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# THE UNIT GRAPH OF A COMMUTATIVE SEMIRING

L. BORO*, M. M. SINGH AND J. GOSWAMI

ABSTRACT. Let $S$ be a commutative semiring with unity and $U(S)$ be the set of all units of $S$. The unit graph of $S$, denoted by $G(S)$ is the undirected graph with vertex set $S$ and two distinct vertices $x$ and $y$ are adjacent in $G(S)$ if and only if $x + y \in U(S)$. In this paper, we concentrate on the unit graph $G(S)$ and look at several properties like the completeness, the bipartiteness, the connectedness, the diameter and the girth. We also obtain necessary and sufficient conditions for $G(S)$ to be traversable under certain conditions.

## 1. Introduction

In 1989, Grimaldi [13] introduced the graphical aspect of algebraic structures, namely unit graph $G(\mathbb{Z}_n)$ of $\mathbb{Z}_n$. The unit graph $G(\mathbb{Z}_n)$ is an undirected graph, whose vertex set is elements of $\mathbb{Z}_n$, and two distinct vertices $x$ and $y$ are adjacent in $G(\mathbb{Z}_n)$ if and only if $x + y$ is a unit of $\mathbb{Z}_n$. Recently, Ashrafi et al. [3] generalized the unit graph $G(\mathbb{Z}_n)$ to $G(R)$ for an arbitrary ring $R$ and obtained various results of finite commutative rings regarding the connectedness, the chromatic index, the diameter, the girth and the planarity of $G(R)$. In recent years, many fundamental papers on unit graphs associated with rings have been appeared, for instance, see [14, 17, 16, 15, 19, 20]. Nowadays, the study of graph structures on semiring theoretical setting is also an interesting area of research. Many research works

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*Corresponding author.

relating to a graph structure associated with semirings has been appeared recently, for instance, see [1, 5, 6, 12, 21].

A semiring $S$ is an algebraic sysytem $(S,\ +,\ .)$ such that $(S, +)$ is a commutative monoid with identity element 0 and $(S, .)$ is a semigroup with identity element 1. In addition, binary operations "+" and "." are connected by distributivity and 0 annihilates $S$. A semiring $S$ with unity is said to be a commutative semiring if $(S, .)$ is a commutative semigroup. A non-empty subset $I$ of $S$ is called an ideal of $S$ if the following two conditions hold: $(i)$ $x + y \in I$ for all $x, y \in I$ $(ii)$ $sx \in I$ for any $s \in S$ and $x \in I$. An ideal $I$ of $S$ is called a $k$-ideal (subtractive ideal) if $x, x + y \in I$, then $y \in I$. Therefore, $\{0\}$ is a $k$-ideal of $S$. A semiring $S$ is said to be a local semiring if and only if $S$ has a unique maximal $k$-ideal. Moreover, $x$ is a unit of $S$ if and only if $x$ lies outside of each maximal $k$-ideal of $S$ [4]. We denote the characteristic, the set of units, the set of non-units, and the Jacobson radical of a semiring $S$ by $char(S)$, $U(S)$, $\overline{U(S)}$ and $J(S)$ respectively. For undefined terminology and concept of semiring theory, we refer to Golan [11].

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Two distinct vertices $u$ and $v$ of $G$ are said to be adjacent $(u \sim v)$ if there is an edge between $u$ and $v$. The degree of a vertex $v$ in $G$ is the number of edges incident on $v$ and it is denoted by $deg(v)$. We denote the maximum degree and the minimum degree of $G$ by $\Delta(G)$ and $\delta(G)$ respectively. $G$ is called regular if every vertex has an equal degree. $G$ is connected if there is a path between every two distinct vertices of $G$; otherwise, it is disconnected. $G$ is totally disconnected if no two vertices of $G$ are adjacent. For $x, y \in V(G)$, the length of the shortest path from $x$ to $y$ is denoted by $d(x, y)$ and the diameter of $G$ is $diam(G)$ $= sup\{d(x,\ y) \mid x, y \in V(G)\}$. The girth $gr(G)$ is defined as the length of the shortest cycle in $G$. We denote $gr(G) = \infty$ if $G$ contains no cycles. $G$ is said to be a complete graph if any two distinct vertices of $G$ are adjacent and we denote the complete graph with $n$ vertices by $K_n$. A complete bipartite graph is one whose vertices are partitioned into two disjoint sets $V_1$ and $V_2$ such that no two vertices of the same partite set are adjacent, but for every $x \in V_1$ and $y \in V_2$ are adjacent. We denote the complete bipartite graph on $m$ and $n$ vertices by $K_{m,n}$. $K_{1,n}$ is called a star graph. A circuit in a graph $G$ is a closed trail of length three or more. A circuit $C$ is called an Eulerian circuit if $C$ contains every edge of $G$. A connected graph $G$ is said to be Eulerian if it contains an Eulerian circuit. A connected graph $G$ is said to be Hamiltonian if it has a circuit that contains all the vertices of $G$. Two graphs $G$ and $H$ are said to be isomorphic to one another, written as

