}_j$, we have $uS_j = S_j$. But $u^k S_j \subseteq u^k(\mathfrak{b}\varprojlim_n Q_n)$, and so $S_j \subseteq u^k(\mathfrak{b}\varprojlim_n Q_n)$. As $u^k Q_k = 0$, it follows that the k-th component of each element of $u^k(\mathfrak{b}\varprojlim_n Q_n)$ is zero. But, $q \in u^k(\mathfrak{b}\varprojlim_n Q_n)$ and the k-th component of q is non-zero, which is a contradiction. □

Theorem 2.10. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module and $t \geq 0$ an integer. If $\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M)$ is representable then $\mathrm{Att}_R(\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M)) \subseteq \mathrm{V}(\mathfrak{a})$.*

Proof. Since $\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M) = \mathfrak{b}\varprojlim_n \mathrm{H}^t_m(M/\mathfrak{a}^n M)$ and $u^k\, \mathrm{H}^t_m(M/\mathfrak{a}^k M) = 0$ for all $u \in \mathfrak{a}$ and $k \in \mathbb{N}$, the result follows by Proposition 2.9. □

Corollary 2.11. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module and $t \geq 0$ an integer. If $\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M)$ is a representable R-module then $\mathfrak{a} \subseteq \sqrt{(0 :_R \mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M))}$. In particular, $\mathfrak{a} \subseteq \sqrt{(0 :_R \mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M))}$ if $\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M)$ is artinian.*

Proof. By [4, Theorem 7.2.11],

$$\sqrt{(0 :_R \mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M))} = \cap_{\mathfrak{p}\in \mathrm{Att}_R(\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M))}\mathfrak{p}$$

and by Theorem 2.10 $\operatorname{Att}_R(\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M)) \subseteq \mathrm{V}(\mathfrak{a})$. Thus $\mathfrak{a} \subseteq \sqrt{(0 :_R \mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M))}$. □

The following result is an extension of [3, Corollary 2.14].

Corollary 2.12. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of a local ring* $(R, \mathfrak{m})$ *and* M *a finitely generated* R*-module. Let* t *be an integer. In each of the following cases,* $\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M)$ *is artinian if and only if* $\mathfrak{a} \subseteq \sqrt{(0 : \mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M))}$*:*
(i) $\mathfrak{a}$ *is principal.*
(ii) $\dim R \leq 2$.
(iii) $\dim R/\mathfrak{a} \leq 1$.

Proof. (⇒) By Corollary 2.11.
(⇐) In the proof of [3, Corollary 2.14], it has shown that in each of these cases we have $\operatorname{Hom}_R(R/\mathfrak{a}, \mathfrak{F}^i_{\mathfrak{a}}(M))$ is artinian for all $i \geq 0$. Thus $\operatorname{Hom}_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M))$ is artinian and so $\operatorname{Hom}_R(R/\mathfrak{a}, \mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M))$ is artinian. On the other hand, by assumption $\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M)$ is an $\mathfrak{a}$-torsion R-module. Hence, the claim follows by [8, Theorem 1.3]. □

The following theorem is a generalization of [3, Corollary 2.8].

Theorem 2.13. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of a local ring* $(R, \mathfrak{m})$ *such that* $\mathfrak{a} \subseteq \mathfrak{b}$ *and* M *a finitely generated* R*-module, and let* $n \in \mathbb{N}$*. Then the following statements are equivalent:*
i) $\mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M)$ *is artinian for all* $i < n$*,*
ii) $\mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M)$ *is representable for all* $i < n$*,*
iii) $\mathfrak{F}^i_{\mathfrak{a}}(M)$ *is artinian for all* $i < n$*.*

Proof. i) ⇒ ii): Any artinian R-module is representable.
ii) ⇒ iii): By Corollary 2.11, $\mathfrak{a} \subseteq \sqrt{(0 :_R \mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M))}$ for all $i < n$. Since $\mathfrak{a} \subseteq \mathfrak{b}$, it follows that $\mathfrak{a} \subseteq \sqrt{(0 :_R \mathfrak{F}^i_{\mathfrak{a}}(M))}$ for all $i < n$. Now, the result follows by [3, Theorem 2.6].
iii) ⇒ i): It is clear. □

Recall that the formal grade of M with respect to $\mathfrak{a}$ is defined by

$$\operatorname{fgrade}(\mathfrak{a}, M) = \inf\{i \in \mathbb{N}_0 : \mathfrak{F}^i_{\mathfrak{a}}(M) \neq 0\}.$$

The next result is a characterization of the artinianness of $\mathfrak{b}\mathfrak{F}^g_{\mathfrak{a}}(M)$, where $g := \operatorname{fgrade}(\mathfrak{a}, M)$.

