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# EXTENSION AND TORSION FUNCTORS WITH RESPECT TO SERRE CLASSES

S. ARDA AND S. O. FARAMARZI*

ABSTRACT. In this paper we generalize the Rigidity Theorem and Zero Divisor Conjecture for an arbitrary Serre subcategory of modules. For this purpose, for any regular $M$-sequence $x_1, ..., x_n$ with respect to $\mathcal{S}$ we prove that if $\mathrm{Tor}_2^R(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$, then $\mathrm{Tor}_i^R(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$, for all $i \geq 1$. Also we show that if $\mathrm{Ext}_R^{n+2}(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$, then $\mathrm{Ext}_R^i(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$, for all integers $i \geq 0$ $(i \neq n)$.

## 1. Introduction

Throughout this paper, $R$ denotes a commutative and Noetherian ring with non-zero identity, $I$ denotes an arbitrary ideal and $M$ denotes a finitely generated $R$-module. Let $\mathcal{S}$ be a Serre subcategory of the category of $R$-modules. In 1961, M. Auslander proposed the Zero Divisor Conjecture in [2] as follows:

**Zero divisor conjecture.** Let $R$ be a local ring and $M$ be a finitely generated $R$-module of finite projective dimension. If $x \in R$ is a non-zerodivisor on $M$, then $x$ is a non-zerodivisor of $R$.

This conjecture was proved by M. Hochster, L. Szpiro, C. Peskin, and P. Robert (see [6]), in special cases. Also M. Auslander introduced rigidity concept as a generalization of Zero Divisor Conjecture.

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*Corresponding author.

**Definition.** Let $(R, \mathfrak{m})$ be a local ring. An $R$-module $M$ is called rigid if $\mathrm{Tor}_i^R(M, N) = 0$ for some finitely generated $R$-module $N$, then $\mathrm{Tor}_j^R(M, N) = 0$ for any $j \geq i$ (see [2]).

He also stated the following theorem.

**Rigidity Theorem.** Let $(R, \mathfrak{m})$ be a regular local ring and $M$ be a finitely generated $R$-module. Then $M$ is rigid.

The Rigidity Theorem was proved by M. Auslander in unramified case. S. Lichtenbaum proved the theorem for arbitrary regular local rings in 1966 (see [5]). In this paper, we generalize the Zero Divisor Conjecture and Rigidity Theorem for an arbitrary Serre subcategory of modules. An $R$-module $M$ is called $\mathcal{S}$-rigid if $\mathrm{Tor}_i^R(M, N) \in \mathcal{S}$ for some finitely generated $R$-module $N$, then $\mathrm{Tor}_j^R(M, N) \in \mathcal{S}$ for any $j \geq i$. Also for an $R$-module $M$, Generalized Zero Divisor Conjecture holds if every regular $M$-sequence with respect to $\mathcal{S}$ is a regular $R$-sequence with respect to $\mathcal{S}$. For this purpose, we prove the following two main theorems.

**Theorem 1.** Let $R$ be a Noetherian (not necessary local) ring and $M$ be a non-zero finitely generated $R$-module. Let $x_1, ..., x_n$ be a poor regular $M$-sequence with respect to $\mathcal{S}$. If $\mathrm{Tor}_2^R(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$, then $\mathrm{Tor}_i^R(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$, for any $i \geq 1$ (see theorem 3.4).

**Theorem 2.** Let $R$ be a Noetherian (not necessary local) ring, $M$ be a non-zero finitely generated $R$-module, and $I$ be an ideal of $R$ with $\mathcal{S} - \mathrm{E}.grad_R(I, M) = n \geqslant 1$. Assume that $x_1, ..., x_n \in I$ is a maximal regular $M$-sequence with respect to $\mathcal{S}$. If $\mathrm{Ext}_R^{n+2}(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$, then $\mathrm{Ext}_R^i(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$, for all integers $i \geq 0$ ($i \neq n$) (see theorem 3.9). Finally, as a consequence of the above theorems, we prove some corollaries for top local cohomology modules (see theorems 3.5 and 3.10).

## 2. Preliminaries

A subcategory of the category of $R$-modules and $R$-homomorphisms $\mathcal{S}$ is said to be a *Serre class* (or Serre subcategory), if for any exact sequence of $R$-modules

$$0 \to L \to M \to N \to 0,$$

the $R$-module $M$ belongs to $\mathcal{S}$ if and only if each of $L$ and $N$ belong to $\mathcal{S}$.

**Definition 2.1.** [1, Definition 2·2] Suppose that $M$ is an $R$-module. A sequence $x_1, ..., x_n$ of elements of $R$ is called a poor regular $M$-sequence with respect to $\mathcal{S}$ if for each $i = 1, ..., n$ the $R$-module $(0 :_{\frac{M}{(x_1,...,x_{i-1})M}} x_i)$

belongs to $\mathcal{S}$. If in addition $\frac{M}{(x_1,...,x_n)M} \notin \mathcal{S}$, we say that $x_1, ..., x_n$ is a regular $M$-sequence with respect to $\mathcal{S}$.

