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UNIFORMLY N-IDEALS OF COMMUTATIVE RINGS

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Abstract. In this paper, we introduce the concept of uniformly n-ideal of commutative rings which is a special type of n-ideal. We call a proper ideal I of R a uniformly n-ideal if there exists a positive integer k for $a, b \in R$ whenever $ab \in I$ and $a \notin I$ implies that $b^k = 0$. The basic properties of uniformly n-ideals are investigated in detail. Moreover, some characterizations of uniformly n-ideals are obtained for some special rings.

1. Introduction

Throughout this paper, R denotes a commutative ring with $1 \neq 0$. The radical of R is given by $\sqrt{I} = \{a \in R : a^n \in I$ for some positive integer $n\}$. In particular, the nilradical of R is denoted by $\sqrt{0}$ which is the set of all nilpotent elements. Let A be a nonempty subset of a ring R. By $(I : A)$, we mean the ideal $\{r \in R : rA \subseteq I\}$ containing I. Since prime ideals appear in many ring theoretical situations, many authors generalize this concept, see [1] and [5]. It is well-known that a proper ideal I of R is called *primary* if $a, b \in R$ and $ab \in I$, then $a \in I$ or $b \in \sqrt{I}$. In [3], a proper ideal I of R is called *2-absorbing primary* if $a, b, c \in R$ with $abc \in I$, then either $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Recall that from [4] a proper ideal I of R is said to be *uniformly primary*, if there exists a positive integer n such that whenever $r, s \in R$ satisfying $rs \in I$ and $r \notin I$, then $s^n \in I$. We say that a uniformly primary ideal I has order N and write $ord(I) = N$,

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if N is the smallest positive integer for which the property holds. In 2015, as a generalization of uniformly primary ideals, the concept of uniformly 2-absorbing primary ideals is introduced in [8]: A proper ideal I of R is a *uniformly 2-absorbing primary* ideal if there exists a positive integer n such that whenever $a, b, c \in R$ satisfy $abc \in I$, $ab \notin I$ and $ac \notin \sqrt{I}$, then $(bc)^n \in I$. They call that N is order of I if N is the smallest positive integer for which the above property holds and it is denoted by $2 - ord(I) = N$. A different from these concepts, the concept of n-ideals is introduced and studied in [10]. They call a proper ideal I of R an *n-ideal* if whenever $a, b \in R$ with $ab \in I$ and $a \notin I$, then $b \in \sqrt{0}$. Observe that prime ideals needs not to be n-ideals; for instance $I = 2\mathbb{Z}$ of $\mathbb{Z}$ is not n-ideal as $1 \cdot 2 \in I$ but neither $1 \in I$ nor $2 \in \sqrt{0}$.

A ring R is said to be reduced if there is no nonzero nilpotent element; i.e.$\sqrt{0} = 0$. By $J(R)$, we denote the intersection of all maximal ideals of R. In this paper, the special type of n-ideals in commutative rings, namely uniformly n-ideals are introduced. In section 2, the basic algebraic properties of uniformly n-ideals are studied and among many results the characterization of uniformly n-ideals is given in Theorem 2.11. Also, Theorem 2.13 and Corollary 2.14 give another characterizations for uniformly n-ideals in terms of some ideals of a ring. It is shown in Theorem 2.12 that if $\sqrt{0}$ is nilpotent in a ring, then the concepts of uniformly n-ideals and n-ideals are coincide. In Theorem 2.9, we establish the (Krull) dimension of R if every nonzero ideal of R is a uniformly n-ideal. It is shown that if R is not a reduced ring whose every nonzero ideal is a uniformly n-ideal, then R is a local ring (see Theorem 2.10). In section 3, we determine under which condition a noetherian ring has a uniformly n-ideal (see Theorem 3.5).

2. Uniformly n-ideals

In this section, we study the basic properties of uniformly n-ideals and give some characterizations of them.

