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DERIVATIONS OF PRIME FILTER THEOREMS GENERATED BY VARIOUS $\cap$-STRUCTURES IN TRANSITIVE GE-ALGEBRAS

M. SAMBASIVA RAO

Abstract. Properties of prime filters and maximal filters of transitive GE-algebras are investigated. An element-wise characterization is derived for the smallest GE-filter containing a given set. It is proved that the set of all GE-filters of a transitive GE-algebra forms a complete distributive lattice. Four different versions of a prime filter theorem are generalized in transitive GE-algebras. A necessary and sufficient condition is derived for a proper filter of a commutative GE-algebra to become a prime filter.

Introduction

In 1966, Imai and Iseki introduced BCK-algebras [5] as the algebraic semantics for a non classical logic possessing only implication. Since then, the generalized concepts of BCK-algebras have been studied by many researchers. H.S. Kim and Y.H. Kim introduced the notion of a BE-algebra as a generalization of a dual BCK-algebra [6]. Hilbert algebras were introduced by Henkin and Skolem in the fifties for investigations in intuitionistic and other non classical logics. R.A. Borzooei and J. Shohani [3] introduced the notion of a generalized Hilbert algebra and studied its properties. As a generalization of Hilbert algebras, R. Bandaru et al., introduced the notion of

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GE-algebras and investigated several properties of GE-filters of GE-algebras [10, 1]. Recently in 2021, M.A. Öztürk et.al., [9] investigated the properties of strong GE-filters and strong GE-ideals of transitive GE-algebras. In [11], Rezaei et.al studied the properties of prominent GE-filters with respect to GE-morphisms of GE-algebras. In [12] A. B. Saeid et.al studied certain properties of balanced GE-filters and voluntary GE-filters are investigated. In [2], R.K. Bandaru et.al studied the relationships between a GE-filter and a belligerent GE-filter. In [4], S. M. Hong and Y. B. Jun characterized the deductive systems and maximal deductive systems of Hilbert algebras. In [7], Meng introduced the notion of prime filters in BCK-algebras, and then gave a description of the filter generated by a set, and obtained some of fundamental properties of prime filters. In [6], some properties of prime ideals in BCK-algebras were investigated. In 2016, A. S. Nasab and A. B. Saeid [8] introduced the notions of prime filters of the first kind, prime filters of the second kind and prime filters of the third kind in Hilbert algebras. They made an extensive study to characterize the prime filters of three kinds and to establish interconnections among these classes of prime filters.

GE-filters are important substructures in a GE-algebra and play an important role. It is well understood that GE-filters are the kernels of congruences. From a logical stand point, different filters correspond to different sets of valid formulas in an appropriate logic. The theory of prime filters is crucial in the study of any class of logical algebras. Designing various types of $\cap$-structures in some logical algebra is algebraically interesting in proving prime filter theorems. With this motivation, we introduce various types of $\cap$-structures in GE-algebras and derive four versions of prime filter theorem in GE-algebra.

The main objective of this paper is to derive four versions of prime filter separation theorems in GE-algebras. For this purpose, the notions of $\cap$-structures are introduced in transitive GE-algebras by the names of $\cap$-closed subset, finite $\cap$-structure, commutative closed set, and $\vee$-closed subset. Some significant properties of prime filters and maximal filters of transitive GE-algebras are investigated. Prime filters of subalgebras of transitive GE-algebras are characterized. An element-wise characterization is derived for the smallest GE-filter that is containing a given set. It is proved that the set of all GE-filters of a transitive GE-algebra forms a complete distributive lattice. A necessary and sufficient condition is derived for a proper filter of a commutative GE-algebra to become a prime filter.

1. Preliminaries

In this section, we present certain definitions and results which are taken mostly from the research paper [10] for the ready reference.

Definition 1.1. [10] An algebra $(X, *, 1)$ of type $(2, 0)$ is called a GE-algebra if it satisfies the following properties:
(GE1) $x * x = 1$,
(GE2) $1 * x = x$,
(GE3) $x * (y * z) = x * (y * (x * z))$ for all $x, y, z \in X$.

Introduce a relation $\leq$ on a GE-algebra $(X, *, 1)$ which is defined by $x \leq y$ if and only if $x * y = 1$ for all $x, y \in X$. Clearly $\leq$ is reflexive.

Theorem 1.2. [10] *Let X be a GE-algebra. For any $x, y, z \in X$,*

(1) $x * 1 = 1$,
(2) $1 \leq x$ *implies* $x = 1$,
(3) $x * (x * y) = x * y$,
(4) $x \leq y * x$,
(5) $x \leq (x * y) * y$,
(6) $x \leq (y * x) * x$,
(7) $x \leq (x * y) * x$,
(8) $x \leq y * (y * x)$,
(9) $x * (y * z) \leq y * (x * z)$,
(10) $x \leq y * z$ *if and only if* $y \leq x * z$.

Theorem 1.3. [10] *Let $(X, *, 1)$ be a GE-algebra. For any $x, y, z \in X$, the following assertions are equivalent:*

(1) $x * y \leq (z * x) * (z * y)$;
(2) $x * y \leq (y * z) * (x * z)$.

Definition 1.4. [10] A GE-algebra $(X, *, 1)$ is called *transitive* if

$$y * z \leq (x * y) * (x * z)$$

for all $x, y, z \in X$.

Theorem 1.5. [10] *Let X be a transitive GE-algebra. For $x, y, z \in X$,*

(1) $x \leq y$ *implies* $z * x \leq z * y$,
(2) $x \leq y$ *implies* $y * z \leq x * z$,
(3) $((x * y) * y) * z \leq x * z$.

Definition 1.6. [10] A non-empty subset F of a GE-algebra X is called a *filter* if, for all $x, y \in X$, it satisfies the following properties:

(GEF1) $1 \in F$,

(GEF2) $x \in F$ and $x * y \in F$ imply that $y \in F$.

A GE-algebra X is called *self-distributive* if $x*(y*z) = (x*y)*(x*z)$ for all $x, y, z \in X$. A GE-algebra X is called *commutative* if

$$(x * y) * y = (y * x) * x$$

$(x*y)*y = (y*x)*x$ for all $x, y \in X$. Every commutative GE-algebra is transitive. If X is commutative, then $\leq$ is transitive, anti-symmetric and hence a partial order on X.

2. Lattice of GE-filters of GE-algebras

In this section, the notion of the smallest GE-filter generated by a non-empty subset of a transitive GE-algebra is introduced. It is proved that the class of all GE-filters of a transitive GE-algebra forms a complete distributive lattice.

Theorem 2.1. *Let B be a filter of a transitive GE-algebra X. For any non-empty subset A of X, the set*

$$\begin{aligned} \langle B \cup A \rangle = \; &\{x \in X \mid a_1 * (a_2 * (\cdots * (a_n * x) \cdots)) \in B \\ &\textit{for some } a_1, a_2, \ldots, a_n \in A;\; n \in \mathbb{N}\} \end{aligned}$$

is the smallest filter of X containing A.