For an $R$-module $L$, we denote

$$\mathcal{S}-\operatorname{Supp}_R L := \{\mathfrak{p} \in \operatorname{Supp}_R L : \frac{R}{\mathfrak{p}} \notin \mathcal{S}\}$$

and

$$\mathcal{S}-\operatorname{Ass}_R L := \{\mathfrak{p} \in \operatorname{Ass}_R L : \frac{R}{\mathfrak{p}} \notin \mathcal{S}\}.$$

**Lemma 2.2.** [1, Lemma $2 \cdot 1$] *Let $M$ be a finitely generated $R$-module. Then $M \in \mathcal{S}$ if and only if $\frac{R}{\mathfrak{p}} \in \mathcal{S}$ for all $\mathfrak{p} \in \operatorname{Supp}_R M$. In particular, for any two finitely generated $R$-modules $N$ and $L$ with $\operatorname{Supp}_R N = \operatorname{Supp}_R L$, we have $N \in \mathcal{S}$ if and only if $L \in \mathcal{S}$.*

The following statements are equivalent by the definition.

**Lemma 2.3.** [1, Lemma $2 \cdot 3$] *Let $M$ be a finitely generated $R$-module and $x_1, ..., x_n$ a sequence of elements of $R$. Then the following are equivalent:*

(1) $x_i \notin \bigcup_{\mathfrak{p} \in \mathcal{S}-\operatorname{Ass}_R \frac{M}{(x_1,...,x_{i-1})}} \mathfrak{p}$ *for all $i = 1, ..., n$.*
(2) *The sequence $x_1, ..., x_n$ is a poor regular $M$-sequence with respect to $\mathcal{S}$.*
(3) *For any $\mathfrak{p} \in \mathcal{S} - \operatorname{Supp}_R M$, the elements $\frac{x_1}{1}, ..., \frac{x_n}{1}$ of the local ring $R_\mathfrak{p}$ form a poor regular $M_\mathfrak{p}$-sequence.*
(4) *The sequence $x_1^{t_1}, ..., x_n^{t_n}$ is a poor regular $M$-sequence with respect to $\mathcal{S}$ for all positive integers $t_1, ..., t_n$.*

**Definition 2.4.** [1, Definition $2 \cdot 6$] Let $M$ be an $R$-module and $\mathfrak{a}$ be an ideal of $R$. The notation of Ext grade of $\mathfrak{a}$ on $M$ with respect to $\mathcal{S}$ is defined as follows:

$$\mathcal{S}-\operatorname{E}.grade_R(\mathfrak{a}, M) := \inf\{i \in \mathbb{N}_0 : \operatorname{Ext}^i_R(\frac{R}{\mathfrak{a}}, M) \notin \mathcal{S}\}.$$

## 3. Main results

Similar to the property of regular sequences we have the following.

**Lemma 3.1.** *Let $x_1, ..., x_n$ be a poor regular $M$-sequence with respect to $\mathcal{S}$, then*

$$\operatorname{Tor}^R_1(\frac{R}{(x_1, ..., x_n)}, M) \in \mathcal{S}.$$

*Proof.* Let $x_1, ..., x_n$ is a poor $M$-sequence with respect to $\mathcal{S}$, then for every $\mathfrak{p} \in \mathcal{S}-\mathrm{Supp}_R(M)$, $\frac{x_1}{1}, ..., \frac{x_n}{1}$ is a poor regular $M_\mathfrak{p}$-sequence. Thus $\mathrm{Tor}_1^{R_\mathfrak{p}}(\frac{R_\mathfrak{p}}{(\frac{x_1}{1},...,\frac{x_n}{1})}, M_\mathfrak{p}) = 0$, by [4, Exercise 1 · 1.12]. This implies that $\mathcal{S} - \mathit{Supp}\,\mathrm{Tor}_1^R(\frac{R}{(x_1,...,x_n)}, M) = \emptyset$. Hence $\mathrm{Tor}_1^R(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$. □

**Lemma 3.2.** *Let $R$ be a Noetherian (not necessary local) ring and $M$ be a non-zero finitely generated $R$-module. Let $x$ be a poor regular $M$-sequence with respect to $\mathcal{S}$. If $\mathrm{Tor}_2^R(\frac{R}{(x)}, M) \in \mathcal{S}$, then $(0{:}_R x)\otimes_R M \in \mathcal{S}$.*

*Proof.* The exact sequence

$$0 \to Rx \to R \to \frac{R}{Rx} \to 0 \tag{3.1}$$

implies that $\mathrm{Tor}_2^R(\frac{R}{Rx}, M) \cong \mathrm{Tor}_1^R(Rx, M)$ and hence

$$\mathrm{Tor}_1^R(Rx, M) \in \mathcal{S}. \tag{3.2}$$

Also, the exact sequence

$$0 \to (0{:}_R x) \to R \to Rx \to 0 \tag{3.3}$$

induces the exact sequence

$$0 \to \mathrm{Tor}_1^R(Rx, M) \to (0{:}_R x)\otimes_R M \to R\otimes_R M \xrightarrow{h} Rx\otimes_R M \to 0.$$

Now, we have the short exact sequence

$$0 \to \mathrm{Tor}_1^R(Rx, M) \to (0{:}_R x)\otimes_R M \to \mathit{Ker} h \to 0 \tag{3.4}$$

where $\mathit{Ker} h \cong (0{:}_R x)M$, and $(0{:}_R x)M \in \mathcal{S}$. Thus by (3.2) and exact sequence (3.4), we get $(0 :_R x)\otimes_R M \in \mathcal{S}$.