Definition 2.1. Let R be a ring and I be a proper ideal of R. We call I a uniformly n-ideal if there exists a positive integer n such that whenever $a, b \in R$ with $ab \in I$ and $a \notin I$, then $b^n = 0$. The smallest integer N which satisfies this property is called the order of I, and is denoted by $ord(I) = N$.

The following diagram shows the relations among n-ideal, primary ideal, uniformly primary ideal, uniformly n-ideal, uniformly 2-absorbing primary ideal and 2-absorbing primary ideal. Note that the converse of these implications does not hold in general.

$$\begin{array}{ccc} \text{uniformly } n\text{-ideal} & \Longrightarrow & n\text{-ideal} \\ \Downarrow & & \Downarrow \\ \text{uniformly primary} & \Longrightarrow & \text{primary} \\ \Downarrow & & \Downarrow \\ \text{uniformly 2-absorbing primary} & \Longrightarrow & \text{2-absorbing primary} \end{array}$$

Example 2.2. (1) A reduced ring R which is not an integral domain has no uniformly n-ideal. For instance, in $R = \mathbb{Z}_6$ (see Proposition 3.6): every nonzero proper ideal of $\mathbb{Z}_6$ is prime. The zero ideal of $\mathbb{Z}_6$ is a uniformly 2-absorbing primary ideal. Indeed, since 0 is a 2-absorbing ideal of $\mathbb{Z}_6$, by [8, Remark 2.9] it is a uniformly 2-absorbing primary ideal of $\mathbb{Z}_6$.

(2) Consider the ring $\mathbb{Z}_8$. Then $I =< 4 >$ is a uniformly n-ideal of order 3, but it is not prime as $2 \cdot 2 \in I$ but $2 \notin I$.

(3) Let K be a field and

$$X = \{X_1, X_2, X_3, ...\}$$

a set of indeterminates over K. Consider the ring

$$R = K[X]/(\{X_i^i\}_{i=1}^{\infty}).$$

Then the ideal $I = (\{X_1X_i\}_{i=2}^{\infty})R$ of R is clearly an n-ideal. However, since for each $k \geq 1$, $X_1X_{k+1} \in I$ but neither $X_1 \in I$ nor $X_{k+1}^k = 0$, it is not a uniformly n-ideal.

Proposition 2.3. *Let R be a ring.*

(1) *If R is an integral domain, then $I = \{0\}$ is a uniformly n-ideal of order 1.*

(2) *$\{0\}$ is a uniformly n-ideal of R if and only if $\{0\}$ is a uniformly primary ideal of R.*

Proof. Trivial. □

Lemma 2.4. *Let I be a proper ideal of R. The following statements hold.*

(1) *If I is an n-ideal of R, then $\sqrt{I} = \sqrt{0}$ is a prime ideal of R.*

(2) *If I is a uniformly n-ideal, then $I \subseteq \sqrt{0}$.*

Proof. (1) Suppose that I is an n-ideal of R. Then $I \subseteq \sqrt{0}$ by [10, Proposition 2.3]. Thus $\sqrt{I} = \sqrt{0}$ and we conclude from [10, Proposition 2.3] that $\sqrt{0}$ is prime.

(2) Suppose that I is a uniformly n-ideal. Then it is an n-ideal and the result follows from [10, Proposition 2.3]. □

We note that the converse of Lemma 2.4(2) is not true in general. For instance, let $R = \mathbb{Z}_6$. Then $I = \{0\} \subseteq \sqrt{0}$ and $2 \cdot 3 \in I$, $2 \notin I$, but there is no positive integer n such that $3^n = 0$.