Corollary 2.14. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of a local ring* $(R, \mathfrak{m})$ *such that* $\mathfrak{a} \subseteq \mathfrak{b}$ *and* M *a finitely generated* R*-module, and let* $g := \operatorname{fgrade}(\mathfrak{a}, M)$*. Then* $\mathfrak{b}\mathfrak{F}^g_{\mathfrak{a}}(M)$ *is artinian, if and only if* $\mathfrak{b}\mathfrak{F}^g_{\mathfrak{a}}(M)$ *is representable.*

Proof. Since $\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$ for all $i < g$, the result follows by Theorem 2.13. □

Theorem 2.15. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R,\mathfrak{m})$ such that $\mathfrak{a}\subseteq\mathfrak{b}$ and M a finitely generated R-module, and let $n\in\mathbb{N}$. Then the following statements are equivalent:*
i) $\mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M)$ is artinian for all $i>n$,
ii) $\mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M)$ is representable for all $i>n$,
iii) $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is artinian for all $i>n$.

Proof. i) ⇒ ii): Any artinian R-module is representable.
ii) ⇒ iii): Since $\mathfrak{a}\subseteq\mathfrak{b}$, by Corollary 2.11, we conclude that

$$\mathfrak{a}\subseteq\sqrt{(0:_R\mathfrak{F}^i_{\mathfrak{a}}(M))}$$

for all $i>n$ and so the result follows by [3, Theorem 2.9].
iii) ⇒ i): It is clear. □

Corollary 2.16. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R,\mathfrak{m})$ such that $\mathfrak{a}\subseteq\mathfrak{b}$ and M a finitely generated R-module, and let $l:=\dim(M/\mathfrak{a}M)$. Then $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ is artinian, if and only if $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ is representable.*

Proof. By [12, Theorem 4.5] $\mathfrak{F}^i_{\mathfrak{a}}(M)=0$ for all $i>l$ and so the result follows by Theorem 2.15. □

Proposition 2.17. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R,\mathfrak{m})$ and M a finitely generated R-module and $t\geq 0$ an integer. Let $\mathfrak{a}\subseteq\mathfrak{b}$. If $\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M))\subseteq \mathrm{V}(\mathfrak{a})$ then $\mathrm{Coass}_R(\mathfrak{F}^t_{\mathfrak{a}}(M))\subseteq\mathrm{V}(\mathfrak{a})$.*

Proof. By [13, Theorem 1.21],

$$\begin{aligned}\mathrm{Coass}(\mathfrak{F}^t_{\mathfrak{a}}(M)/\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M)) &= \mathrm{Supp}(R/\mathfrak{b})\cap\mathrm{Coass}_R(\mathfrak{F}^t_{\mathfrak{a}}(M))\\ &\subseteq \mathrm{V}(\mathfrak{b})\\ &\subseteq \mathrm{V}(\mathfrak{a}),\end{aligned}$$

and by assumption $\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M))\subseteq\mathrm{V}(\mathfrak{a})$. Since by [13, Theorem 1.10] we have

$$\mathrm{Coass}_R(\mathfrak{F}^t_{\mathfrak{a}}(M))\subseteq\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M))\cup\mathrm{Coass}_R(\mathfrak{F}^t_{\mathfrak{a}}(M)/\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M)),$$

we conclude that $\mathrm{Coass}_R(\mathfrak{F}^t_{\mathfrak{a}}(M))\subseteq\mathrm{V}(\mathfrak{a})$, as required. □

Theorem 2.18. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R,\mathfrak{m})$ and M a finitely generated R-module. If either R is complete or $\mathfrak{a}\subseteq\mathfrak{b}$ then $\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)/\mathfrak{a}^k\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)$ is artinian for any integer k.*