□

We now generalize the rigid concept to an arbitrary Serre subcategory as follows.

**Definition 3.3.** An $R$-module $M$ is called $\mathcal{S}$-rigid if $\mathrm{Tor}_i^R(M, N) \in \mathcal{S}$ for some finitely generated $R$-module $N$, then $\mathrm{Tor}_j^R(M, N) \in \mathcal{S}$ for any $j \geq i$.

In the following theorem, we introduce and prove conditions for $\mathcal{S}-$rigidity.

**Theorem 3.4.** *Let $R$ be a Noetherian (not necessary local) ring and $M$ be a non-zero finitely generated $R$-module. Let $x_1, ..., x_n$ be a poor regular $M$-sequence with respect to $\mathcal{S}$. If $\mathrm{Tor}_2^R(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$, then $\mathrm{Tor}_i^R(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$, for any $i \geq 1$.*

*Proof.* It is enough to show that $\mathcal{S} - Supp\,\mathrm{Tor}_i^R(\frac{R}{(x_1,\dots,x_n)}, M) = \emptyset$. If $\mathfrak{p} \in \mathcal{S} - Supp_R(M) - V(x_1,\dots,x_n)$, then $(\frac{x_1}{1},\dots,\frac{x_n}{1}) = R_\mathfrak{p}$, hence $\mathrm{Tor}_i^{R_\mathfrak{p}}(\frac{R_\mathfrak{p}}{(\frac{x_1}{1},\dots,\frac{x_n}{1})}, M_\mathfrak{p}) = 0$. Therefore without loss of generality, we may assume that $\mathcal{S} - \mathrm{Supp}_R M \subseteq V(x_1,\dots,x_n)$ and $M \notin \mathcal{S}$. We use induction on n. Assume that $n = 1$ and set $x := x_1$. By Lemma 3.2, we have $(0:_R x)\otimes_R M \in \mathcal{S}$.

On the other hand, $\mathrm{Supp}\,\mathrm{Tor}_i^R((0:_R x), M) \subseteq \mathrm{Supp}\,((0:_R x)\otimes_R M)$ for all $i \geq 0$. Thus by Lemma 3.2, for all $i \geqslant 0$

$$\mathrm{Tor}_i^R((0:_R x), M) \in \mathcal{S}.$$

Also, using the exact sequences (3.1) and (3.2), we have

$$\mathrm{Tor}_i^R\left((0:_R x), M\right) \cong \mathrm{Tor}_{i+1}^R\left(Rx, M\right) \cong \mathrm{Tor}_{i+2}^R(\frac{R}{Rx}, M)$$

for all $i \geq 1$. Therefore, by Lemma 3.1, $\mathrm{Tor}_i^R(\frac{R}{Rx}, M) \in \mathcal{S}$, for any $i \geq 1$.
Now assume that $n > 1$ and the result has been proved for smaller values of $n$. Set $I := (x_1,\dots,x_{n-1})$ and $J := (x_1,\dots,x_n)$. Let $\mathfrak{p} \in \mathcal{S} - Supp_R M$. By Lemma 2.3, we have the exact sequence

$$0 \to \frac{R_\mathfrak{p}}{IR_\mathfrak{p}} \overset{\frac{x_n}{1}}{\to} \frac{R_\mathfrak{p}}{IR_\mathfrak{p}} \to \frac{R_\mathfrak{p}}{JR_\mathfrak{p}} \to 0,$$

which induces the following exact sequence

$$\mathrm{Tor}_2^{R_\mathfrak{p}}(\frac{R_\mathfrak{p}}{IR_\mathfrak{p}}, M_\mathfrak{p}) \overset{\frac{x_n}{1}}{\to} \mathrm{Tor}_2^{R_\mathfrak{p}}(\frac{R_\mathfrak{p}}{IR_\mathfrak{p}}, M_\mathfrak{p}) \to \mathrm{Tor}_2^{R_\mathfrak{p}}(\frac{R_\mathfrak{p}}{JR_\mathfrak{p}}, M_\mathfrak{p}).$$

Thus, we obtain

$$\mathrm{Tor}_2^{R_\mathfrak{p}}(\tfrac{R_\mathfrak{p}}{IR_\mathfrak{p}}, M_\mathfrak{p}) = \tfrac{x_n}{1}\,\mathrm{Tor}_2^{R_\mathfrak{p}}(\tfrac{R_\mathfrak{p}}{IR_\mathfrak{p}}, M_\mathfrak{p}),$$

and then by Nakayama's Lemma $\mathrm{Tor}_2^{R_\mathfrak{p}}(\frac{R_\mathfrak{p}}{IR_\mathfrak{p}}, M_\mathfrak{p}) = 0$. This implies that $\mathrm{Tor}_2^R(\frac{R}{I}, M) \in \mathcal{S}$. Now, by the inductive hypothesis,

$$\mathrm{Tor}_i^R(\frac{R}{I}, M) \in \mathcal{S} \tag{3.5}$$

for all $i \geq 1$. The exact sequence

$$0 \to (0:_{\frac{R}{I}} x_n) \to \frac{R}{I} \to \frac{J}{I} \to 0$$

induces the exact sequence

$$\mathrm{Tor}_{i+1}^R(\frac{R}{I}, M) \to \mathrm{Tor}_{i+1}^R(\frac{J}{I}, M) \to \mathrm{Tor}_i^R((0:_{\frac{R}{I}} x_n), M).$$