Theorem 2.5. *Let R be a ring such that $J(R) = 0$. Then R has no nonzero uniformly n-ideal. In particular, if R is a semisimple ring, then R has no nonzero uniformly n-ideal.*

Proof. Suppose that I is a uniformly n-ideal. Then

$$I \subseteq \sqrt{0} \subseteq J(R) = 0.$$

The "in particular" case is clear as $J(R) = 0$ for every semisimple ring. □

Proposition 2.6. *Let A be a nonempty subset of a ring R and I be a uniformly n-ideal of R. Then $(I : A)$ is a uniformly n-ideal of R with $ord((I : A)) \leq ord(I)$.*

Proof. Suppose that I is a uniformly n-ideal with $ord(I) = n$ and $bc \in (I : A)$ such that $b \notin (I : A)$. Then there exists $a \in A$ such that $ab \notin I$. Since $abc \in I$, $ab \notin I$ and $ord(I) = n$, we have $c^n = 0$. Thus $(I : A)$ is uniformly n-ideal with $ord((I : A)) \leq n$. □

Corollary 2.7. *Let I be a uniformly n-ideal of R. Then $(I : a)$ is a uniformly n-ideal of R for all $a \in R$ and $ord((I : a)) \leq ord(I)$.*

Proof. It follows from Proposition 2.6. □

Theorem 2.8. *Let I be a proper ideal of a ring R.*

(1) *If I is a prime and uniformly n-ideal of R, then $I = \sqrt{0}$.*
(2) *If I is a maximal uniformly n-ideal of R, then $I = \sqrt{0}$.*

Proof. (1) If I is a prime and uniformly n-ideal of R, then $I = \sqrt{0}$. Now apply Lemma 2.4 (2).

(2) Suppose that I is a maximal uniformly n-ideal and $ab \in I$ with $a \notin I$ for some $a, b \in R$. Then by Corollary 2.7 $(I : a)$ is a uniformly n-ideal. Since I is maximal between uniformly n-ideals and $I \subseteq (I : a)$, $I = (I : a)$. On the other hand, $b \in (I : a)$ and so $b \in I$. Therefore I is prime and so by (1) $I = \sqrt{0}$. □

Theorem 2.9. *Let R be a ring. If every nonzero cyclic ideal of R is a uniformly n-ideal, then $dim R \leq 1$.*

Proof. Suppose that $dim R > 1$. Then there exists a chain of prime ideals $P_1 \subset P_2 \subset P_3$. Let $a \in P_2 \setminus P_1$ and $b \in P_3 \setminus P_2$. Now $ab \notin P_1$. Then $ab \neq 0$. We have $ab \in < ab > \subset P_2$. Since the ideal $< ab >$ is a uniformly n-ideal and $b^n \neq 0$ for every positive integer n, $a \in < ab >$.

So $a(1-rb)=0\in P_1$ for some $r\in R$. Then $1-rb\in P_1\subset P_3$. Hence $1\in P_3$ and this is a contradiction. □

Theorem 2.10. *Let R be a ring, the following conditions are equivalent:*

(1) *Every nonzero ideal of R is uniformly n-ideal.*
(2) *Every prime ideal of R is uniformly n-ideal.*
(3) *R is a local ring with maximal nil ideal.*

Proof. (1) ⇒ (2) Trivial.

(2) ⇒ (3) By Theorem 2.8.

(3) ⇒ (1) Let M be a unique maximal ideal of R. By Our assumption there exist a number k such that $M^k=0$ and for every ideal I of R, $I\subseteq M$. If for some $a,b\in R$, $ab\in I$ and $a\notin I$, then $b\in M$, which implies that $b^k=0$. □

An important characterization of uniformly n-ideals is given by the next theorem:

Theorem 2.11. *Let I be a proper ideal of a ring R. Then I is a uniformly n-ideal and 0 is a uniformly primary ideal of order N if and only if the following conditions hold.*

(1) *I is an n-ideal of R.*
(2) *There exists a positive integer n such that*
$$\sqrt{0}=\{a\in R: a^n=0\}.\ ord_0(I)=N$$
if and only if N is the smallest integer which satisfies this property.