Proof. Let $x \in B$. Since B is a GE-filter, we get

$$a_1 * (a_2 * (\cdots * (a_n * x) \cdots)) \in B$$

for any $a_1, \ldots, a_n \in X$. Hence $x \in \langle B \cup A \rangle$. Therefore $B \subseteq \langle B \cup A \rangle$ and hence $1 \in \langle B \cup A \rangle$. Let $x, x * y \in \langle B \cup A \rangle$. Then

$$a_1 * (a_2 * (\cdots * (a_n * x) \cdots)) \in B$$

and $b_1 * (b_2 * (\cdots * (b_m * (x * y)) \cdots)) \in B$ for some $a_1, a_2, \ldots, a_n \in A$ and $b_1, b_2, \ldots, b_m \in A$. Clearly $b_m * (x * y) \leq x * (b_m * y)$. Since X is transitive, by Theorem 1.5(1), we get

$$\begin{aligned} &b_1 * (b_2 * (\cdots * (b_{m-1} * (b_m * (x * y))) \cdots)) \\ &\leq b_1 * (b_2 * (\cdots * (b_{m-1} * (x * (b_m * y))) \cdots)) \\ &\leq b_1 * (b_2 * (\cdots * (x * (b_{m-1} * (b_m * y))) \cdots)) \\ &\leq \cdots \\ &\leq x * (b_1 * (b_2 * (\cdots * (b_{m-1} * (b_m * y)) \cdots))) \end{aligned}$$

Since B is a GE-filter, we get $x*(b_1*(b_2*(\cdots*(b_{m-1}*(b_m*y))) \cdots)) \in B$. Since X is a transitive, by the sequential application of the GE-ordering

$\leq$, we get that

$$\begin{aligned}
&x * (b_1 * (b_2 * (\cdots * (b_m * y) \cdots)))\\
&\leq (a_1 * (\cdots * (a_n * x) \cdots))\\
&*(a_1 * (\cdots * (a_n * (b_1 * (\cdots * (b_m * y) \cdots))) \cdots)).
\end{aligned}$$

Hence

$$(a_1 * (\cdots * (a_n * x) \cdots)) * (a_1 * (\cdots * (a_n * (b_1 * (\cdots * (b_m * y) \cdots))) \cdots)) \in B.$$

Since $a_1 * (\cdots * (a_n * x) \cdots) \in B$, we get

$$a_1 * (\cdots * (a_n * (b_1 * (\cdots * (b_m * y) \cdots))) \cdots) \in B$$

where $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in A$. Thus $y \in \langle B \cup A \rangle$. Therefore $\langle B \cup A \rangle$ is a GE-filter of X. For any $x \in A$, we get

$$x * (\cdots * (x * x) \cdots) = 1 \in B.$$

Hence $x \in \langle B \cup A \rangle$. Thus $A \subseteq \langle B \cup A \rangle$. Therefore $\langle B \cup A \rangle$ is a GE-filter containing $B \cup A$.

We now prove that $\langle B \cup A \rangle$ is the smallest filter containing $B \cup A$. Let F be a GE-filter of X containing $B \cup A$. Let $x \in \langle B \cup A \rangle$. Then there exist $a_1, a_2, \ldots, a_n \in B \cup A$ such that

$$a_1 * (a_2 * (\cdots * (a_n * x) \cdots)) \in B \subseteq B \cup A \subseteq F.$$

Since $B \cup A \subseteq F$, we get that $a_1, a_2, \ldots, a_n \in F$. Since $a_1 \in F$, $a_1 * (a_2 * (\cdots * (a_n * x) \cdots)) \in F$ and F is a GE-filter, we get

$$a_2 * (\cdots * (a_n * x) \cdots) \in F.$$

Since $a_2 \in F$, $a_2 * (\cdots * (a_n * x) \cdots) \in F$ and F is a GE-filter, we get $a_3 * (\cdots * (a_n * x) \cdots) \in F$. Continuing in this way, we finally get that $x \in F$. Hence $\langle B \cup A \rangle \subseteq F$. Therefore $\langle B \cup A \rangle$ is the smallest GE-filter containing $B \cup A$. □

Example 2.2. Let $X = \{1, a, b, c, d\}$. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c	d
1	1	a	b	c	1
a	1	1	b	c	d
b	1	a	1	1	1
c	1	a	b	1	d
d	1	a	b	c	1

Observe that $(X, *, 1)$ is a GE-algebra. Clearly $B = \{1, a\}$ is a GE-filter of X. Take the subset $A = \{c\}$ of X. Clearly $1, a, c \in \langle B \cup A \rangle$. Now, $c * d = d \notin B$ and $c * b = b \notin B$. Hence $b, d \notin \langle B \cup A \rangle$. Therefore $\langle B \cup A \rangle = \{1, a, c\}$ is the smallest GE-filter containing $B \cup A$.

For any GE-algebra X and $a, x \in X$, let us denote

$$a^n * x = a * (\cdots * (a * x) \cdots)$$

and a occurs n times. Then the following result is a direct consequence because of $a * (a * x) = a * x$:

Corollary 2.3. *Let B be a GE-filter of a transitive GE-algebra X. For any $a \in X$, the set*

$$\langle B \cup \{a\} \rangle = \{x \in X \mid a * x \in B \textit{ for some } n \in \mathbb{N}\}$$

is the smallest GE-filter containing $B \cup \{a\}$.

Corollary 2.4. *Let X be a transitive GE-algebra X. Then we have*

(i) For any $\emptyset \neq A \subseteq X$, the set

$$\begin{aligned} \langle A \rangle = \; & \{x \in X \mid a_1 * (a_2 * (\cdots * (a_n * x) \cdots)) = 1 \\ & \textit{for some } a_1, a_2, \ldots, a_n \in A; \; n \in \mathbb{N}\} \end{aligned}$$

is the smallest GE-filter containing A.

(ii) For any $a \in X$, $\langle \{a\} \rangle = \{x \in X \mid a * x = 1\}$ is the smallest GE-filter containing a.

Proof. (i) Taking of $B = \{1\}$ in the main theorem, it is clear.
(ii) Taking $A = \{a\}$ and $B = \{1\}$ in the main theorem, it is clear. $\square$

It is obvious that $\langle A \cup \{1\} \rangle = \langle A \rangle$. For any non-empty subset A of a transitive GE-algebra, $\langle A \rangle$ is called the GE-filter generated by A. Obviously we have $\langle A \rangle \subseteq \langle B \rangle$ whenever $A \subseteq B$ for any two subsets A and B of a transitive GE-algebra X. We denote $\langle \{a\} \rangle$ simply by $\langle a \rangle$ and call this a *principal GE-filter* generated by a.

Proposition 2.5. *Let F and G be two GE-filters of a transitive GE-algebra X. Then*

$$\langle F \cup G \rangle = \{x \in X \mid a * (b * x) = 1 \textit{ for some } f \in F \textit{ and } g \in G \}$$

Proof. Let

$$A = \{x \in X \mid a * (b * x) = 1 \text{ for some } f \in F \text{ and } g \in G \}.$$

We now prove that $A = \langle F \cup G \rangle$. Let $x \in A$. Then $f * (g * x) \in G$ for some $f \in F$ and $g \in G$. Since $f, g \in F \cup G$, by Corollary 2.3, we get $x \in \langle F \cup G \rangle$. Therefore $A \subseteq \langle F \cup G \rangle$.