$G \cong H$, if there exists a bijection $f : V(G) \longrightarrow V(H)$ such that for each pair $u, v$ of vertices of $G$, $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. For undefined terminology and concept of graph theory, we refer to Diestel [9].

In this paper, we generalize the unit graph of a commutative ring to the unit graph of a commutative semiring under semiring theoretic settings. By following [13], we define the undirected unit graph $G(S)$ of $S$ by setting all the elements of $S$ to be the vertices and two distinct vertices $x$ and $y$ are adjacent in $G(S)$ if and only if $x+y$ is a unit of $S$. If we omit the word "distinct" in the definition, we obtain the closed unit graph of $S$, denoted by $\overline{G}(S)$, and this graph may have loops also.

The organization of this paper is as follows: in Section 2, we study and examine the properties of the unit graph $G(S)$ of $S$ such as the completeness, the bipartiteness, and the regularity for additive group $T$ in semiring $S$. We also prove that $G(R) \cong G(S)$ if $R \cong S$, and finally we discuss the connectedness of $G(S)$. In Section 3, we study and determine the diameter and the girth of the unit graph $G(S)$ of semiring $S$. Finally, we show that $G(S)$ is traversable under some conditions.

## 2. Some Basic Properties of $G(S)$

In this section, first we look at a relation between the unit graph $G(S)$ and the closed unit graph $\overline{G}(S)$ of a commutative semiring $S$ with unity.

**Lemma 2.1.** *Let $S$ be a commutative semiring with unity. Then $\overline{G}(S) = G(S)$ if and only if $2 \notin U(S)$.*

*Proof.* Assume that $\overline{G}(S) = G(S)$. Hence, $\overline{G}(S)$ has no loop at any $x \in S$. Note that $1 + 1 = 2$. Since $\overline{G}(S)$ has no loop at 1, this implies that $2 \notin U(S)$.

Conversely, let $2 \notin U(S)$, i.e. $1 + 1 = 2 \notin U(S)$. Then there is no loop at 1. Now, we will show that there is no loop at any $x \in S$. On the contrary, suppose that there is a loop at $x \in S$, then

$$x + x = 2x \in U(S).$$

Now, $2x \in U(S)$, then there exists an element $x' \in S$ such that $(2x)x' = 1 = x'(2x)$ and so $2(xx') = 1 = (xx')2$. Thus 2 is a unit in $S$, a contradiction. Therefore, there is no loop at any $x \in S$, and so $\overline{G}(S) = G(S)$. □

**Example 2.2.** Let $S = \mathbb{N} \cup \{0\}$. Then $(S, +, .)$ is a semiring, where $2 \notin U(S)$ and $G(S) = \overline{G}(S)$.

**Proposition 2.3.** *Let $S$ be a commutative semiring with unity. Then $G(S)$ is a complete graph if and only if $S$ is a semifield with $char(S) = 2$.*

*Proof.* Let $G(S)$ be a complete graph and let $x$ be any non-zero element of $S$. So, we have $x + 0 = 0 + x = x \in U(S)$, for all non-zero $x \in S$. Therefore, every non-zero element of $S$ has a multiplicative inverse. Thus, $S$ is a semifield and by Lemma 2.1, we have $char(S) = 2$.