Proof. Suppose that I is a uniformly n-ideal of R of order N. Then (1) is satisfied. Let $a\in\sqrt{0}$. Hence $a^n=0$ but $a^{n-1}\neq 0$ for some positive integer n. Since $a^{n-1}a=0$ and 0 is uniformly primary with $ord(0)=N$, we have $a^N=0$. Conversely, suppose that both conditions (1) and (2) hold. Let $a,b\in R$ with $ab\in I$ and $a\notin I$. It follows $b\in\sqrt{0}$ by (1). On the other hand, there exists a positive integer n which is independent of elements of R such that $b^n=0$ by (2). Thus I is a uniformly n-ideal of R. □

Theorem 2.12. *Let R be a ring and $\sqrt{0}$ a nilpotent ideal of R. The following statements are equivalent.*

(1) *I is an n-ideal.*
(2) *For all ideals J,K of R with $JK\subseteq I$ and $K\nsubseteq I$, then there exists a positive integer n such that $J^n=0$.*
(3) *I is a uniformly n-ideal.*

Proof. (1)⇒(2) Suppose that $JK \subseteq I$ and $K \nsubseteq I$ for some ideals J, K of R. Then $J \cap (R - \sqrt{0}) = \emptyset$ by [10, Theorem 2.7]. Since $\sqrt{0}$ is nilpotent, there exists a positive integer n such that $J^n \subseteq \sqrt{0}^n = 0$, as needed.

(2)⇒(3) Let $a, b \in R$ with $ab \in I$ and $b \notin I$. Put $J := (a)$ and $K := (b)$. Then we conclude the result from our assumption (2).

(3)⇒(1) It is clear. □

It is well-known that if R is an Artinian ring, then $\sqrt{0}$ is a nilpotent ideal. Therefore, we note that Theorem 2.12 holds true for Artinian rings.

Let I be a proper ideal of a ring R. According to [2], the ideal $< \{i^n : i \in I\} >$ of R which is generated by n-th powers of elements of I denoted by I_n.

Theorem 2.13. *Let I be a proper ideal of R. The following statements are equivalent.*

1. *I is a uniformly n-ideal of R of order n.*
2. *There exists a positive integer n such that for every $a \in R$, either $(I : a) = R$ or $(I : a)_n = \{0\}$.*
3. *There exists a positive integer n for every $a \in R$, $aJ \subseteq I$, implies that either $a \in I$ or $J_n = \{0\}$.*

Proof. (1)⇒(2) Suppose that I is a uniformly n-ideal of R of order n and $(I : a) \neq R$. Hence $a \notin I$. Let $b \in (I : a)$. Since $ab \in I$ and $ord(I) = n$, we have $b^n = 0$. Therefore $(I : a)_n = \{0\}$.

(2)⇒(3) On the contrary, suppose that $aJ \subseteq I$ but neither $a \in I$ nor $J_n = \{0\}$. Then there exists nonzero $b^n \in J_n$ where $b \in J$. Hence we conclude $(I : a) \neq R$ and $(I : a)_n \neq \{0\}$, a contradiction. Thus $a \in I$ or $J_n = \{0\}$.

(3)⇒(1) It is clear. □

In [2, Theorem 5], it has been shown $I_n = I^n$ provided that $n!$ is a unit in R. Then we obtain the following result.

Corollary 2.14. *Let R be a ring, I be a proper ideal of R and n be a positive integer number. If $n!$ is a unit element in R then the following statements are equivalent.*

1. *I is a uniformly n-ideal of R of order n.*
2. *There exists a positive integer n such that for every nonunit elements $a, b \in R$, either $(I : a) = R$ or $(I : a)^n = \{0\}$.*
3. *There exists a positive integer n such that for every nonunit elements $a, b \in R$ such that $aJ \subseteq I$, either $a \in I$ or $J^n = \{0\}$.*

Proof. It follows from Theorem 2.13. □

Theorem 2.15. *For a ring R, the following statements are equivalent.*

(1) *Every proper principal ideal of R is a uniformly n-ideal of order k.*
(2) *Every proper ideal of R is a uniformly n-ideal of order k.*

Proof. (1)⇒(2) Suppose that $a, b \in I$, $a \notin I$ for some $a, b \in R$. Then by our assumption, (ab) is a uniformly n-ideal . Since $ab \in (ab)$ and $a \notin (ab)$, it implies that $b^k = 0$. Thus I is a uniformly n-ideal of order k.