Conversely, let $x \in \langle F \cup G \rangle$. Then there exists

$$a_1, a_2, \ldots, a_i, \ldots, a_n \in F \cup G$$

such that $a_n * (\cdots * (a_1 * x) \cdots) = 1$. By rearranging the $a_i^\prime s$ as seen in Theorem 2.1, we get

$$a_n * (\cdots * (a_{i+1} * (a_i \cdots * (a_1 * x) \cdots)) \cdots) = 1 \in G$$

such that $a_1, \ldots, a_i \in F$ and $a_{i+1}, \ldots, a_n \in G$. Since $a_n \in G$ and G is a filter, we get that

$$a_{n-1} * (\cdots * (a_{i+1} * (a_i \cdots * (a_1 * x) \cdots)) \cdots) \in G.$$

By continuing this, we get $a_i * (\cdots * (a_1 * x) \cdots) \in G$. Put $g = a_i * (\cdots * (a_1 * x) \cdots)$. Then

$$\begin{aligned} 1 &= g * g \\ &= g * (a_i * (\cdots * (a_1 * x) \cdots)) \\ &\leq a_i * (\cdots * (a_1 * (g * x)) \cdots) \end{aligned}$$

Hence $a_i * (\cdots * (a_1 * (g * x)) \cdots) = 1 \in F$. Since $a_i \in F$ and F is a GE-filter, we get $a_{i-1} * (\cdots * (a_1 * (g * x))) \in F$. By continuing this argument, we get $g * x \in F$. Put $f = g * x$. Then $f * (g * x) = (g * x) * (g * x) = 1$. Since $f \in F$ and $g \in G$, it implies that $x \in A$. Hence $\langle F \cup G \rangle \subseteq A$. Therefore $A = \langle F \cup G \rangle$. $\square$

In what follows, $\mathcal{F}(X)$ denotes the class of all GE-filters of a transitive GE-algebra X. Then, for any two GE-filters F and G of a transitive GE-algebra, it can be easily seen that $F \cap G$ is the infimum of both F and G. Now, in the following theorem, we obtain that $\mathcal{F}(X)$ forms a complete distributive lattice.

Theorem 2.6. *For any transitive GE-algebra X, $\mathcal{F}(X)$ forms a complete distributive lattice.*

Proof. For any two GE-filters F, G of a transitive GE-algebra, define

$$F \vee G = \langle F \cup G \rangle = \{a \in X \mid x * (y * a) = 1 \text{ for some } x \in F \text{ and } y \in G\ \}$$

By Proposition 2.5, $\langle F \cup G \rangle$ is the supremum of both F and G. Then clearly $(\mathcal{F}(X), \cap, \vee)$ is a complete lattice with respect to set inclusion. Let $F, G, H \in \mathcal{F}(X)$. Then clearly

$$(F \cap G) \vee (F \cap H) \subseteq F \cap (G \vee H).$$

Conversely, let $x \in F \cap (G \vee H)$. Then $x \in F$ and $x \in G \vee H$. Then there exists $g \in G$ and $h \in H$ such that $g * (h * x) = 1$. Now letting

$$b_1 = h * x \text{ and } b_2 = b_1 * x,$$

it is clear that $b_1 \in F$ and $b_2 \in F$. Now $g * b_1 = g * (h * x) = 1 \in G$. Since $g \in G$ and G is a GE-filter, we get that $b_1 \in G$. Hence $b_1 \in F \cap G$.

By Theorem 1.2(9), we get

$$\begin{aligned} 1 &= (h * x) * (h * x) \\ &\leq h * ((h * x) * x) \\ &= h * (b_1 * x) \\ &= h * b_2 \end{aligned}$$

Hence $h * b_2 = 1 \in H$. Since $h \in H$, we get that $b_2 \in H$. Thus $b_2 \in F \cap H$. Now

$$\begin{aligned} 1 &= ((h * x) * x) * ((h * x) * x) \\ &\leq (h * x) * (((h * x) * x) * x) \\ &= (h * x) * ((b_1 * x) * x) \\ &= b_1 * (b_2 * x) \end{aligned}$$

which gives that $b_1 * (b_2 * x) = 1$. Since $b_1 \in F \cap G$ and $b_2 \in F \cap H$, we get that $x \in (F \cap G) \vee (F \cap H)$. Hence $F \cap (G \vee H) \subseteq (F \cap G) \vee (F \cap H)$. Thus $F \cap (G \vee H) = (F \cap G) \vee (F \cap H)$. Therefore $(\mathcal{F}(X), \cap, \vee)$ is a complete distributive lattice. $\square$

Corollary 2.7. *Let F be a GE-filter of a transitive GE-algebra X. For any $a \in X$,*

$$\langle F \cup \{a\} \rangle = F \vee \langle a \rangle$$

Corollary 2.8. *Let X be a transitive GE-algebra. Then the class $\mathcal{F}(X)$ of all filters of X is a complete lattice with respect to the inclusion ordering $\subseteq$ in which for any set $\{F_\alpha\}_{\alpha \in \Delta}$ of filters of X, $inf\{F_\alpha\}_{\alpha \in \Delta} = \bigcap\limits_{\alpha \in \Delta} F_\alpha$ and $sup\{F_\alpha\}_{\alpha \in \Delta} = \langle \bigcup\limits_{\alpha \in \Delta} F_\alpha \rangle$.*

$\inf\{F_\alpha\}_{\alpha \in \Delta}$ and $\sup\{F_\alpha\}_{\alpha \in \Delta}$ are also denoted by $\bigwedge\limits_{\alpha \in \Delta} F_\alpha$ and $\bigvee\limits_{\alpha \in \Delta} F_\alpha$ respectively. Also, it can be easily observed that the lattice $\mathcal{F}(X)$ of all filters of X is an algebraic lattice too, in which the compact elements are precisely the finitely generated filters of X.

3. Prime filters of GE-algebras

In this section, the notion of prime GE-filters is introduced in GE-algebra. Some characterizations of prime filters are derived in transitive GE-algebras. Finally, two different versions of prime filter theorem are established in transitive GE-algebras.

Lemma 3.1. *Let X be a transitive GE-algebra. For any $a, b \in X$,*

(1) $\langle 1 \rangle = \{1\}$,
(2) $a \leq b$ *implies* $\langle b \rangle \subseteq \langle a \rangle$,

(3) *For any GE-filter F, $a \in F$ implies $\langle a\rangle \subseteq F$.*

Proof. It can be proved by routine verification. $\square$

Definition 3.2. A proper GE-filter P of a GE-algebra X is called a *prime filter* if $F \cap G \subseteq P$ implies that $F \subseteq P$ or $G \subseteq P$ for any two GE-filters F and G of X.

Example 3.3. Let $X = \{1, a, b, c, d, e, f\}$. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c	d	e	f
1	1	a	b	c	d	e	f
a	1	1	1	c	c	e	f
b	1	a	1	d	d	e	f
c	1	a	1	1	1	e	1
d	1	a	1	1	1	e	f
e	1	a	1	c	c	1	f
f	1	1	b	d	d	e	1

Clearly $(X, *, 1)$ is a GE-algebra and $P = \{1, a, b, c, d, f\}$ is a proper GE-filter of X. It can be easily seen that P is a prime filter of X. Take the proper filter $Q = \{1, a, b\}$. Now $F = \{1, b, c, d, f\}$ and $G = \{1, b, e\}$ are two GE-filters of X such that $F \cap G = \{1, b\} \subseteq Q$ but neither $F \subseteq Q$ nor $B \subseteq Q$. Therefore Q is not a prime filter of X.