By (3.5)

$$\mathrm{Tor}_i^R(\frac{J}{I}, M) \in \mathcal{S} \tag{3.6}$$

for all $i \geq 1$. Finally the exact sequence

$$0 \to \frac{J}{I} \to \frac{R}{I} \to \frac{R}{J} \to 0$$

induces the exact sequence

$$\mathrm{Tor}_{i+1}^R(\frac{R}{I}, M) \to \mathrm{Tor}_{i+1}^R(\frac{R}{J}, M) \to \mathrm{Tor}_i^R(\frac{J}{I}, M).$$

By (3.6) and (3.5), we have $\mathrm{Tor}_i^R(\frac{R}{J}, M) \in \mathcal{S}$, for all $i > 1$. Hence $\mathrm{Tor}_i^R(\frac{R}{J}, M) \in \mathcal{S}$, for all $i \geq 1$, by Lemma 3.1.

□

Bahmanpour in [3, Corollary $2 \cdot 5$] proved that if $x_1, ..., x_n$ is a poor regular $M$-regular sequence, then

$$\mathrm{Tor}_{n+i}^R(\tfrac{R}{(x_1,...,x_n)}, \mathrm{H}^n_{(x_1,...,x_n)}(M)) \cong \mathrm{Tor}_i^R(\tfrac{R}{(x_1,...,x_n)}, M),$$

for all $i \geq 0$. Therefore, if $x_1, ..., x_n$ is a poor regular $M$-sequence with respect to $\mathcal{S}$, then $\mathrm{Tor}_i^R(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$ if and only if

$$\mathrm{Tor}_{n+i}^R(\tfrac{R}{(x_1,...,x_n)}, \mathrm{H}^n_{(x_1,...,x_n)}(M)) \in \mathcal{S},$$

for all $i \geq 0$. Hence we have the following equivalent statements.

**Theorem 3.5.** *Let $R$ be a Noetherian ring and $M$ be a non-zero finitely generated $R$-module. Let $n \geq 1$ be an integer and $x_1, ..., x_n$ be a poor regular $M$-sequence with respect to $\mathcal{S}$. Then the following statements are equivalent:*

(1) $\mathrm{Tor}_i^R(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$ *for every* $i \geq 1$*;*
(2) $\mathrm{Tor}_2^R(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$*;*
(3) $\mathrm{Tor}_i^R(\frac{R}{(x_1,...,x_n)}, \mathrm{H}^n_{(x_1,...,x_n)}(M)) \in \mathcal{S}$ *for all integers* $i \geq n+1$*;*
(4) $\mathrm{Tor}_{n+2}^R(\frac{R}{(x_1,...,x_n)}, \mathrm{H}^n_{(x_1,...,x_n)}(M)) \in \mathcal{S}$*.*

By Zero Divisor Conjecture any regular $M$-sequence is a regular $R$-sequence. We generalize the Zero Divisor Conjecture as follows.
Zero Divisor Conjecture with respect to $\mathcal{S}$. Every regular $M$-sequence with respect to $\mathcal{S}$ is a regular $R$-sequence with respect to $\mathcal{S}$.
In the following, we provide some conditions in which the conjecture is established.

**Lemma 3.6.** *Let $x_1, ..., x_n$ be a poor regular $M$-sequence with respect to $\mathcal{S}$. Then*

$$\mathrm{Ext}_R^{n+1}(\frac{R}{(x_1, ..., x_n)}, M) \in \mathcal{S}.$$

*Proof.* Let $x_1, ..., x_n$ is a poor $M$-sequence with respect to $\mathcal{S}$, then for every $\mathfrak{p} \in \mathcal{S} - \operatorname{Supp}_R(M)$, $\frac{x_1}{1}, ..., \frac{x_n}{1}$ is a poor regular $M_{\mathfrak{p}}$-sequence. Thus $\operatorname{Ext}_R^{n+1}(\frac{R_{\mathfrak{p}}}{(\frac{x_1}{1},...,\frac{x_n}{1})}, M_{\mathfrak{p}}) = 0$, by [3, Lemma $3 \cdot 3$]. This implies that $\mathcal{S} - Supp \operatorname{Ext}_R^{n+1}(\frac{R}{(x_1,...,x_n)}, M) = \emptyset$. Hence

$$\operatorname{Ext}_R^{n+1}(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}.$$

□

*Remark* 3.7. The concept of $\mathcal{S} - C.grade(I, M)$ is defined as the supremum length of poor $M$-sequences with respect to $\mathcal{S}$ in $I$. It is shown that any two maximal regular $M$-sequences in $I$ with respect to $\mathcal{S}$ have the same length. In [1, Theorem $2 \cdot 8$] it is shown that the concepts $\mathcal{S} - C.grade(I, M)$ and $\mathcal{S} - E.grade(I, M)$ are the same.