(2)⇒(1) It is clear. □

In the following, we obtain some elementary properties of uniformly n-ideals. The first property allows us to compare the orders of the elements of a chain of uniformly n-ideals.

Proposition 2.16. *Let I_1 and I_2 be uniformly n-ideals of R with $I_1 \subseteq I_2$. Then $ord(I_1) \geq ord(I_2)$.*

Proof. Put $ord(I_1) = m$ and $ord(I_2) = n$ for some $n, m \geq 1$. Then there exist $r, s \in R$ such that $rs \in I_2$, $r \notin I_2$ and $s^n = 0$, $s^{n-1} \neq 0$. Now $rs^{n-1} \cdot s = 0 \in I_1$. If $rs^{n-1} \notin I_1$, then $s^m = 0$; so $m \geq n$. Now suppose that $rs^{n-1} \in I_1$. Hence $rs^{n-2} \cdot s \in I_1$. Again we have two cases: if $rs^{n-2} \notin I_1$, then $s^m = 0$; so $m \geq n$. Assume that $rs^{n-2} \in I_1$. Hence $rs^{n-3} \cdot s \in I_1$. Repeating this process, we get $rs \in I_1$, $r \notin I_1$ which implies $s^m \in I_1$. Thus $m \geq n$. □

Proposition 2.17. *Let $\{I_i\}_{i\in\Lambda}$ be a chain of uniformly n-ideals of R with maximum order is $n \geq 1$. Then $I = \cap_{i\in\Lambda} I_i$ is a uniformly n-ideal of R with $ord(I) \leq n$.*

Proof. Suppose that $a, b \in R$ with $ab \in I$, $a \notin I$. Then $a \notin I_k$ for some $k \in \Lambda$. Since I_k is uniformly n-ideal with $ord_n(I_k) \leq n$, we have $b^n = 0$. Thus I is a uniformly n-ideal of R of order at most n. □

Proposition 2.18. *Let $I_1, ..., I_n$ be a chain of uniformly n-ideals of R. Then $I = \cup_{i=1}^{n} Ii$ is a uniformly n-ideal of R.*

Proof. Suppose that each I_i $(i = 1, ..., n)$ is a uniformly n-ideal with $ord(I_i) = k_i$. Let $a, b \in R$ with $ab \in I$ and $a \notin I$. Then $ab \in I_j$ for some $j \in \{1, ..., n\}$ and $a \notin I_j$. It implies that $b^{k_j} = 0$. Then I is a uniformly n-ideal of order at most $k = \sum_{i=1}^{n} k_i$. □

Theorem 2.19. *Let R_1 and R_2 be commutative rings and $f : R_1 \to R_2$ a homomorphism. The following statements hold.*

(1) *Let f be a monomorphism. If I_2 is a uniformly n-ideal of R_2, then $f^{-1}(I_2)$ is a uniformly n-ideal of R_1 with*
$$ord_{R_1}(f^{-1}(I_2)) \leq ord_{R_2}(I_2).$$
(2) *Let f be an epimorphism. If I_1 is a uniformly n-ideal of R_1 containing $Kerf$, then $f(I_1)$ is a uniformly n-ideal of R_2 with $ord_{R_2}(f(I_1)) \leq ord_{R_1}(I_1)$.*

Proof. (1) Suppose that $ab \in f^{-1}(I_2)$ and $a \notin f^{-1}(I_2)$ for $a, b \in R_1$. Then $f(ab) = f(a)f(b) \in I_2$. Put $ord_{R_2}(I_2) = n$. Then $f(a) \in I_2$ or $f(b)^n = 0$. Hence $a \in f^{-1}(I_2)$ or $b^n \in$Ker$f = 0$. Thus $f^{-1}(I_2)$ is a uniformly n-ideal of R_1 with $ord_{R_1}(f^{-1}(I_2)) \leq n$.