Theorem 3.4. *A proper GE-filter P of a transitive GE-algebra X is prime if and only if $\langle x\rangle \cap \langle y\rangle \subseteq P$ implies $x \in P$ or $y \in P$ for all $x, y \in X$.*

Proof. Assume that P is prime. Let $x, y \in X$ be such that $\langle x\rangle \cap \langle y\rangle \subseteq P$. Since P is prime, it implies that $x \in \langle x\rangle \subseteq P$ or $y \in \langle y\rangle \subseteq P$.

Conversely, assume that the condition holds. Let F and G be two GE-filters of X such that $F \cap G \subseteq P$. Let $x \in F$ and $y \in G$ be the arbitrary elements. Then $\langle x\rangle \subseteq F$ and $\langle y\rangle \subseteq G$. Hence

$$\langle x\rangle \cap \langle y\rangle \subseteq F \cap G \subseteq P.$$

Then by the assumed condition, we get $x \in P$ or $y \in P$. Thus $F \subseteq P$ or $G \subseteq P$. Therefore P is a prime filter of X. $\square$

Theorem 3.5. *Let X be a transitive GE-algebra and F be a GE-filter of X. Then $A \cap B \subseteq F$ if and only if $\langle F \cup A\rangle \cap \langle F \cup B\rangle = F$ for any two GE-filters A and B of X.*

Proof. Assume that $\langle F \cup A\rangle \cap \langle F \cup B\rangle = F$ for any two GE-filters A and B of X. Since $A \subseteq F \cup A$ and $B \subseteq F \cup B$, we get

$$A \cap B \subseteq \langle F \cup A\rangle \cap \langle F \cup B\rangle.$$

Hence $A \cap B \subseteq F$.

Conversely, assume that $A \cap B \subseteq F$. Clearly

$$F \subseteq \langle F \cup A \rangle \cap \langle F \cup B \rangle.$$

Let $x \in \langle F \cup A \rangle \cap \langle F \cup B \rangle$. Since F is a GE-filter, there exist $a_1, a_2, \dots, a_n \in A$ and $b_1, b_2, \dots, b_m \in B$ such that

$$a_1 * (\cdots * (a_n * x) \cdots) \in F$$

and $b_1 * (\cdots * (b_m * x) \cdots) \in F$. Then we get

$$a_1 * (\cdots * (a_n * x) \cdots) = h_1$$

and $b_1 * (\cdots * (b_m * x) \cdots) = h_2$ for some $h_1, h_2 \in F$. Since X is transitive, by Theorem 1.2(9), we get

$$\begin{aligned} 1 &= h_1 * h_1 \\ &= h_1 * (a_1 * (\cdots * (a_n * x) \cdots)) \\ &\leq a_1 * (h_1 * (\cdots * (a_n * x) \cdots)) \\ &\cdots \\ &\cdots \\ &\leq a_1 * (\ldots * (a_n * (h_1 * x)) \cdots) \end{aligned}$$

which yields $a_1 * (\ldots * (a_n * (h_1 * x)) \cdots) = 1 \in A$. Since A is a GE-filter and $a_1, a_2, \dots, a_n \in A$, we get $h_1 * x \in A$. By the similar argument, we get $h_2 * x \in B$. Since $h_1, h_2 \in F$, we get

$$h_1 * x \leq h_2 * (h_1 * x) \leq h_1 * (h_2 * x) \text{ and } h_2 * x \leq h_1 * (h_2 * x).$$

Since $h_1 * x \in A$ and A is a GE-filter, we get $h_1 * (h_2 * x) \in A$. Since $h_2 * x \in B$ and B is a GE-filter, we get $h_1 * (h_2 * x) \in B$. Hence $h_1 * (h_2 * x) \in A \cap B \subseteq F$. Since $h_1, h_2 \in F$ and F is a GE-filter, we get $x \in F$. Hence $\langle F \cup A \rangle \cap \langle F \cup B \rangle \subseteq F$. Therefore $\langle F \cup A \rangle \cap \langle F \cup B \rangle = F$. □

Corollary 3.6. *Let X be a transitive GE-algebra and F a GE-filter of X. Then $\langle a \rangle \cap \langle b \rangle \subseteq F$ if and only if $\langle F \cup \{a\} \rangle \cap \langle F \cup \{b\} \rangle = F$ for any $a, b \in X$.*

Definition 3.7. A proper GE-filter M of a GE-algebra X is called *maximal* if there exists a GE-filter F such that $M \subseteq F \subseteq X$, then $M = F$ or $F = X$.

Example 3.8. Consider the GE-algebra X given in Example 3.3. Clearly $M = \{1, a, b, e, f\}$ is a proper filter of X. It can be easily verified that M is a maximal filter of X.

In the following, we derive a necessary and sufficient condition for every proper GE-filter of a transitive GE-algebra to become maximal.

Theorem 3.9. *A proper GE-filter M of a transitive GE-algebra X is maximal if and only if $\langle M \cup \{x\} \rangle = X$ for any $x \in X - M$.*

Proof. Assume that M is a maximal filter of X. Let $x \in X - M$. Suppose $\langle M \cup \{x\} \rangle \neq X$. Choose $a \notin \langle M \cup \{x\} \rangle$ and $a \in X$. Hence $M \subseteq \langle M \cup \{x\} \rangle \subset X$. Since M is maximal, we get $M = \langle M \cup \{x\} \rangle$. Hence $x \in M$, which is a contradiction. Therefore M is maximal.

Conversely, assume the condition. Suppose there is a filter F such that $M \subseteq F \subseteq X$. Let $M \neq F$. Suppose $F \neq X$. Choose $x \in X$ such that $x \notin F$. Then clearly $x \notin M$ because of $F \subseteq M$. Since $x \notin M$, by the assumed condition, we get that $\langle M \cup \{x\} \rangle = X$. Let $a \in X$. Then $a \in \langle M \cup \{x\} \rangle$. Hence $x * a \in M \subseteq F$. Since $x \in F$ and F is a GE-filter, we get $a \in F$. Therefore $X \subseteq F$, which means $F = X$. □

Theorem 3.10. *Every maximal filter of a transitive GE-algebra is a prime filter.*

Proof. Let M be a maximal filter of a transitive GE-algebra X. Let $\langle x \rangle \cap \langle y \rangle \subseteq M$ for some $x, y \in X$. Suppose $x \notin M$ and $y \notin M$. Then $\langle M \cup \{x\} \rangle = X$ and $\langle M \cup \{y\} \rangle = X$. Hence

$$\langle M \cup \{x\} \rangle \cap \langle M \cup \{y\} \rangle = X \neq M$$

By Corollary 3.6, we get $\langle x \rangle \cap \langle y \rangle \nsubseteq M$, which is a contradiction. Thus $x \in M$ or $y \in M$. Therefore M is a prime filter of X. □

Example 3.11. Let $X = \{1, a, b, c, d, e, f\}$. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c	d	e	f
1	1	a	b	c	d	e	f
a	1	1	1	d	d	f	f
b	1	a	1	c	c	e	e
c	1	a	1	1	1	e	f
d	1	a	1	1	1	e	f
e	1	a	b	d	d	1	1
f	1	a	b	c	d	1	1

Clearly $(X, *, 1)$ is a GE-algebra and $P = \{1, b, c, d\}$ is a prime filter of X. Observe that P is a not a maximal filter of X because of $F = \{1, a, b, c, d\}$ is a proper filter of X such that $P \subset F \subset X$.