Conversely, assume that $S$ is a semifield with $char(S) = 2$. Hence, each $x \in S$, $x + x = 0$, and so $x$ is the additive inverse of $x$ in $S$. Therefore, $(S, +)$ is an abelian group, and so $(S, +, .)$ is a field. Let $x, y \in S$ with $x \neq y$. Since $x$ is the additive inverse of $x$, it follows that $x + y \neq 0$, and so $x + y \in U(S)$. Therefore, $G(S)$ is complete. □

For the additive group $T$ in semiring $S$, we obtain the following generalization result for $G(S)$ from [3, Proposition 2.4].

**Proposition 2.4.** *Let $S$ be a finite commutative semiring with unity and $T$ be an additive group in $S$ with multiplicative identity. Then the following results hold for the unit graph $G(S)$:*

(1) *If $2 \notin U(T)$, then the unit graph $G(S)$ is $|U(T)|$-regular.*

(2) *If $2 \in U(T)$, then for every $x \in U(T)$ we have*

$$deg(x) = |U(T)| - 1,$$

*and for every $x \in \overline{U(T)}$ we have $deg(x) = |U(T)|$.*

*Proof.* For the proof of both (1) and (2), we assume that the vertex $x \in T$ is given. We have $T + x = T$, therefore, for every $u \in U(T)$, there exists an element $x_u \in T$ such that $x_u + x = u$. Clearly, $x_u$ is uniquely determined by $u$.

(1) Let $2 \notin U(T)$. Then $x_u \neq x$, therefore, $x_u$ is adjacent to $x$ in $G(S)$. Therefore, $f : U(T) \longrightarrow N_{G(S)}(x)$ given by $f(u) = x_u$ is a well-defined function. Now, it is easy to see that $f$ is a bijection and therefore, $\deg(x) = |N_{G(S)}(x)| = |U(T)|$, which yields that $G(S)$ is regular for every $x \in V(G(S))$. Thus we have $deg(x) = |U(T)|$.

(2) Let $2 \in U(T)$. Then we have the following two cases:

**Case 1.** If $x \in \overline{U(T)}$, then we have $x_u \neq x$, therefore, $x_u$ is adjacent to $x$ in $G(S)$. Thus, the above result (1) is still valid, which yields that $\deg(x) = |U(T)|$.

**Case 2.** If $x \in U(T)$, then $2x \in U(T)$, and we have $x_u \neq x$ for $u \neq 2x$, and so $x_{2x} = x$. Now, $x_u$ is adjacent to $x$ in $G(S)$ for $u \neq 2x$. Therefore, $f : U(T) \longrightarrow N_{G(S)}[x]$ given by $f(u) = x_u$, is a well-defined function. It is easy to see that $f$ is a bijection. Therefore,

$$\deg(x) = |N_{G(S)}[x]| - 1 = |U(T)| - 1.$$

□

We discuss the bipartiteness criterion of $G(S)$ in the following results.

**Proposition 2.5.** *Let $S$ be a semifield. Then $G(S)$ is a star graph if and only if $|S| \leq 3$.*

*Proof.* Let $G(S)$ be a star graph. Then there exists a vertex of degree one, and so $U(S)$ is finite and non-empty. Suppose that $|S| > 3$ and $G(S)$ is a tree, then every non-zero element $x$ of $S$ is adjacent to 0 since $S$ is a semifield. Again, for some $x, y \neq 0$ of $S$, we have $x + y \in U(S)$, which is a contradiction. This yields that $G(S)$ is a star graph if and only if $G(S)$ is either $K_{1,1}$ or $K_{1,2}$. Note that $G(S)$ is $K_{1,1}$ if and only if $|S| = 2$. If $|S| = 3$, then by Proposition 2.3, $G(S)$ is not a complete graph since $char(S) \neq 2$, and so it is $K_{1,2}$. This yields that $G(S)$ is a star graph if and only if $|S| \leq 3$. □

*Remark* 2.6. If semiring $S$ is a semifield, then $G(S)$ has pendant vertex if and only if $|S| \leq 3$. There are some more semirings that are not rings but have a pendant vertex in the unit graphs. For example $S = (P(X), \cup, \cap)$, where $X = \{a, b\}$ and $P(X)$ is a power set of $X$. Then it is easy to see that $deg(\phi) = 1$.