**Theorem 3.8.** *Let $R$ be a Noetherian (not necessary local) ring, $M$ be a non-zero finitely generated $R$-module, and $I$ be an ideal of $R$ with $\mathcal{S}-\mathrm{E}.grade_R(I, M) = n$. Let $x_1, ...x_n$ be a maximal regular $M$-sequence in $I$ with respect to $\mathcal{S}$. If $\operatorname{Ext}_R^{n+2}(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$, then $x_1, ..., x_n \in I$ is a regular $R$-sequence with respect to $\mathcal{S}$.*

*Proof.* We use induction on n. Assume that n=1 and set $x := x_1$. The exact sequences (3.1) and (3.3) imply that

$$\operatorname{Ext}_R^i\left((0:_R x), M\right) \cong \operatorname{Ext}_R^{i+1}\left(Rx, M\right) \cong \operatorname{Ext}_R^{i+2}(\frac{R}{Rx}, M)$$

for all $i \geq 1$. By assumption, $\operatorname{Ext}_R^3(\frac{R}{Rx}, M) \in \mathcal{S}$ and so

$$\operatorname{Ext}_R^1((0:_R x), M) \in \mathcal{S}. \tag{3.7}$$

Since $x$ is a regular $M$-sequence with respect to $\mathcal{S}$ and

$$\operatorname{Supp}(0:_R x) \subseteq V(x),$$

thus

$$\operatorname{Hom}_R((0:_R x), M) \in \mathcal{S} \tag{3.8}$$

by Lemma 2.2. We claim that $(0:_R x) \in \mathcal{S}$. Assume the opposite, then there exists $\mathfrak{q} \in \operatorname{Ass}(0:_R x)$ such that $\frac{R}{\mathfrak{q}} \notin \mathcal{S}$. Thus $x \in \mathfrak{q}$ and $\mathfrak{q} \in \operatorname{Ass} R$. Since $x$ is a regular $M$-sequence with respect to $\mathcal{S}$, $\mathfrak{q} \notin \operatorname{Ass} M$ and so $\mathfrak{q}R_{\mathfrak{q}} \notin \operatorname{Ass} M_{\mathfrak{q}}$. Therefore depth $M_{\mathfrak{q}} \geq 1$, and so $M_{\mathfrak{q}} \neq 0$ and $\mathfrak{q} \in \mathcal{S}-Supp_R M$. The exact sequences

$$0 \to (0:_M x) \to M \to xM \to 0$$

and

$$0 \to xM \to M \to \frac{M}{xM} \to 0$$

and (3.7) and (3.8) imply that $\operatorname{Hom}_R((0:_R x), \frac{M}{xM}) \in \mathcal{S}$. So, by Lemma 3.2, $\operatorname{Hom}_{R_\mathfrak{q}}((0:_{R_\mathfrak{q}} \frac{x}{1}), \frac{M_\mathfrak{q}}{\frac{x}{1}M_\mathfrak{q}}) = 0$. Since $Supp_{R_\mathfrak{q}}\left(0:_{R_\mathfrak{q}} \frac{x}{1}\right) \subseteq V(\frac{x}{1})$ and $\frac{x}{1}$ is a regular $M_\mathfrak{q}$-sequence, we have $\operatorname{Hom}_{R_\mathfrak{q}}(\frac{R_\mathfrak{q}}{(\frac{x}{1})}, \frac{M_\mathfrak{q}}{\frac{x}{1}M_\mathfrak{q}}) = 0$ which is a contradiction. Therefore $x$ is a regular $R$-sequence with respect to $\mathcal{S}$. Now assume, inductively, that $n > 1$ and the assertion has been proved for smaller values of $n$.

Set $\mathfrak{a} := (x_1, ..., x_{n-1})$ and $\mathfrak{b} := (x_1, ..., x_n)$, and assume that $x_1, ..., x_n$ is an regular $M$-sequence in $I$ with respect to $\mathcal{S}$. We show that $\operatorname{Ext}_R^{n+1}(\frac{R}{\mathfrak{a}}, \frac{M}{x_n M}) \in \mathcal{S}$. For this purpose, we can assume that

$$\mathcal{S} - \operatorname{Supp}_R M \subseteq V(x_1, ..., x_n).$$

Let $\mathfrak{p} \in \mathcal{S} - Supp_R M$. The exact sequence

$$0 \to \frac{\mathfrak{b}}{\mathfrak{a}} \to \frac{R}{\mathfrak{a}} \to \frac{R}{\mathfrak{b}} \to 0$$

induces the exact sequence

$$\operatorname{Ext}_R^{n+1}(\frac{R}{\mathfrak{b}}, M) \to \operatorname{Ext}_R^{n+1}(\frac{R}{\mathfrak{a}}, M) \to \operatorname{Ext}_R^{n+1}(\frac{\mathfrak{b}}{\mathfrak{a}}, M) \to \operatorname{Ext}_R^{n+2}(\frac{R}{\mathfrak{b}}, M).$$