4. Prime filter theorems

In this section, we now generalise and present four versions of the famous *prime filter theorem* of various algebraic structures in transitive GE-algebras. Let us define a $\cap$-closed subset of a GE-algebra as the subset S of X in which $\langle a\rangle \cap \langle b\rangle \subseteq S$ for all $a, b \in S$.

Proposition 4.1. *Let P be a prime filter of a transitive GE-algebra X and $a \in X$. Then the set*

$$S = \{x \in X \mid \langle x\rangle \subseteq \langle a\rangle \vee F \text{ for some filter } F \text{ with } F \nsubseteq P\}$$

is a $\cap$-closed subset of X.

Proof. Let P be a prime filter of X and $x, y \in X$. Suppose $x, y \in S$. Then there exist filters F_1 and F_2 of X with $F_1 \nsubseteq P, F_2 \nsubseteq P$ such that $\langle x\rangle \subseteq \langle a\rangle \vee F_1$ and $\langle y\rangle \subseteq \langle a\rangle \vee F_2$. Hence

$$\langle x\rangle \cap \langle y\rangle \subseteq \big\{\langle a\rangle \vee F_1\big\} \cap \big\{\langle a\rangle \vee F_2\big\} = \langle a\rangle \vee \big(F_1 \cap F_2\big).$$

Since P is prime, we get $F_1 \cap F_2 \nsubseteq P$. Let $t \in \langle x\rangle \cap \langle y\rangle$. Then $\langle t\rangle \subseteq \langle x\rangle \cap \langle y\rangle \subseteq \langle a\rangle \vee \big(F_1 \cap F_2\big)$. Hence $t \in S$, which gives that $\langle x\rangle \cap \langle y\rangle \subseteq S$. Therefore S is $\cap$-closed subset of X. □

Theorem 4.2. ***(First prime filter theorem)*** *Let F be a GE-filter and S be a $\cap$-closed subset of a transitive GE-algebra X such that $F \cap S = \emptyset$. Then there exists a prime filter P of X such that $F \subseteq P$ and $P \cap S = \emptyset$.*

Proof. Let F be a GE-filter and S be a $\cap$-closed subset of X such that $F \cap S = \emptyset$. Consider

$$\mathcal{F} = \{G \in \mathcal{F}(X) \mid F \subseteq G \text{ and } G \cap S = \emptyset\}.$$

Clearly $F \in \mathcal{F}$ and so $\mathcal{F} \neq \emptyset$. Let $\{G_\alpha\}_{\alpha\in\Delta}$ be a chain of elements of $\mathcal{F}$. Then clearly $\bigcup_{\alpha\in\Delta} G_\alpha$ is an upper bound of $\{G_\alpha\}_{\alpha\in\Delta}$. Hence by the Zorn's Lemma, $\mathcal{F}$ has a maximal element, say M. Clearly M is a GE-filter such that $F \subseteq M$ and $M \cap S = \emptyset$. We now prove that M is prime. Let $x, y \in X$ be such that $x \notin M$ and $x \notin M$. Then $M \subset M \vee \langle x\rangle$ and $M \subset M \vee \langle y\rangle$. By the maximality of M, we should have $\{M\vee\langle x\rangle\}\cap S \neq \emptyset$ and $\{M\vee\langle y\rangle\}\cap S \neq \emptyset$. Choose $a \in \{M\vee\langle x\rangle\}\cap S$ and $b \in \{M \vee \langle y\rangle\} \cap S$. Since $a, b \in S$, we get $\langle a\rangle \cap \langle b\rangle \subseteq S$ because of S is $\cap$-closed. Now

$$\begin{aligned} \langle a\rangle \cap \langle b\rangle &\subseteq \{M \vee \langle x\rangle\} \cap \{M \vee \langle y\rangle\} \\ &= M \vee \{\langle x\rangle \cap \langle y\rangle\}. \end{aligned}$$

If $\langle x\rangle \cap \langle y\rangle \subseteq M$, then $\langle a\rangle \cap \langle b\rangle \subseteq M$. Hence $\langle a\rangle \cap \langle b\rangle \subseteq M \cap S$, which is a contradiction. Thus $\langle a\rangle \cap \langle b\rangle \nsubseteq M$. Therefore M is prime. □

Definition 4.3. A nonempty subset S of a GE-algebra X is called a *finite $\cap$-structure*, if $\{\langle x\rangle \cap \langle y\rangle\} \cap S \neq \emptyset$ for all $x, y \in S$.

Clearly every GE-filter of a transitive GE-algebra is a finite $\cap$-structure. It can also be observed that every $\cap$-closed subset is a finite $\cap$-structure but the converse is not true.

Example 4.4. Let $X = \{1, a, b, c, d\}$. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c	d
1	1	a	b	c	1
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then it can be easily verified that $(X, *, 1)$ is a GE-algebra. Consider the set $S = \{b, d\}$. Then $\langle b\rangle = \{1, b\}$ and $\langle d\rangle = \{1, a, b, c, d\}$. Hence $\{\langle b\rangle \cap \langle d\rangle\} \cap S = \{b\}$, which means S is a finite $\cap$-structure. For $b, d \in S$, we have $\langle b\rangle \cap \langle d\rangle \nsubseteq S$. Hence S is not $\cap$-closed.

Proposition 4.5. *Let P be a proper GE-filter of a transitive GE-algebra X. Then P is prime if and only if $X - P$ is finite $\cap$-structure.*

Proof. Let P be a GE-filter of X. Assume that P is prime. Let $x, y \in X - P$. Then $x \notin P$ and $y \notin P$. Suppose $\{\langle x\rangle \cap \langle y\rangle\} \cap (X - P) = \emptyset$. Then $\langle x\rangle \cap \langle y\rangle \subseteq P$. Since P is prime, we get $x \notin P$ or $y \notin P$, which is a contradiction. Hence $\{\langle x\rangle \cap \langle y\rangle\} \cap (X - P) \neq \emptyset$.