**Proposition 2.7.** *Let $S$ be a commutative non-local semiring with unity such that $|S/\mathfrak{m}| = 2$, where $\mathfrak{m}$ is a maximal $k$-ideal with maximal cardinality of semiring $S$. Then $G(S)$ is a bipartite graph.*

*Proof.* Let $S$ be a commutative non-local semiring with unity. Then $S$ has more than one maximal $k$-ideal. Let $\mathfrak{m}$ be a maximal $k$-ideal with maximal cardinality of semiring $S$. Then we can partition the vertex set of $G(S)$ as $V_1 = \mathfrak{m}$ and $V_2 = S \backslash \mathfrak{m}$. Now, we have $V(G(S)) = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$. Clearly, any two distinct elements of $V_1$ are not adjacent. To prove the Proposition, it is enough to show that no two elements of $V_2$ are adjacent. Let $a$ be a fixed element of $V_2$ and let $x, y$ be any two distinct elements of $V_2$. Now, by assumption $S = \mathfrak{m} \cup (\mathfrak{m} + a)$. Therefore, we can write $x = b_1 + a$ and $y = b_2 + a$, where $b_1, b_2 \in \mathfrak{m}$. This implies that $x + y = b_1 + b_2 + 2a$. If $x + y \in U(S)$, then $b_1 + b_2 + 2a \in U(S)$, which implies that $V_1$ has a unit, a contradiction. Therefore, any two distinct elements of $V_2$ are not adjacent, which yields that $G(S)$ is a bipartite graph. □

**Example 2.8.** (1) For $S = (\mathbb{Z}_{10}, +, .)$ semiring, $\mathfrak{m}_1 = \{0, 2, 4, 6, 8\}$ and $\mathfrak{m}_2 = \{0, 5\}$ are two maximal ideals of $S$. Therefore, $\mathfrak{m}_1$ and $S \setminus \mathfrak{m}_1$ are two partite sets of $G(S)$; moreover $G(S)$ is a 4-regular bipartite graph.

(2) An inspection will shows that the set $SP_4 = \{0, 1, 2, b\}$ equipped with operations $+$ and . defined by:

| $+$ | 0 | 1 | 2 | $b$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | $b$ |
| 1 | 1 | 2 | 1 | 2 |
| 2 | 2 | 1 | 2 | 1 |
| $b$ | $b$ | 2 | 1 | 0 |

| . | 0 | 1 | 2 | $b$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | $b$ |
| 2 | 0 | 2 | 2 | 0 |
| $b$ | 0 | $b$ | 0 | $b$ |

is a semiring (which is not a ring) with unity. Here, $\mathfrak{m}_1 = \{0, 2\}$ and $\mathfrak{m}_2 = \{0, b\}$ are two maximal $k$-ideals of $SP_4$ and so $\mathfrak{m}_2$ and $S \setminus \mathfrak{m}_2$ are two partite sets of $G(SP_4)$; moreover $G(SP_4)$ is a tree.

**Proposition 2.9.** *Let $S$ be a commutative local semiring with unity. Then $G(S)$ is a complete bipartite graph if and only if either $(S, \mathfrak{m} \neq 0)$ or $|S| \leq 3$.*

*Proof.* Let $G(S)$ be a complete bipartite graph. If $G(S)$ is a tree, then by Proposition 2.5, $|S| \leq 3$. Now, we assume that $G(S)$ is not a tree, and let $V_1, V_2$ be two partite sets of $G(S)$. Without loss of generality, we can assume that $0 \in V_1$. Let $u \in U(S)$. Note that $0+u \in U(S)$. Hence, 0 and $u$ are adjacent in $G(S)$. As $0 \in V_1$, it follows that $u \in V_2$. Let $s \in V_2$. Since $G(S)$ is a complete bipartite with partite sets $V_1$ and $V_2$, 0 and $s$ are adjacent in $G(S)$. Therefore, $s = 0 + s \in U(S)$. The above arguments imply that $V_2 = U(S)$. Thus $S = \overline{U(S)} \cup U(S) = V_1 \cup V_2$, and so it follows that $V_1 = \overline{U(S)} = \mathfrak{m}$. This yields that $S$ has a unique maximal ideal $\mathfrak{m}$. Therefore, $S$ is a local semiring.