Also the exact sequence

$$0 \to (0:_{\frac{R}{\mathfrak{a}}} \mathfrak{b}) \to \frac{R}{\mathfrak{a}} \to \frac{\mathfrak{b}}{\mathfrak{a}} \to 0$$

induces the exact sequence

$$\operatorname{Ext}_R^{n}(\frac{R}{\mathfrak{a}}, M) \to \operatorname{Ext}_R^{n}((0:_{\frac{R}{\mathfrak{a}}} \mathfrak{b}), M) \to \operatorname{Ext}_R^{n+1}(\frac{\mathfrak{b}}{\mathfrak{a}}, M) \to \operatorname{Ext}_R^{n+1}(\frac{R}{\mathfrak{a}}, M).$$

If $\operatorname{Ext}_R^{n+1}(\frac{R}{\mathfrak{a}}, M) \in \mathcal{S}$, then by Lemma 3.6, (3.9) and (3.9),

$$\operatorname{Ext}_R^{n}((0:_{\frac{R}{\mathfrak{a}}} \mathfrak{b}), M) \in \mathcal{S}.$$

If $\operatorname{Ext}_R^{n+1}(\frac{R}{\mathfrak{a}}, M) \notin \mathcal{S}$, then by Lemma 3.6 and hypothesis,

$$\operatorname{Ext}_{R_\mathfrak{p}}^{n+1}(\frac{R_\mathfrak{p}}{\mathfrak{a}R_\mathfrak{p}}, M_\mathfrak{p}) \cong \operatorname{Ext}_{R_\mathfrak{p}}^{n+1}(\frac{\mathfrak{b}R_\mathfrak{p}}{\mathfrak{a}R_\mathfrak{p}}, M_\mathfrak{p}). \tag{3.9}$$

Thus by (3.9) and (3.9) $\operatorname{Ext}_R^{n}((0:_{\frac{R}{\mathfrak{a}}} \mathfrak{b}), M) \in \mathcal{S}$. On the other hand, by assumption, $\operatorname{Ext}_R^{i}(\frac{R}{\mathfrak{b}}, M) \in \mathcal{S}$ for all integers $0 \leq i \leq n-1$. Thus

$$\operatorname{Ext}_R^{i}((0:_{\frac{R}{\mathfrak{a}}} \mathfrak{b}), M) \in \mathcal{S} \tag{3.10}$$

for all integers $0 \leq i \leq n$. We conclude that $\mathrm{Ext}^i_R(\frac{R}{\mathfrak{b}}, M) \in \mathcal{S}$, for all integers $0 \leq i \leq n$, by Lemma 2.2. Now, we claim that $(0:_{\frac{R}{\mathfrak{a}}} \mathfrak{b}) \in \mathcal{S}$. Assume the opposite, then there exists $\mathfrak{q} \in \mathrm{Ass}(0:_{\frac{R}{\mathfrak{a}}} \mathfrak{b})$ such that $\frac{R}{\mathfrak{q}} \notin \mathcal{S}$. Since $\mathfrak{q} \in \mathrm{Ass}(\frac{R}{\mathfrak{a}})$, there is $\mathfrak{r} \in \mathrm{Ass}\, R$ such that $\mathfrak{r} \subseteq \mathfrak{q}$ and $\frac{R}{\mathfrak{r}} \notin \mathcal{S}$. Since $x_1$ is a regular $M$-sequence with respect to $\mathcal{S}$ and $\mathcal{S}-\mathrm{Supp}(M) \subseteq V(x_1)$, so $\mathfrak{r} \notin \mathrm{Ass}\, M$. This implies that $M_{\mathfrak{r}} \neq 0$ and so $\mathfrak{r} \in \mathcal{S} - \mathrm{Supp}(M)$. The exact sequences

$$0 \to (0:_M x_1) \to M \to x_1 M \to 0$$

and

$$0 \to x_1 M \to M \to \frac{M}{x_1 M} \to 0$$

and (3.10) imply that

$$\mathrm{Ext}^{i-1}_R((0:_{\frac{R}{\mathfrak{a}}} \mathfrak{b}), \tfrac{M}{x_1 M}) \in \mathcal{S}$$

for all integers $0 \leq i < n$. Therefore

$$\mathrm{Ext}^{i-1}_{R_{\mathfrak{r}}}((0:_{\frac{R_{\mathfrak{r}}}{\mathfrak{a}R_{\mathfrak{r}}}} \mathfrak{b}R_{\mathfrak{r}}), \tfrac{M_{\mathfrak{r}}}{\frac{x_1}{1} M_{\mathfrak{r}}}) = 0$$

for all integers $0 \leq i \leq n$, specially $\mathrm{Hom}_{R_{\mathfrak{r}}}((0:_{\frac{R_{\mathfrak{r}}}{\mathfrak{a}R_{\mathfrak{r}}}} \mathfrak{b}R_{\mathfrak{r}}), \frac{M_{\mathfrak{r}}}{\frac{x_1}{1} M_{\mathfrak{r}}}) = 0$. Since $\mathrm{Supp}_{R_{\mathfrak{r}}} \left(0:_{\frac{R_{\mathfrak{r}}}{\mathfrak{a}R_{\mathfrak{r}}}} \mathfrak{b}R_{\mathfrak{r}}\right) \subseteq V\left(\frac{x_1}{1}\right)$, implies that