Conversely, assume that $X - P$ is finite $\cap$-structure. Let $x, y \in X$ be such that $\langle x\rangle \cap \langle y\rangle \subseteq P$. Suppose $x \notin P$ and $y \notin P$. Then $x, y \in X - P$. Since $X - P$ is finite $\cap$-structure, we get $\{\langle x\rangle \cap \langle y\rangle\} \cap (X - P) \neq \emptyset$. Hence $\langle x\rangle \cap \langle y\rangle \nsubseteq P$, which is a contradiction. Thus $x \in P$ or $y \in P$. Therefore P is a prime filter of X. □

Theorem 4.6. ***(Second prime filter theorem)*** *Let F be a GE-filter of a transitive GE-algebra X. If S is a finite $\cap$-structure such that $F \cap S = \emptyset$, then there exists a prime filter P of X such that $F \subseteq P$ and $P \cap S = \emptyset$.*

Proof. Let F be a GE-filter of X and S be a finite $\cap$-structure such that $F \cap S = \emptyset$. Consider

$$\mathcal{F} = \{G \in \mathcal{F}(X) \mid F \subseteq G \text{ and } G \cap S = \emptyset\}.$$

Clearly $F \in \mathcal{F}$ and so $\mathcal{F} \neq \emptyset$. Let $\{G_\alpha\}_{\alpha\in\Delta}$ be a chain of elements of $\mathcal{F}$. Then clearly $\cup_{\alpha\in\Delta} G_\alpha$ is an upper bound of $\{G_\alpha\}_{\alpha\in\Delta}$. Hence the hypothesis of Zorn's Lemma is satisfied. Thus $\mathcal{F}$ has a maximal element, say M. Clearly M is a GE-filter such that $F \subseteq M$ and $M \cap S = \emptyset$. We now prove that M is prime. Let H and K be two GE-filters of X such that $H \nsubseteq M$ and $K \nsubseteq M$. Then $M \subset \langle M \cup H \rangle$ and $M \subset \langle M \cup K \rangle$. By the maximality of M, we should have

$$\langle M \cup H \rangle \cap S \neq \emptyset$$

and $\langle M \cup K \rangle \cap S \neq \emptyset$. Choose $a \in \langle M \cup H \rangle \cap S$ and $b \in \langle M \cup K \rangle \cap S$. Since $a \in \langle M \cup H \rangle$ and $b \in \langle M \cup K \rangle$, we get

$$\langle a \rangle \cap \langle b \rangle \;\subseteq\; \langle M \cup H \rangle \cap \langle M \cup K \rangle.$$

Since $a, b \in S$, we get $\{\langle a \rangle \cap \langle b \rangle\} \cap S \neq \emptyset$ because of S is finite $\cap$-structure. Hence

$$\{\langle M \cup H \rangle \cap \langle M \cup K \rangle\} \cap S \neq \emptyset.$$

Since $M \in \mathcal{F}$, we get $M \cap S = \emptyset$. Comparing this with the last relation, we get $M \neq \langle M \cup H \rangle \cap \langle M \cup K \rangle$. By Theorem 3.5, gives $H \cap K \nsubseteq M$. Therefore M is a prime filter of X. $\square$

Theorem 4.7. *Let X ba a transitive GE-algebra and $a \in X$. If F is a GE-filter of X such that $a \notin F$, then there exists a prime filter P such that $a \notin P$ and $F \subseteq P$.*

Proof. Let F be a GE-filter of X such that $a \notin F$. Consider the set $[a] = \{x \in X \mid x \leq a\}$. We first show that $[a]$ is a finite $\cap$-structure. Clearly $a \in [a]$. Let $x, y \in [a]$. Then $x \leq a$ and $y \leq a$. Hence $a \in \langle x \rangle$ and $a \in \langle y \rangle$, which gives $a \in \big\{\langle x \rangle \cap \langle y \rangle\big\} \cap [a]$. Therefore $[a]$ is a finite $\cap$-structure. We now claim that $[a] \cap F = \emptyset$. Suppose $x \in [a] \cap F$. Then $x \leq a$ and $x \in F$. Since F is a filter, we get $a \in F$ which is a contradiction. Thus $[a] \cap F = \emptyset$. Therefore, by Theorem 4.6, there exists a prime filter P such that $F \subseteq P$ and $[a] \cap P = \emptyset$. Since $a \in [a]$, we must have $a \notin P$. Therefore the theorem is proves. $\square$

Corollary 4.8. *Let F be a proper GE-filter of a transitive GE-algebra X. Then*

$$F = \bigcap\{P \mid P \text{ is a prime filter of } X \text{ such that } F \subseteq P\}.$$

Proof. Let F be a proper GE-filter and $x \in F$. For any prime filter P with $F \subseteq P$, we must have $x \in P$. Hence

$$x \in \bigcap\{P \mid P \text{ is a prime filter of } X \text{ such that } F \subseteq P\}.$$

Therefore $F \subseteq \bigcap\{P \mid P$ is a prime filter of X such that $F \subseteq P\}$. Conversely, let $x \notin F$. Then by the main theorem, there exists a prime filter P_x such that $x \notin P_x$ and $F \subseteq P_x$. Therefore

$$x \notin \bigcap\{P \mid P \text{ is a prime filter of } X \text{ such that } F \subseteq P\}.$$

Therefore $\bigcap\{P \mid P$ is a prime filter of X such that $F \subseteq P\} \subseteq F$. $\square$

Corollary 4.9. *Let X be a transitive GE-algebra and $1 \neq x \in X$. Then there exists a prime filter P such that $x \notin P$.*

Proof. Let $1 \neq x \in X$ and $F = \{1\}$. Then F is a GE-filter and $x \notin F$. By the main theorem, there exists a prime filter P such that $x \notin P$. $\square$

The following corollary is a direct consequence of the above results.

Corollary 4.10. *The intersection of all prime filters of a transitive GE-algebra is equal to $\{1\}$.*

Theorem 4.11. *Let X_1 be a subalgebra of a transitive GE-algebra X and P_1 is a prime filter of X_1. Then there exists a prime filter P of X such that $P \cap X_1 = P_1$.*

Proof. Let P_1 be a prime filter of X_1. Then $X_1 - P_1$ is a $\cap$-closed subset of X. Write $F = \langle P_1 \rangle$, the filter generated by P_1. Then $P_1 \subseteq F \cap X_1$. Suppose $F \cap (X_1 - P_1) \neq \emptyset$. Choose $x \in F \cap (X_1 - P_1)$. Then $x \in F$ and $x \in (X_1 - P_1)$. Since $x \in F = \langle P_1 \rangle$, there exists $a_1, a_2, \ldots, a_n \in P_1, n \in \mathbb{N}$ such that $a_1 * (a_2 * (\ldots (a_n * x) \ldots)) = 1$. Since $a_1, a_2, \ldots, a_n \in P_1$, we get $x \in P_1$. Since $x \in (X_1 - P_1)$, we have arrived at a contradiction. Hence $F \cap (X_1 - P_1) = \emptyset$. Then by the second prime filter theorem, there exists a prime filter P of X such that $F \subseteq P$ and $P \cap (X_1 - P_1) = \emptyset$. Since $F \subseteq P$, we get $F \cap X_1 \subseteq P \cap X_1$. Since $P \cap (X_1 - P_1) = \emptyset$, we get $P \subseteq P_1$. Both observations lead to

$$P_1 \subseteq F \cap X_1 \subseteq P \cap X_1 \subseteq P_1 \cap X_1 \subseteq P_1.$$

Therefore $P_1 = P \cap X_1$. $\square$

Definition 4.12. A subset S of a GE-algebra X is called commutative closed if $(x * y) * y = (y * x) * x \in S$ for all $x, y \in S$

In a commutative GE-algebra, it can be observed that every GE-filter is a commutative closed subset.