Conversely, let semiring $S$ be either $(S, \mathfrak{m} \neq 0)$ or $|S| \leq 3$. If $|S| \leq 3$, then the result holds from the Proposition 2.5. Now, we assume that $\mathfrak{m} = \overline{U(S)} \neq 0$ is a unique maximal $k$-ideal of $S$, and so we obtain $V_1 = \overline{U(S)}$ and $V_2 = U(S)$ as partite sets of $G(S)$. Let $x \in V_1$ and $y \in V_2$ be given. If $x + y \notin U(S)$, a contradiction. Therefore, $x + y \in U(S)$, which yields that $x$ and $y$ are adjacent and each vertex of $V_1$ is joined to every vertex of $V_2$. Therefore, $G(S)$ is a complete bipartite graph. □

**Proposition 2.10.** *Let $R$ and $S$ be two commutative semirings with unity. If $R \cong S$, then $G(R) \cong G(S)$.*

*Proof.* Let $R \cong S$, then clearly $|R| = |S|$. Thus, for $G(R)$ and $G(S)$ we have $|V(G(R))| = |V(G(S))|$. Now to prove that the adjacency of vertices are also preserved. First we shall show that image of a unit is also a unit under isomorphism between $R$ and $S$. Let $f : R \longrightarrow S$ be an isomorphism of semirings. For any $r \in R$, we denote $f(r)$ by $r_S$. Let $x$ be a unit of $R$. Then $xy = 1 = yx$ for some $y \in R \setminus \{0\}$. Therefore, $f(xy) = f(1)$ and so $f(x)f(y) = f(1)$. Thus $x_S\, y_S = 1_S$, where $1_S$ is a unity of $S$. This shows that $x_S \in U(S)$ and $f(U(R)) = U(S)$.

Now, to check the edges, let $x, y \in R$ be such that $xy$ is an edge of $G(R)$. Then $x + y \in U(R)$, and so for $f(x), f(y) \in S$, we have $f(x) + f(y) = f(x + y) \in U(S)$, which yields that adjacency of the vertices are preserved. Therefore, $G(R) \cong G(S)$. □

We discuss the connectedness property of $G(S)$ in the following results.

**Proposition 2.11.** *Let $S$ be a commutative semiring without unity and let $|S| \geq 2$. Then $G(S)$ is totally disconnected.*

*Proof.* By hypothesis, the semiring $S$ has no unity. Hence, $U(S) = \phi$. Therefore, $G(S)$ has no edges, and so it follows that $G(S)$ is totally disconnected. □

**Example 2.12.** Consider the set $S = \{0, 1\}$. On $S$ we define the operations as follows: $0 + 0 = 1 + 1 = 0$, $1 + 0 = 0 + 1 = 1$ and $0.0 = 0.1 = 1.0 = 1.1 = 0$. Then $(S, +, .)$ forms a commutative semiring without unity and so $G(S)$ is totally disconnected.

**Proposition 2.13.** *Let $S$ be a commutative semiring with unity. If $\overline{U(S)}$ is a $k$-ideal, then $G(S)$ is connected.*

*Proof.* Let $S$ be a commutative semiring with unity, and let $\overline{U(S)}$ be a $k$-ideal of $S$. Then $\overline{U(S)}$ is a unique maximal $k$-ideal of semiring $S$, and so $J(S) = \overline{U(S)}$. Therefore, for any $x \in \overline{U(S)}$ and $y \in U(S)$, we have $x + y \in U(S)$. Therefore, $G(S)$ is connected. □

**Proposition 2.14.** [4] *Let $S$ be a semiring. Then $S$ is a local semiring if and only if $\overline{U(S)}$ is a $k$-ideal.*

From Propositions 2.13 and 2.14, we can easily conclude the following result:

**Corollary 2.15.** *Let $S$ be a local semiring with unity. Then $G(S)$ is always connected.*

## 3. Diameter, Girth and Traversability of $G(S)$

In this section, first we study and determine the diameter of the unit graph $G(S)$ for local semiring $S$ with unity.