Proof. At first, assume that R is complete. By [1, Theorem 2.6], $\mathfrak{F}^0_{\mathfrak{a}}(M)$ is a finitely generated R-module and by [12, Lemma 4.1]

$$\mathrm{Ass}_R(\mathfrak{F}^0_{\mathfrak{a}}(M))=\{\mathfrak{p}\in\mathrm{Ass}_R M:\dim R/(\mathfrak{a}+\mathfrak{p})=0\}$$

and so $\mathrm{Supp}_R(\mathfrak{F}^0_{\mathfrak{a}}(M))\cap\mathrm{V}(\mathfrak{a})\subseteq\{\mathfrak{m}\}$. Therefore

$$\mathrm{Supp}_R(\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)/\mathfrak{a}^k\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M))=\mathrm{Supp}_R(\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M))\cap\mathrm{V}(\mathfrak{a})\subseteq\mathrm{V}(\mathfrak{m}).$$

But $\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)/\mathfrak{a}^k\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)$ is finitely generated and so $\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)/\mathfrak{a}^k\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)$ has finite length and the result follows in this case.

Now, assume that $\mathfrak{a} \subseteq \mathfrak{b}$. By [1, Theorem 3.8] $\mathfrak{F}^0_{\mathfrak{a}^k}(M)/\mathfrak{a}^k\mathfrak{F}^0_{\mathfrak{a}^k}(M)$ is artinian for all $k \in \mathbb{N}$. But, by [12, Lemma 3.8] $\mathfrak{F}^0_{\mathfrak{a}^k}(M) \simeq \mathfrak{F}^0_{\mathfrak{a}}(M)$ and so $\mathfrak{F}^0_{\mathfrak{a}}(M)/\mathfrak{a}^k\mathfrak{F}^0_{\mathfrak{a}}(M)$ is artinian for any integer k. Since $\mathfrak{a} \subseteq \mathfrak{b}$ we conclude that $\mathfrak{F}^0_{\mathfrak{a}}(M)/\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)$ is artinian. From the exact sequence

$$0 \to \mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M) \to \mathfrak{F}^0_{\mathfrak{a}}(M) \to \mathfrak{F}^0_{\mathfrak{a}}(M)/\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M) \to 0$$

we have the following exact sequence

$$\operatorname{Tor}^R_1(R/\mathfrak{a}^k, \mathfrak{F}^0_{\mathfrak{a}}(M)/\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)) \to R/\mathfrak{a}^k \otimes_R \mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M) \to R/\mathfrak{a}^k \otimes_R \mathfrak{F}^0_{\mathfrak{a}}(M).$$

Since $\operatorname{Tor}^R_1(R/\mathfrak{a}^k, \mathfrak{F}^0_{\mathfrak{a}}(M)/\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M))$ and $R/\mathfrak{a}^k \otimes_R \mathfrak{F}^0_{\mathfrak{a}}(M)$ are artinian, from the above sequence we conclude that $\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)/\mathfrak{b}\mathfrak{a}^k\mathfrak{F}^0_{\mathfrak{a}}(M)$ is artinian for all $k \in \mathbb{N}$. □

Corollary 2.19. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ such that $\mathfrak{a} \subseteq \mathfrak{b}$ and M a finitely generated R-module. If $\operatorname{Coass}_R(\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)) \subseteq \operatorname{V}(\mathfrak{a})$ then $\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)$ is minimax.*

Proof. Since $\operatorname{Coass}_R(\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)) \subseteq \operatorname{V}(\mathfrak{a})$ by [15, Satz 2.4] there exists an integer k such that $\mathfrak{a}^k\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)$ is finitely generated. But, by Theorem 2.18 $\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)/\mathfrak{a}^k\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)$ is artinian. Therefore we conclude that $\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)$ is minimax, as required. □

Theorem 2.20. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ such that $\mathfrak{a} \subseteq \mathfrak{b}$ and M a finitely generated R-module. Then the following statements are equivalent:*

i) $\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)$ is finitely generated,
ii) $\operatorname{Cosupp}_R(\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{m}\}$.