(2) Suppose that $a_2b_2 \in f(I_1)$ and $a_2 \notin f(I_1)$ for $a_2, b_2 \in R_2$. Put $m = ord_{R_1}(I_1)$. Since f is onto, there exists $a_1, b_1 \in R_1$ such that $a_2 = f(a_1)$, $b_2 = f(b_1)$. Hence $f(a_1)f(b_1) = f(a_1b_1) \in f(I_1)$, $f(a_1) \notin f(I_1)$, which means $a_1b_1 \in I_1$ and $a_1 \notin I_1$ as Ker$f \subseteq I_1$. It follows $b_1^m = 0$; so $f(b_1)^m = f(0) = 0$. Thus $f(I_1)$ is a uniformly n- ideal of R_2 with $ord_{R_2}(f(I_1)) \leq m$. □

Theorem 2.20. *For a uniformly n-ideal I of R of order N, the following statements hold.*

(1) *If R_1 is a subring of a ring R, then $I \cap R_1$ is a uniformly n-ideal of R_1 with $ord(I \cap R_1)_{R_1} \leq N$.*
(2) *If J is an ideal of R with $J \subseteq I$, then I/J is a uniformly n-ideal of R/J with $ord(I/J) \leq N$.*

Proof. It is an application of Theorem 2.19. □

Corollary 2.21. *Let I be a proper ideal of a ring R and X an indeterminate. If (I, X) is a uniformly n-ideal of $R[X]$, then I is a uniformly n-ideal of R*

Proof. Define a function $\Pi : R[X] \to R$ by $f(x) \mapsto f(0)$. It is easily seen that Π is an epimorphism and Ker$\Pi = X \subset (I, X)$. Thus $I = \Pi((I, X))$ is a uniformly n-ideal of R by Theorem 2.19 (2). □

Proposition 2.22. *Let $R = R_1 \times R_2$ where R_1 and R_2 are commutative rings with $1 \neq 0$. Then there is no uniformly n-ideal in R.*

Proof. As a uniformly n-ideal is an n-ideal, we are done from [10, Proposition 2.26]. □

Let R be a ring and I an ideal of R. By $Z(R)$ and $Z_I(R)$, we denote the set of zero divisors of R, and the set
$$\{a \in R \mid ab \in I \quad \text{for some} \quad b \notin I\},$$
respectively.

Theorem 2.23. *Let S be a multiplicatively closed subset of R and I a proper ideal of R. Then the following statements are satisfied.*

(1) *If I is a uniformly n-ideal of R such that $I \cap S = \emptyset$, then $S^{-1}I$ is a uniformly n-ideal of $S^{-1}R$ with $ord_{s^{-1}R}(S^{-1}I) \leq ord_R(I)$.*

(2) *If $S^{-1}I$ is a uniformly n-ideal of $S^{-1}R$, $S \cap Z_I(R) = \emptyset$ and $S \cap Z(R) = \emptyset$, then I is a uniformly n-ideal of R with*

$$ord(I) \leq ord(S^{-1}I).$$

Proof. (1) Let $\frac{a}{s_1}\frac{b}{s_2} \in S^{-1}I$ for some $a, b \in I$ and $s_1, s_2 \in S$. Put $ord_R(I) = n$. Suppose that $\frac{a}{s_1} \notin S^{-1}I$. Then $uab \in I$ for some $u \in S$ and $ua \notin I$. Since I is uniformly n-ideal, we have $b^n = 0$. So, $\left(\frac{b}{s_2}\right)^n = 0$. Thus $S^{-1}I$ is a uniformly n-ideal of $S^{-1}R$ with

$$ord_{s^{-1}R}(S^{-1}I) \leq n.$$

(2) Let $ab \in I$ for some $a, b \in R$. Put $ord(S^{-1}I) = m$. Hence $\frac{ab}{1} = \frac{a}{1}\frac{b}{1} \in S^{-1}I$. Since $S^{-1}I$ is uniformly n-ideal, we have either $\frac{a}{1} \in S^{-1}I$ or $\left(\frac{b}{1}\right)^m = 0_{S^{-1}R}$. If $\frac{a}{1} \in S^{-1}I$, then $ua \in I$ for some $u \in S$. Since $u \notin Z_I(R)$, we conclude that $a \in I$. If $\left(\frac{b}{1}\right)^m = 0_{S^{-1}R}$, then $(tb)^m = 0 \in I$ for some $t \in S$. Since $S \cap Z(R) = \emptyset$, we have $b^m = 0$. Therefore I is a uniformly n-ideal of R with $ord(I) \leq m$. □