**Proposition 3.1.** *Let $S$ be a commutative semiring with unity. If $S$ is a semifield with $char(S) = 2$, then $diam(G(S)) = 1$.*

*Proof.* Let $S$ be a semifield with $char(S) = 2$. Then by Proposition 2.3, $G(S)$ is a complete graph. This yields that $diam(G(S)) = 1$. □

**Proposition 3.2.** *Let $S$ be a commutative local semiring with unity. If $|S| \geq 3$ and $char(S) \neq 2$, then $diam(G(S)) = 2$.*

*Proof.* By hypothesis, $S$ is a commutative local semiring with unity. Hence, $\overline{U(S)}$ is a unique maximal $k$-ideal. Let $x \in U(S)$ and $y \in \overline{U(S)}$. As $\overline{U(S)}$ is a $k$-ideal of $S$, it follows that $x + y \in U(S)$, and so $xy$ is an edge of $G(S)$. This shows that $G(S)$ is a connected graph with $diam(G(S)) \leq 2$.

Now, the following two cases arise:

**Case 1:** We assume that $S$ is not a semifield. Therefore, there exists $x \in S \setminus \{0\}$ such that $x \in \overline{U(S)}$. Note that $x + 0 \in \overline{U(S)}$ and hence, $x$ and 0 are not adjacent in $G(S)$. Therefore, $diam(G(S)) \geq 2$, and so $diam(G(S)) = 2$.

**Case 2:** We assume that $S$ is a semifield. Now by hypothesis, $|S| \geq 3$ and $char(S) \neq 2$. It follows from the Proposition 2.3 that $G(S)$ is not a complete graph. Hence, $diam(G(S)) \geq 2$, and so $diam(G(S)) = 2$. □

We discuss the diameter of unit graph $G(S)$ for non-local semiring $S$ with unity in the next result.

**Proposition 3.3.** *Let $S$ be a non-local commutative semiring with unity. Then $diam(G(S)) \in \{2, 3, \infty\}$.*

*Proof.* If $G(S)$ is disconnected, then $diam(G(S)) = \infty$. Let $S$ be a non-local semiring with unity, and so there exist more than one maximal $k$-ideals. Let $I_1, ..., I_n$ be non-trivial maximal $k$-ideals of $S$. Then $\overline{U(S)} = I_1 \cup I_2 \cup ... \cup I_n$ and $\overline{U(S)}$ is not a $k$-ideal. Next, we assume that $x, y \neq 0 \in \overline{U(S)}$ such that $x + y \in U(S)$. Again, let $z \in U(S)$ such that $y + z \in U(S)$. Then $diam(G(S)) \leq 3$. If $x + z \in U(S)$, then there exists a path $x - z - 0$ in $G(S)$. If $x + z \notin U(S)$, then there exists a path $x - y - z - 0$ in $G(S)$. Since $S$ is a non-local commutative semiring, and so $S$ is not a semifield with $char(S) = 2$. Therefore, by Proposition 2.3, $G(S)$ is not a complete graph, and so $diam(G(S)) \neq 1$. Hence, the result follows. □

**Proposition 3.4.** *Let $S$ be a commutative semiring with unity. Then $diam(G(S)) \in \{1, 2, 3, \infty\}$.*

*Proof.* The proof follows by Propositions 3.1, 3.2 and 3.3. □

In the following result, we study and determine the girth of $G(S)$ for local semiring $S$ with unity.

**Proposition 3.5.** *Let $S$ be a commutative local semiring with unity. Then $gr(G(S)) \in \{3, 4, \infty\}$.*

*Proof.* If $|S| \leq 3$, then characteristic of $S$ is either 2 or 3. If $char(S) = 2$, then it is easy to see that $G(S)$ is $K_2$. If $char(S) = 3$, then $G(S)$ is not a complete graph by the Proposition 2.3, which shows that $G(S)$ has no cycle. Therefore, let $G(S)$ has a cycle and $|S| \geq 4$, then $|U(S)| \geq 2$. If $|S| = 4$ and $S$ is not a semifield, then $U(S) = \{1, u\}$ and $u^2 = 1$, and so there exists a cycle $0 \longrightarrow 1 \longrightarrow x \longrightarrow u \longrightarrow 0$ of shortest length 4 in $G(S)$. Again, if $S$ is a local semiring with $\mathfrak{m} \neq 0$, then $G(S)$ is a complete bipartite graph by Proposition 2.9. Therefore, $gr(G(S)) = 4$. If $S$ is a semifield and $G(S)$ contains a cycle, then for some $x, y \in S \setminus \{0\}$, there exists a cycle $0 \longrightarrow x \longrightarrow y \longrightarrow 0$ of shortest length 3. Therefore, $gr(G(S)) \in \{3, 4, \infty\}$. □