$$\mathrm{Hom}_{R_{\mathfrak{r}}}(\tfrac{R_{\mathfrak{r}}}{(\frac{x_1}{1})}, \tfrac{M_{\mathfrak{r}}}{\frac{x_1}{1} M_{\mathfrak{r}}}) = 0,$$

which is a contradiction. Therefore $x_n$ is a regular $\frac{R}{\mathfrak{a}}$-sequence with respect to $\mathcal{S}$. Now, the exact sequence

$$0 \to \frac{R_{\mathfrak{p}}}{\mathfrak{a}R_{\mathfrak{p}}} \xrightarrow{\frac{x_n}{1}} \frac{R_{\mathfrak{p}}}{\mathfrak{a}R_{\mathfrak{p}}} \to \frac{R_{\mathfrak{p}}}{\mathfrak{b}R_{\mathfrak{p}}} \to 0$$

induces the exact sequence

$$\mathrm{Ext}^{n+1}_{R_{\mathfrak{p}}}(\frac{R_{\mathfrak{p}}}{\mathfrak{a}R_{\mathfrak{p}}}, M_{\mathfrak{p}}) \xrightarrow{\frac{x_n}{1}} \mathrm{Ext}^{n+1}_{R_{\mathfrak{p}}}(\frac{R_{\mathfrak{p}}}{\mathfrak{a}R_{\mathfrak{p}}}, M_{\mathfrak{p}}) \to \mathrm{Ext}^{n+2}_{R_{\mathfrak{p}}}(\frac{R_{\mathfrak{p}}}{\mathfrak{b}R_{\mathfrak{p}}}, M_{\mathfrak{p}})$$
$$\to \mathrm{Ext}^{n+2}_{R_{\mathfrak{p}}}(\frac{R_{\mathfrak{p}}}{\mathfrak{a}R_{\mathfrak{p}}}, M_{\mathfrak{p}}) \xrightarrow{\alpha} \mathrm{Ext}^{n+2}_{R_{\mathfrak{p}}}(\frac{R_{\mathfrak{p}}}{\mathfrak{a}R_{\mathfrak{p}}}, M_{\mathfrak{p}}).$$

By hypothesis $\mathrm{Ext}^{n+2}_{R_{\mathfrak{p}}}(\frac{R_{\mathfrak{p}}}{\mathfrak{b}R_{\mathfrak{p}}}, M_{\mathfrak{p}}) = 0$ and so $\alpha$ is monomorphism. On the other hand, the exact sequence

$$0 \to M_{\mathfrak{p}} \xrightarrow{\frac{x_n}{1}} M_{\mathfrak{p}} \to \frac{M_{\mathfrak{p}}}{\frac{x_n}{1} M_{\mathfrak{p}}} \to 0$$

induces the exact sequence

$$\mathrm{Ext}^{n+1}_{R_\mathfrak{p}}(\frac{R_\mathfrak{p}}{\mathfrak{a}R_\mathfrak{p}}, M_\mathfrak{p}) \to \mathrm{Ext}^{n+1}_{R_\mathfrak{p}}(\frac{R_\mathfrak{p}}{\mathfrak{a}R_\mathfrak{p}}, \frac{M_\mathfrak{p}}{\frac{x_n}{1}M_\mathfrak{p}}) \to \mathrm{Ext}^{n+2}_{R_\mathfrak{p}}(\frac{R_\mathfrak{p}}{\mathfrak{a}R_\mathfrak{p}}, M_\mathfrak{p}) \xrightarrow{\alpha} \mathrm{Ext}^{n+2}_{R_\mathfrak{p}}(\frac{R_\mathfrak{p}}{\mathfrak{a}R_\mathfrak{p}}, M_\mathfrak{p}).$$

Since $\alpha$ is monomorphism, we get $\mathrm{Ext}^{n+1}_{R_\mathfrak{p}}(\frac{R_\mathfrak{p}}{\mathfrak{a}R_\mathfrak{p}}, \frac{M_\mathfrak{p}}{\frac{x_n}{1}M_\mathfrak{p}}) = 0$, by Nakayama. Thus $\mathrm{Ext}^{n+1}_R(\frac{R}{\mathfrak{a}}, \frac{M}{x_nM}) \in \mathcal{S}$. Now, since

$$\mathcal{S}\!-\mathrm{E}\,.grad_R(I, \frac{M}{x_nM}) = \mathcal{S}\!-\mathrm{E}\,.grad_R(I, M) - 1,$$

it follows from the inductive hypothesis that $x_1, ..., x_{n-1}$ is a regular $R$-sequence with respect to $\mathcal{S}$. But we have already proved that $x_n$ is a regular $\frac{R}{\mathfrak{a}}$-sequence with respect to $\mathcal{S}$. Therefore $x_1, ..., x_{n-1}, x_n$ is a regular $R$-sequence with respect to $\mathcal{S}$. □

Next, we prove that for any maximal regular $M$-sequence $x_1, ..., x_n$ in $I$ with respect to $\mathcal{S}$, if $\mathrm{Ext}^{n+2}_R(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$, then

$$\mathrm{Ext}^i_R\left(\frac{R}{(x_1,...,x_n)}, M\right) \in \mathcal{S}$$

for all $i \geq 0$ $(i \neq n)$.