Let R be a ring and M an R-module. Consider

$$R(+)M = R \times M = \{(r, m) : r \in R, m \in M\}$$

and let (r, m) and (s, n) be two elements of $R(+)M$. Then $R(+)M$ is a commutative ring with identity under addition and multiplication defined by $(r, m)+(s, n) = (r+s, m+n)$ and $(r, m)(s, n) = (rs, rn+sm)$. For more detail information about idealization refer to [6].

Proposition 2.24. *Let I be a proper ideal of R. If I is a uniformly n-ideal, then $I(+)M$ is a uniformly n-ideal of $R(+)M$ such that*

$$ord(I) + 1 \geq ord(I + (M)) \geq ord(I).$$

In particular if M is a torsion free R-module, then $ord(I+(M)) = n+1$.

Proof. Let I be uniformly n-ideal of order n. Suppose that

$$(a, m)(b, n) \in I(+)M \text{ and } (a, m) \notin I(+)M.$$

Then $ab \in I$ but $a \notin I$. Since $ord(I) = n$, we have $b^n = 0$ and

$$ord(I) + 1 \geq ord(I + (M)) \geq ord(I).$$

□

3. Uniformly n-ideals over Noetherian rings

In this section, we characterized uniformly n-ideals in Noetherian rings. Let R be a ring. Recall that an element a of R is nilpotent if $a^n = 0$ for some positive integer n. A proper ideal I of R is said to be nil if every element of I is nilpotent; I is nilpotent ideal if $I^n = 0$ for some positive integer n. We denote the least positive integer which satisfies the property $I^n = 0$ by $e(I)$. It is clear that every nilpotent ideal is nil but the converse is not true in general. (for the general background see [7]). Now we state the next lemma which is necessary for the proof of Theorem 3.3.

Lemma 3.1. [7, Proposition 2.13, Remark, p.430] *Let R be a ring. Then*

1. *If R is Noetherian, then every nil ideal is nilpotent.*
2. *If R is Artinian, then the radical $J(R)$ is a nilpotent ideal.*

Lemma 3.2. *Let R be a ring. The following statements hold.*

1. *Let I be a P-primary ideal of R where P is a nilpotent ideal. Then I is a uniformly n-ideal of R with $ord(I) \leq e(P)$.*
2. *If I is a nilpotent prime ideal, then I is a uniformly n-ideal of R.*

Proof. (1) Let $a, b \in R$ with $ab \in I$ and $a \notin I$. Since I is P-primary, it implies $b \in P$. Then $b^{e(P)} \in P^{e(P)} = 0$. Thus I is uniformly n-ideal of order less than $e(P)$.

(2) It is clear by (1). □

Theorem 3.3. *For a comutative ring with nonzero identity R, the following are hold.*

1. *Let R be a Noetherian ring. Then every prime nil ideal of R is a uniformly n-ideal of R. In particular, if $\sqrt{0}$ is prime, then it is a uniformly n-ideal with $ord(\sqrt{0}) = e(\sqrt{0})$.*
2. *Let R be an Artinian ring. Then every prime nil ideal of R is uniformly n-ideal.*
3. *If R is a Noetherian or Artinian ring, then every $\sqrt{0}$-primary ideal is uniformly n-ideal.*

Proof. (1) It is clear by Lemma 3.1 (1) and Lemma 3.2 (2).

(2) It is well-known that in an Artinian ring, every nil ideal is nilpotent. So the result follows from Lemma 3.1 (1) and Lemma 3.2 (2).