**Proposition 3.6.** *Let $S$ be a finite commutative semiring with unity and $T$ be an additive group in $S$ with multiplicative identity. Suppose $2 \notin U(T)$. Then $G(S)$ is Eulerian if and only if $|U(T)|$ is even.*

*Proof.* Let $T$ be an additive group in $S$ with multiplicative identity and $2 \notin U(T)$. Let $G(S)$ be Eulerian. Then by Proposition 2.4, $G(S)$ is $|U(T)|$-regular. Therefore, $|U(T)|$ is even.

Conversely, let $|U(T)|$ be even and $2 \notin U(T)$. Then by Proposition 2.4, $G(S)$ is $|U(T)|$-regular graph, which yields that $G(S)$ is Eulerian. □

In order to prove the existence of Hamiltonian in unit graph $G(S)$, we recall the following Theorem.

**Theorem 3.7.** [9, Ore] *Let $G$ be a graph of order $n \geq 3$ and for every pair $u$ and $v$ of nonadjacent vertices, $deg(u) + deg(v) \geq n$, then $G$ is Hamiltonian.*

**Proposition 3.8.** *Let $S$ be a commutative local semiring with unity. If $|S| \geq 4$ with $|\overline{U(S)}| = |U(S)|$, then $G(S)$ is Hamiltonian.*

*Proof.* Let $S$ be a commutative local semiring with unity. Then $\overline{U(S)}$ is a unique maximal $k$-ideal of $S$, and so $J(S) = \overline{U(S)}$. Therefore, for each $x \in \overline{U(S)}$ and $y \in U(S)$, we have $x + y \in U(S)$. Since $|\overline{U(S)}| = |U(S)|$, and so for $x \in S$, we have $deg(x) = |\overline{U(S)}| = |U(S)|$. Thus for any two non-adjacent vertices $x$ and $y$ in $G(S)$, we have $deg(x) + deg(y) = |S|$. This yields that $G(S)$ is Hamiltonian by Theorem 3.7. □

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**Laithun Boro**
Department of Mathematics, North-Eastern Hill University P.O. Box 793022, Shillong, India.
laithunb@gmail.com

**Madan Mohan Singh**
Department of Basic Sciences & Social Sciences, North-Eastern Hill University, P.O. Box 793022, Shillong, India.
mmsingh2004@gmail.com

**Jituparna Goswami**
Department of Mathematics, Gauhati University, P.O. Box 781014, Guwahati, India.
jituparnagoswami18@gmail.com

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THE UNIT GRAPH OF A COMMUTATIVE SEMIRING

L. BORO, M. M. SINGH AND J. GOSWAMI

گراف یکه نیم‌حلقه‌ی جابه‌جایی

لیتون بورو[۱]، مادان موهان سینگ[۲]، و جیتوپارنا گوسوامی[۳]

[۱]گروه ریاضیات، دانشگاه شمال شرقی هیل، شیلونگ، هند

[۲]گروه علوم پایه و علوم اجتماعی، دانشگاه شمال شرقی هیل، شیلونگ، هند

[۳]گروه ریاضیات، دانشگاه گاوهاتی، گواهاتی، هند

فرض می‌کنیم $S$ نیم‌حلقه‌ی جابه‌جایی یکدار و $U(S)$ مجموعه‌ی همه‌ی عناصر یکه آن باشد. گراف یکه $S$ که با $G(S)$ نشان داده می‌شود، گرافی غیرجهتی است که مجموعه‌ی رئوس آن $S$ می‌باشد و دو رأس متمایز $x$ و $y$ در آن مجاورند هرگاه $x + y \in U(S)$. در این مقاله، ما به مطالعه‌ی گراف یکه $G(S)$ می‌پردازیم و برخی خواص این گراف مانند کامل بودن، دوبخشی بودن، همبندی، قطر و کمر را بررسی می‌کنیم. همچنین، شرطی لازم و کافی ارائه می‌دهیم که تحت آن، $G(S)$ گراف قابل پیمایش است.

کلمات کلیدی: گراف یکه، همبندی، قطر، کمر، قابل پیمایش.