Proof. i) ⇒ ii): By [13, Theorem 2.10].

ii) ⇒ i): Assume that $\operatorname{Coass}_R(\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{m}\}$. Thus we have $\operatorname{Coass}_R(\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)) \subseteq \operatorname{V}(\mathfrak{a})$. Now Corollary 2.19 implies that $\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)$ is minimax and so there exists a finitely generated submodule N such that $L := \dfrac{\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)}{N}$ is artinian. Hence

$$\operatorname{Att}_R L = \operatorname{Coass}_R L \subseteq \operatorname{Cosupp}_R L \subseteq \operatorname{Cosupp}_R(\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{m}\}.$$

Since L is artinian and $\operatorname{Att}_R L \subseteq \{\mathfrak{m}\}$, [4, Corollary 7.2.12] implies that L is finitely generated. Since L and N are finitely generated, we conclude that $\mathfrak{b}\mathfrak{F}^0_{\mathfrak{a}}(M)$ is finitely generated. □

Theorem 2.21. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ such that $\mathfrak{a} \subseteq \mathfrak{b}$ and M a finitely generated R-module, and let $t \in \mathbb{N}$. If $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is artinian for all $i > t$ then $\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M)/\mathfrak{a}^k\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M)$ is artinian for all $k \in \mathbb{N}$.*

Proof. By [12, Lemma 3.8] $\mathfrak{F}^i_{\mathfrak{a}^k}(M) \simeq \mathfrak{F}^i_{\mathfrak{a}}(M)$ for all integers i, k and so by using [1, Theorem 3.8] it follows that $\mathfrak{F}^t_{\mathfrak{a}}(M)/\mathfrak{a}^k\mathfrak{F}^t_{\mathfrak{a}}(M)$ is artinian for all $k \in \mathbb{N}$. Since $\mathfrak{a} \subseteq \mathfrak{b}$ we have $\mathfrak{F}^t_{\mathfrak{a}}(M)/\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M)$ is artinian. But, the exact sequence

$$0 \to \mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M) \to \mathfrak{F}^t_{\mathfrak{a}}(M) \to \mathfrak{F}^t_{\mathfrak{a}}(M)/\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M) \to 0$$

induces the following exact sequence:

$$\mathrm{Tor}^R_1(R/\mathfrak{a}^k, \mathfrak{F}^t_{\mathfrak{a}}(M)/\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M)) \to R/\mathfrak{a}^k \otimes_R \mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M) \to R/\mathfrak{a}^k \otimes_R \mathfrak{F}^t_{\mathfrak{a}}(M).$$

From the above sequence we conclude that $\mathfrak{b}\mathfrak{F}^t_{\mathfrak{a}}(M)/\mathfrak{b}\mathfrak{a}^k\mathfrak{F}^t_{\mathfrak{a}}(M)$ is artinian. □

Corollary 2.22. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ such that $\mathfrak{a} \subseteq \mathfrak{b}$ and M a finitely generated R-module. Let $l := \dim(M/\mathfrak{a}M)$. Then $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)/\mathfrak{a}^k\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ is artinian for all $k \in \mathbb{N}$.*

Proof. Since $\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$ for all $i > l$ by [12, Theorem 4.5], the result follows by Theorem 2.21. □

Corollary 2.23. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ such that $\mathfrak{a} \subseteq \mathfrak{b}$ and M a finitely generated R-module. Let $l := \dim(M/\mathfrak{a}M)$. If $\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \mathrm{V}(\mathfrak{a})$ then $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ is minimax.*

Proof. Since $\mathrm{Coass}_R(\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)) \subseteq \mathrm{V}(\mathfrak{a})$ there exists an integer k such that $\mathfrak{a}^k\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ is finitely generated. By Corollary 2.22,

$$\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)/\mathfrak{a}^k\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$$

is artinian. Therefore $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ is minimax, as required. □

Theorem 2.24. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module, and let $n \in \mathbb{N}$. Then the following statements are equivalent:*

i) $\mathfrak{m}\mathfrak{F}^i_{\mathfrak{a}}(M)$ is finitely generated for all $i > n$,
ii) $\mathrm{Coass}_R(\mathfrak{m}\mathfrak{F}^i_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{m}\}$ for all $i > n$,
iii) $\mathrm{Coass}_R(\mathfrak{F}^i_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{m}\}$ for all $i > n$,
iv) $\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$ for all $i > n$.