(3) It is well-known that if R is Noetherian or Artinian, $\sqrt{0}$ is a nilpotent ideal of R. Then we conclude the result by Lemma 3.2 (1). □

Proposition 3.4. *Let R be a Noetherian ring. If I is an n-ideal of R, then $\sqrt{I}$ is a uniformly n-ideal of R.*

Proof. Suppose that I is an n-ideal of R. Then $\sqrt{I} = \sqrt{0}$ is prime by Lemma 2.4 (1). Thus $\sqrt{I}$ is uniformly n-ideal by Theorem 3.3 (1). □

Theorem 3.5. *For a Noetherian ring R, the following statements are equivalent.*

(1) *There exists a uniformly n-ideal of R.*
(2) *$\sqrt{0}$ is a prime ideal of R.*

Proof. (1)⇒(2) Suppose that I is a uniformly n-ideal of R. Then it is n-ideal, so we have the result by [10, Theorem 2.12].

(2)⇒(1) Assume that $\sqrt{0}$ is a prime ideal of R. Hence $\sqrt{0}$ is a uniformly n-ideal by Theorem 3.3 (1). □

In the next proposition, we give equivalent conditions for that every ideal of the ring $\mathbb{Z}_n$ is a uniformly n-ideal.

Proposition 3.6. *Consider the ring $\mathbb{Z}_n$ where $n \geq 2$ is a positive integer n. The following are equivalent.*

(1) *There exists a uniformly n-ideal of $\mathbb{Z}_n$ of order k.*
(2) *$n = p^k$ for some prime number p and positive integer k.*
(3) *Every ideal of $\mathbb{Z}_n$ is a uniformly n-ideal of order k.*

Proof. (1)⇒ (2) By Theorem 3.5, $\sqrt{0}$ is a prime ideal of $\mathbb{Z}_n$. Therefore $n = p^k$ where p is a prime number and k is a positive integer number.

(2)⇒(3) Suppose that I is a uniformly n-ideal of $\mathbb{Z}_{p^k}$. We need to show that $ord(I) = k$. Observe that $I = (p^t)$ for some positive integer $1 \leq t \leq k$. Since $p^{t-1}p \in I$ and $p^{t-1} \notin I$, we have $p^{ord(I)} = 0$. Then since $p^{ord(I)} \equiv 0 \pmod{p^k}$ and $ord(I) \leq k$, we have $ord(I) = k$.

(3)⇒(1) It is clear. □

Theorem 3.7. *Let R be a Noetherian ring. Then R is an integral domain if and only if the only uniformly n-ideal of R is 0.*

Proof. Suppose that R is an integral domain and $I \neq 0$ is a uniformly n-ideal of R. Then $I \subseteq \sqrt{0}$ by Lemma 2.4 (2). But since R is an integral domain, we conclude $\sqrt{0} = 0$, so $I = 0$, a contradiction. Conversely, suppose that 0 is the only uniformly n-ideal of R. Then $\sqrt{0}$ is prime by Theorem 3.5, and so $\sqrt{0}$ is a uniformly n-ideal by Theorem 3.3

(1). Hence our assumption implies that $\sqrt{0} = 0$. Thus 0 is prime, and therefore R is an integral domain. □

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UNIFORMLY n-IDEALS OF COMMUTATIVE RINGS

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n-ایده‌آل‌های یکنواخت حلقه‌های جابجایی

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در این مقاله مبحث n-ایده‌آل یکنواخت را که نمونه خاصی از n-ایده‌آل می‌باشد را معرفی می‌کنیم. یک ایده‌آل محض I از حلقه R را n-ایده‌آل یکنواخت گوئیم هرگاه عدد صحیح و مثبت k موجود باشد به‌طوری‌که برای هر $a, b \in R$ وقتی که $ab \in I$ و $a \notin I$ باعث شود که $b^k \in I$. همچنین، خواص اساسی n-ایده‌آل‌های یکنواخت بررسی شده است. به‌علاوه، مشخص‌سازی از n-ایده‌آل‌های یکنواخت برای بعضی حلقه‌های خاص ارائه گردیده است.

کلمات کلیدی: n-ایده‌آل اولیه یکنواخت، n-ایده‌آل، n-ایده‌آل یکنواخت.