Proposition 4.13. *Every commutative closed set of a transitive GE-algebra is a finite $\cap$-structure.*

Proof. Let S be a commutative closed subset of a transitive GE-algebra X. Let $x, y \in S$. Since S is commutative closed, we get

$$(x * y) * y = (y * x) * x \in S.$$

Since $\langle x \rangle$ is a filter and $x \in \langle x \rangle$, we get $(y * x) * x \in \langle x \rangle$. Similarly, we get $(x * y) * y \in \langle y \rangle$. Hence $(x * y) * y = (y * x) * x \in \langle x \rangle \cap \langle y \rangle$. Thus $\{\langle x \rangle \cap \langle y \rangle\} \cap S \neq \emptyset$. Therefore S is a finite $\cap$-structure in X. $\square$

The converse of the above proposition is not true. For, consider the following example:

Example 4.14. In the GE-algebra considered in Example 4.4, the set $S = \{a, c\}$ is a finite $\cap$-structure because of $\{\langle a\rangle \cap \langle c\rangle\} \cap S = \{a\} \neq \emptyset$. However, for this choice of a, c, the set S is not commutative closed because of $(a * c) * c = c * c \neq a = 1 * a = (c * a) * a$.

In the following couple of examples, we can see the independency between $\cap$-closed subsets and commutative closed subsets.

Example 4.15. (a) In the GE-algebra considered in Example 4.4, the set $S = \{a, c\}$ is a finite $\cap$-structure because of

$$\{\langle a\rangle \cap \langle c\rangle\} \cap S = \{a\} \neq \emptyset.$$

However, for this choice of a and c, the set S is not commutative closed because of $(a * c) * c = c * c \neq a = 1 * a = (c * a) * a$.
(b) Let $X = \{1, a, b, c\}$. Define a binary operation $*$ on X as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then, it can be easily verified that $(X, *, 1)$ is a GE-algebra. Consider the set $S = \{a, b\}$. Clearly $(a * b) * b = (b * a) * a = a \in S$. Hence S is commutative closed. But S is not $\cap$-closed because of

$$\langle a\rangle \cap \langle b\rangle = \{1, a\} \cap \{1, a, b\} = \{1, a\} \nsubseteq S.$$

Theorem 4.16. ***(Third prime filter theorem)*** *Let X be a transitive GE-algebra and S is a commutative subset of X. If F is a GE-filter of X such that $F \cap S = \emptyset$, then there exists a prime filter P of X such that $F \subseteq P$ and $P \cap S = \emptyset$.*

Proof. Let F be a GE-filter of X such that $F \cap S = \emptyset$. Consider

$$\Im = \{G \in \mathcal{F}(L) \mid F \subseteq G \text{ and } G \cap S = \emptyset\}.$$

Let M be a maximal element of $\Im$. Suppose H and K are two GE-filters of X such that $H \cap K \subseteq M$. Suppose $H \nsubseteq M$ and $K \nsubseteq M$. By the maximality of M, we get that $\langle M \cup H\rangle \cap S \neq \emptyset$ and $\langle M \cup K\rangle \cap S \neq \emptyset$. Choose $a \in \langle M \cup H\rangle \cap S$ and $b \in \langle M \cup K\rangle \cap S$. Since $a, b \in S$, we get $(a * b) * b = (b * a) * a \in S$. For this $a, b \in S$, we get

$$1 = (a * b) * (a * b) \leq a * ((a * b) * b)$$

Hence $a * ((a * b) * b) = 1 \in \langle M \cup H\rangle$. Since $a \in \langle M \cup H\rangle$, we get $(a * b) * b \in \langle M \cup H\rangle$. Similarly, we get $(b * a) * a \in \langle M \cup K\rangle$. Hence

$$(a * b) * b = (b * a) * a \in \big\{\langle M \cup H\rangle \cap \langle M \cup K\rangle\big\} \cap S.$$

Suppose $\langle M \cup H\rangle \cap \langle M \cup K\rangle = M$. Then $(a*b)*b \in M \cap S$, which is a contradiction. $(\langle M \cup H\rangle \cap \langle M \cup K\rangle) \neq M$. Thus by Theorem 3.5, we get $H \cap K \nsubseteq M$, which is a contradiction. Hence $H \subseteq M$ or $K \subseteq M$. Therefore P is prime. □

We now discuss the prime filters of commutative GE-algebras. For any $x, y \in X$, define $x \vee y = (x * y) * y$. If X commutative, then clearly $x \vee y = (y * x) * x = (x * y) * y = y \vee x$. Also $x \vee y$ is the supremum of x and y. Hence $(X, \vee)$ is a semilattice. We now discuss the properties of prime filters in commutative GE-algebras.

Lemma 4.17. *Let X be a commutative GE-algebra. For any $a, b \in X$, $\langle a \vee b\rangle = \langle a\rangle \cap \langle b\rangle$.*

Proof. Since $a, b \leq a \vee b$, we get $\langle a \vee b\rangle \subseteq \langle a\rangle, \langle b\rangle$. Hence

$$\langle a \vee b\rangle \subseteq \langle a\rangle \cap \langle b\rangle.$$

Conversely, let $x \in \langle a\rangle \cap \langle b\rangle$. Then $a * x = 1$ and $b * x = 1$. Hence $a \leq x$ and $b \leq x$, which gives $a \vee b \leq x$. Hence $(a \vee b) * x = 1$, which means $x \in \langle a \vee b\rangle$. □

Theorem 4.18. *Let X be a commutative GE-algebra and P a proper GE-filter of X. Then P is prime if and only if for any $x, y \in X$, $x \vee y \in P$ implies $x \in P$ or $y \in P$.*

Proof. Assume that P is a prime filter of X. Let $x, y \in X$ be such that $x \vee y \in P$. Hence $\langle x\rangle \cap \langle y\rangle = \langle x \vee y\rangle \subseteq P$. Since P is prime, we get $x \in P$ or $y \in P$.

Conversely, assume that P satisfies the condition. Let $x, y \in X$ be such that $\langle x\rangle \cap \langle y\rangle \subseteq P$. Since $\langle x \vee y\rangle = \langle x\rangle \cap \langle y\rangle$, we get $\langle x \vee y\rangle \subseteq P$. Hence $x \vee y \in P$. By the assumed condition, we get $x \in P$ or $y \in P$. Therefore P is a prime filter. □

Definition 4.19. A subset S of a commutative GE-algebra is called $\vee$-closed if $x \vee y \in S$ whenever $x, y \in S$.