**Theorem 3.9.** *Let $R$ be a Noetherian (not necessary local) ring, $M$ be a non-zero finitely generated $R$-module, and $I$ be an ideal of $R$ with $\mathcal{S} - \mathrm{E}\,.grad_R(I, M) = n \geqslant 1$. Assume that $x_1, ..., x_n \in I$ is a maximal regular $M$-sequence with respect to $\mathcal{S}$. If $\mathrm{Ext}^{n+2}_R(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$, then $\mathrm{Ext}^i_R(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$, for all integers $i \geq 0$ $(i \neq n)$.*

*Proof.* Let $\mathfrak{p} \in \mathcal{S}\!-\mathrm{Supp}(M)$. By Theorems 2.3 and 3.8, $\frac{x_1}{1}, ..., \frac{x_n}{1}$ is a poor regular $R_\mathfrak{p}$-sequence. We show that

$$\mathrm{Ext}^i_{R_\mathfrak{p}}(\frac{R_\mathfrak{p}}{(\frac{x_1}{1},...,\frac{x_n}{1})}, M_\mathfrak{p}) = 0$$

for all $i \geq n+1$. For this purpose, we may assume that $\mathfrak{p} \in V(x_1, ..., x_n)$. Since $\mathrm{pd}_{R_p}(\frac{R_p}{(\frac{x_1}{1},...,\frac{x_n}{1})}) = n$, clearly $\mathrm{Ext}^i_{R_\mathfrak{p}}(\frac{R_\mathfrak{p}}{(\frac{x_1}{1},...,\frac{x_n}{1})}, M_\mathfrak{p}) = 0$ for all $i \geq n+1$. So $\mathrm{Ext}^i_R(\frac{R}{(x_1,...,x_n)}, M) \in \mathcal{S}$ for all $i \geq n+1$. □

**Corollary 3.10.** *Let $R$ be a Noetherian (not necessary local) ring, $M$ be a non-zero finitely generated $R$-module, and $I$ be an ideal of $R$ with $\mathcal{S}\!-\mathrm{E}\,.grad_R(I, M) = n \geqslant 1$. Assume that $x_1, ..., x_n \in I$ is a maximal regular $M$-sequence with respect to $\mathcal{S}$. Then the following statements are equivalent:*

(1) *$x_1, ..., x_n$ is an regular $R$-sequence with respect to $\mathcal{S}$;*

(2) $\mathrm{Ext}^i_R(\frac{R}{(x_1,\ldots,x_n)}, M) \in \mathcal{S}$ *for all* $i > n$;
(3) $\mathrm{Ext}^{n+2}_R(\frac{R}{(x_1,\ldots,x_n)}, M) \in \mathcal{S}$;
(4) $\mathrm{Ext}^2_R(\frac{R}{(x_1,\ldots,x_n)}, \mathrm{H}^n_{(x_1,\ldots,x_n)}(M)) \in \mathcal{S}$;
(5) $\mathrm{Ext}^i_R(\frac{R}{(x_1,\ldots,x_n)}, \mathrm{H}^n_{(x_1,\ldots,x_n)}(M)) \in \mathcal{S}$ *for all integers* $i \geq 1$.

*Proof.* This is an immediate consequence of Theorems 3.8 and 3.9. □

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**Sajjad Arda**
Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-4697, Tehran, Iran.
Email: `sajjad.arda@gmail.com`

**Seadat ollah Faramarzi**
Department of Mathematics, Payame Noor University (PNU), P.O. Box 19395-4697, Tehran, Iran.
Email: `s.o.faramarzi@pnu.ac.ir`

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# EXTENSION AND TORSION FUNCTORS WITH RESPECT TO SERRE CLASSES

S. ARDA AND S. O. FARAMARZI

# تابعگون‌های توسیعی و تابدار نسبت به رده‌های سر

سجاد اردا[۱] و سعادت الله فرامرزی[۲]

[۱,۲]دانشکده‌ی ریاضی، مرکز تحصیلات تکمیلی دانشگاه پیام نور، تهران، ایران

در این مقاله، ما قضیه‌ی صفرشونده و حدس مقسوم‌علیه صفر برای یک رسته‌ی سر دلخواه از مدول‌ها تعمیم می‌دهیم. برای این منظور، به ازای هر $M$-رشته‌ی منظم $x_1, \ldots, x_n$ نسبت به $\mathcal{S}$ اگر

$$\mathrm{Tor}_2^R(\frac{R}{(x_1,\ldots,x_n)}, M) \in \mathcal{S},$$

آن‌گاه به ازای هر $i \geqslant 1$، داریم $\mathrm{Tor}_i^R(\frac{R}{(x_1,\ldots,x_n)}, M) \in \mathcal{S}$. هم‌چنین ما نشان می‌دهیم که، اگر $\mathrm{Ext}_R^{n+2}(\frac{R}{(x_1,\ldots,x_n)}, M) \in S$، آن‌گاه به ازای هر عدد صحیح $i \geq 0$ (که $n \neq i$) داریم $\mathrm{Ext}_R^i(\frac{R}{(x_1,\ldots,x_n)}, M) \in \mathcal{S}$.

کلمات کلیدی: رده‌های سر، حدس مقسوم علیه صفر، قضیه‌ی صفرشونده، بالاترین مدول کوهمولوژی موضعی.