Proof. i) ⇒ ii): By [13, Theorem 2.10].

ii) ⇒ iii): Let $t > n$ be an integer. we have

$$\mathrm{Coass}_R(\mathfrak{F}^t_{\mathfrak{a}}(M)) \subseteq \mathrm{Coass}_R(\mathfrak{m}\mathfrak{F}^t_{\mathfrak{a}}(M)) \cup \mathrm{Coass}_R(\mathfrak{F}^t_{\mathfrak{a}}(M)/\mathfrak{m}\mathfrak{F}^t_{\mathfrak{a}}(M)).$$

Since

$$\mathrm{Coass}_R(\mathfrak{F}^t_{\mathfrak{a}}(M)/\mathfrak{m}\mathfrak{F}^t_{\mathfrak{a}}(M)) = \mathrm{V}(\mathfrak{m}) \cap \mathrm{Coass}_R\,\mathfrak{F}^t_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{m}\},$$

by using the assumption we conclude that $\mathrm{Coass}_R(\mathfrak{F}^t_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{m}\}$.

iii) $\Rightarrow$ iv) By [11, Theorem 2.10].

iv) $\Rightarrow$ i): It is clear. □

Theorem 2.25. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ such that $\mathfrak{a} \subseteq \mathfrak{b}$ and M a finitely generated R-module. Let n be an integer. If $\mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M)$ is finitely generated for all $i > n$ then $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is artinian for all $i > n$.*

Proof. Let $l := \dim(M/\mathfrak{a}M)$. Then $\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$ for all $i > l$ by [12, Theorem 4.5]. Thus we can assume that $n < l$. We proceed by descending induction on n. Assume, inductively that the result has been proved for all $i > n+1$. Thus $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is artinian for all $i > n+1$ and it is enough to show that $\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)$ is artinian. By [1, Theorem 3.8] it follows that $\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)/\mathfrak{a}\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)$ is artinian. Since $\mathfrak{a} \subseteq \mathfrak{b}$ we can see that $\mathfrak{a}\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)$ is a submodule of $\mathfrak{b}\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)$ and so $\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)/\mathfrak{b}\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)$ is artinian. Thus, for any ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\mathfrak{p} \neq \mathfrak{m}$ we have $(\mathfrak{F}^{n+1}_{\mathfrak{a}}(M))_{\mathfrak{p}} = \mathfrak{b}_{\mathfrak{p}}\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)_{\mathfrak{p}}$. But, assumption implies that $\mathfrak{b}_{\mathfrak{p}}\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module and so the above equality shows that $\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module. Thus, Nakayama Lemma implies that $\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)_{\mathfrak{p}} = 0$ and so $\mathfrak{b}_{\mathfrak{p}}\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)_{\mathfrak{p}} = 0$. It now follows that $\mathrm{Supp}_R(\mathfrak{b}\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{m}\}$. Since by assumption $\mathfrak{b}\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)$ is a finitely generated R-module we conclude that $\mathfrak{b}\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)$ is artinian. On the other hand, $\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)/\mathfrak{b}\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)$ is artinian and so we conclude that $\mathfrak{F}^{n+1}_{\mathfrak{a}}(M)$ is artinian, as required. □

Corollary 2.26. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ such that $\mathfrak{a} \subseteq \mathfrak{b}$ and M a finitely generated R-module. Let $l := \dim(M/\mathfrak{a}M) > 0$. If $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ is finitely generated then $\mathfrak{F}^l_{\mathfrak{a}}(M)$ is artinian.*

Proof. Assume that $\mathfrak{b}\mathfrak{F}^l_{\mathfrak{a}}(M)$ is a finitely generated R-module. Since $\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$ for all $i > l$ by Theorem 2.25 we conclude that $\mathfrak{F}^l_{\mathfrak{a}}(M)$ is artinian. □

Theorem 2.27. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ such that $\mathfrak{a} \subseteq \mathfrak{b}$ and M a finitely generated R-module. In each of the following cases, $\mathrm{Tor}^R_j(R/\mathfrak{a}, \mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M))$ is artinian for all i and j.*

i) $\dim R \leq 2$.

ii) $\mathfrak{a}$ *is principal.*

iii) $\dim R/\mathfrak{a} \leq 1$.