Proposition 4.20. *Let X be a commutative BE-algebra and $a \in X$. Then the set $[a] = \{x \in X \mid x \leq a\}$ is a $\vee$-closed set.*

Proof. Let $x, y \in [a]$. Then $x \leq a$ and $y \leq a$. Since X is commutative, it is partially ordered. Hence $x \vee y \leq a$, which gives that $x \vee y \in [a]$. Therefore $[a]$ is a $\vee$-closed subset of X. □

Every $\cap$-closed subset of a commutative GE-algebra is $\vee$-closed. For, consider a $\cap$-closed subset S and $x, y \in S$. Since S is $\cap$-closed, we get

$\langle x\rangle \cap \langle y\rangle \subseteq S$. Then $x \vee y \in \langle x \vee y\rangle = \langle x\rangle \cap \langle y\rangle \subseteq S$. Therefore S is $\vee$-closed. However, every $\vee$-closed subset need not be $\cap$-closed which can be seen in the following example:

Example 4.21. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ X as follows:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	c
b	1	1	1	c
c	1	a	b	1

$\vee$	1	a	b	c
1	1	1	1	1
a	1	a	a	1
b	1	a	b	1
c	1	1	1	c

It can be routinely verified that $(X, *, \vee, 1)$ is a commutative GE-algebra. Consider $S = \{a, b\}$. Clearly S is a $\vee$-closed subset of X. It can be easily observed that $\langle a\rangle = \{1, a\}$ and $\langle b\rangle = \{1, a, b\}$. Hence $\langle a\rangle \cap \langle b\rangle = \{1, a\} \nsubseteq S$. Therefore S is not $\cap$-closed.

Every $\vee$-closed subset of a commutative GE-algebra is finite $\cap$-structure. For, consider a $\vee$-closed subset S and $x, y \in S$. Then $x \vee y \in S$. Since $x, y \leq x \vee y$, we get $x \vee y \in \langle x\rangle \cap \langle y\rangle$. Thus

$$x \vee y \in \{\langle x\rangle \cap \langle y\rangle\} \cap S.$$

Therefore S is finite $\cap$-structure. Regarding the converse, we see that every finite $\cap$-structure need not be $\vee$-closed. However, in the following proposition, a necessary and sufficient condition for a subset of a GE-algebra to become a $\vee$-closed:

Proposition 4.22. *Let P be a proper GE-filter of a commutative GE-algebra X. Then P is prime if and only if $X - P$ is $\vee$-closed.*

Proof. Let P be a GE-filter of X. Assume that P is prime. Let $x, y \in X - P$. Then $x \notin P$ and $y \notin P$. Since P is prime, we get $x \vee y \notin P$. Therefore $x \vee y \in X - P$.

Conversely, assume that $X - P$ is $\vee$-closed. Let $x, y \in X$ be such that $\langle x\rangle \cap \langle y\rangle \subseteq P$. Suppose $x \notin P$ and $y \notin P$. Then $x, y \in X - P$. Since $X - P$ is $\vee$-closed, we get $x \vee y \in P$. Hence $\langle x\rangle \cap \langle y\rangle = \langle x \vee y\rangle \subseteq P$, which is a contradiction. Therefore $x \in P$ or $y \in P$. $\square$

Theorem 4.23. ***(Fourth prime filter theorem)*** *Let X be a commutative GE-algebra and S is a $\vee$-closed subset of X. If F is a GE-filter of X such that $F \cap S = \emptyset$, then there exists a prime filter P of X such that $F \subseteq P$ and $P \cap S = \emptyset$.*

Proof. Let F be a GE-filter of X such that $F \cap S = \emptyset$. Consider

$$\Im = \{G \in \mathcal{F}(L) \mid F \subseteq G \text{ and } G \cap S = \emptyset\}.$$

Let M be the maximal element of $\Im$. Let $a, b \in X$ be such that $a \notin M$ and $b \notin M$. Then $M \subseteq \langle M \cup \{a\}\rangle$ and $M \subseteq \langle M \cup \{b\}\rangle$. By the maximality of M, we get that $\langle M\cup\{a\}\rangle \cap S \neq \emptyset$ and $\langle M\cup\{b\}\rangle\cap S \neq \emptyset$. Choose $x \in \langle M \cup \{a\}\rangle \cap S$ and $y \in \langle M \cup \{b\}\rangle \cap S$. Clearly $x \vee y \in S$. Now

$$\begin{aligned} x \vee y &\in \langle M \cup \{a\}\rangle \cap \langle M \cup \{b\}\rangle \\ &= \{M \vee \langle a\rangle\} \cap \{M \vee \langle b\rangle\} \\ &= M \vee \{\langle a\rangle \cap \langle b\rangle\} \\ &= M \vee \langle a \vee b\rangle \end{aligned}$$

If $a \vee b \in M$, then $x \vee y \in M$. Hence $x \vee y \in M \cap S$, which is a contradiction. Thus $a \vee b \notin M$. Therefore M is a prime filter. □

Corollary 4.24. *Let a, b be two distinct elements of a commutative GE-algebra X such that $a * b \neq 1$ or $b * a \neq 1$. Then there exists a prime filter which contains exactly one of a and b.*

Proof. Assume that $a * b \neq 1$. Clearly $[b] = \{x \in X \mid x \leq b\}$ is a $\vee$-closed set. Suppose $[b] \cap \langle a\rangle \neq \emptyset$. Choose $x \in [b] \cap \langle a\rangle$. Then $x \leq b$ and $a * x = 1$. Then $1 = a * x \leq a * b$. Hence $a * b = 1$, which is a contradiction. Hence $[b] \cap \langle a\rangle = \emptyset$. By the main theorem, there exists a prime filter P such that $\langle a\rangle \subseteq P$ and $[b] \cap P = \emptyset$. Therefore $a \in P$ and $b \notin P$. □

5. Conclusion

In this article, an investigation is made to introduce various types of meet structures in the form of typical subsets of *GE*-algebras. Certain properties of prime filters and maximal filters of transitive *GE*-algebras are investigated. Based on the characterization of meet structures, different versions of prime filter theorem are generalized in the case of transitive *GE*-algebras. In the future work, further properties of prime filter of *GE*-algebras are investigated which help in the characterization of several structures of *GE*-algebras.

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Mukkamala Sambasiva Rao
Department of Mathematics, MVGR College of Engineering, Vizianagaram, Andhra Pradesh, India.
Email: mssraomaths35@rediffmail.com

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DERIVATIONS OF PRIME FILTER THEOREMS GENERATED BY VARIOUS $\cap$-STRUCTURES IN TRANSITIVE GE-ALGEBRAS

M. SAMBASIVA RAO

مشتقاتی از قضیه‌های فیلتر اول تولیدشده توسط $\cap$-ساختارهای مختلف در GE-جبرهای متعدی

موکامالا سامباسیوارائو

گروه ریاضی، کالج‌مهندسی $MVGR$، ویزیاناگارام، آندراپرادش، هند

خواص فیلترهای اول و فیلترهای ماکسیمال GE-جبرهای متعدی بررسی شده‌اند. مشخصه‌سازی بر اساس عنصر حلقه، کوچکترین GE-فیلتر است که شامل مجموعه‌ی داده شده می‌باشد. نشان داده شده است که مجموعه‌ی همه‌ی GE-فیلترهای یک GE-جبر متعدی، یک مشبکه‌ی کامل توزیع‌پذیر است. چهار ورژن متفاوت قضیه‌ی فیلتر اول در GE-جبرهای متعدی تعمیم داده شده‌اند. یک شرط لازم و کافی ارائه گردیده که تحت آن، یک فیلتر محض از GE-جبر جابه‌جایی، فیلتر اول می‌باشد.

کلمات کلیدی: GE-جبر، GE-فیلتر، فیلتر اول، فیلتر ماکسیمال، مجموعه‌ی $\cap$-بسته، $\cap$-ساختار متناهی، مجموعه‌ی بسته‌ی جابه‌جایی، مجموعه‌ی $\vee$-بسته.