Proof. By [1, Theorems 3.2, 3.6 and 3.8] in each of these cases

$$\mathrm{Tor}^R_j(R/\mathfrak{a}, \mathfrak{F}^i_{\mathfrak{a}}(M))$$

is artinian for all i and j. Thus $\mathfrak{F}^i_{\mathfrak{a}}(M)/\mathfrak{a}\mathfrak{F}^i_{\mathfrak{a}}(M)$ is artinian for all i. Since $\mathfrak{a} \subseteq \mathfrak{b}$ we conclude that $\mathfrak{F}^i_{\mathfrak{a}}(M)/\mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M)$ is artinian for all i. But, the exact sequence

$$0 \to \mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M) \to \mathfrak{F}^i_{\mathfrak{a}}(M) \to \mathfrak{F}^i_{\mathfrak{a}}(M)/\mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M) \to 0$$

induces the following exact sequence:

$$\operatorname{Tor}^R_{j+1}(R/\mathfrak{a}, \mathfrak{F}^i_{\mathfrak{a}}(M)/\mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M)) \to \operatorname{Tor}^R_j(R/\mathfrak{a}, \mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M)) \to \operatorname{Tor}^R_j(R/\mathfrak{a}, \mathfrak{F}^i_{\mathfrak{a}}(M)).$$

From the above sequence we conclude that $\operatorname{Tor}^R_j(R/\mathfrak{a}, \mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M))$ is artinian for all i and j, as required. □

The next result, can be considered as a generalization of [1, Corollary 3.9].

Corollary 2.28. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a local ring $(R, \mathfrak{m})$ such that $\mathfrak{a} \subseteq \mathfrak{b}$ and M a finitely generated R-module. In each of the following cases, the Betti number $\beta^j(\mathfrak{m}, \mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M))$ is finite for all i and j.*

i) $\dim R \leq 2$.

ii) $\mathfrak{a}$ *is principal.*

iii) $\dim R/\mathfrak{a} \leq 1$.

Proof. By Theorem 2.27, in each of the above cases $\operatorname{Tor}^R_j(R/\mathfrak{a}, \mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M))$ is artinian for all i and j. Thus by [1, Lemma 3.1] it follows that $\operatorname{Tor}^R_j(R/\mathfrak{m}, \mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M))$ is also artinian for all i and j. Hence, we conclude that $\beta^j(\mathfrak{m}, \mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M)) := \dim_{R/\mathfrak{m}} \operatorname{Tor}^R_j(R/\mathfrak{m}, \mathfrak{b}\mathfrak{F}^i_{\mathfrak{a}}(M))$ is finite for all i and j, as required. □

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ON THE FINITENESS OF FORMAL LOCAL COHOMOLOGY MODULES

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نتایجی درباره متناهی بودن مدول‌های کوهمولوژی موضعی صوری

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فرض کنید $\mathfrak{a}$ یک ایده‌آل از حلقه‌ی نوتری موضعی $(R, \mathfrak{m})$ و M یک R-مدول متناهی مولد باشد. در این مقاله چندین نتیجه درباره متناهی مولد بودن، مینیماکس بودن و آرتینی بودن مدول‌های کوهمولوژی موضعی صوری به‌دست می‌آوریم. همچنین آخرین مدول ناصفر کوهمولوژی موضعی صوری $\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M)$ را مورد بررسی قرار داده و مجموعه‌های

$$\mathrm{Ann}_R(\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M)) \text{ و } \mathrm{Cos}_R(\mathfrak{F}_{\mathfrak{a}}^{\dim M/\mathfrak{a}M}(M))$$

را به‌دست می‌آوریم. در یکی دیگر از نتایج بدست آمده نشان می‌دهیم که اگر $\mathfrak{a}$ و $\mathfrak{b}$ دو ایده‌آل حلقه باشند و $\mathfrak{a} \subseteq \mathfrak{b}$ در این‌صورت برای هر دو عدد i و j، عدد بتی $\beta^j(\mathfrak{m}, \mathfrak{b}\mathfrak{F}_{\mathfrak{a}}^i(M))$ در برخی شرایط خاص متناهی است.

کلمات کلیدی: کوهمولوژی موضعی صوری، کوهمولوژی موضعی، آرتینی.