Journal of Algebraic Systems
Vol. 12, No. 1, (2024), pp 163-177

ON (α,τ)-P-DERIVATIONS OF NEAR-RINGS

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Abstract. The relationship between derivations and algebraic structures of quotient near-rings has become a fascinating topic in modern algebra in recent decades. Assume $\mathcal{N}$ is a near-ring and P is its prime ideal. In this paper we introduce the notion of (α,τ)-P-derivation in near-rings. Also, we study the structure of the quotient near-rings $\mathcal{N}/P$ that satisfies certain algebraic identities involving (α,τ)-P-derivation.

1. Introduction

Throughout the paper, $\mathcal{N}$ will denote a left near-ring with multiplicative center $Z(\mathcal{N})$ and additive center $C(\mathcal{N})$. A near-ring $\mathcal{N}$ is said to be zero-symmetric if $0x = 0$ for all $x \in \mathcal{N}$ (recall that a left distributivity in $\mathcal{N}$ yields that $x0 = 0$). Also, $\mathcal{N}$ is said to be 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in \mathcal{N}$. Recall that $\mathcal{N}$ is called a 3-prime near-ring, if for $x, y \in \mathcal{N}$, $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. For all $x, y \in \mathcal{N}$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$ shall denote the Lie product and the Jordan product, respectively. The symbol (x, y) will denote the additive-group commutator $x+y-x-y$. A normal subgroup P of $(\mathcal{N},+)$ is called a left ideal (resp. a right ideal) if $P\mathcal{N} \subseteq P$ (resp. $(x + p)y - xy \in P$ for all $x, y \in \mathcal{N}$ and $p \in P$), and if P is both a left ideal and a right ideal, then P is said to be an ideal of $\mathcal{N}$. Following Groenewald [4]; an ideal P is a 3-prime

DOI: 10.22044/JAS.2023.12352.1661.
MSC(2010): Primary: 16U60; Secondary: 16W25.
Keywords: Near-rings; Prime ideals; (α,τ)-derivations; Commutativity.
Received: 20 October 2022, Accepted: 4 January 2023.
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if for $a, b \in \mathcal{N}$, $a\mathcal{N}b \subseteq P \Rightarrow a \in P$ or $b \in P$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is a (α, τ)-derivation if there exist automorphisms $\alpha, \tau : \mathcal{N} \to \mathcal{N}$ such that $d(xy) = \tau(x)d(y) + d(x)\alpha(y)$ for all $x, y \in \mathcal{N}$, or equivalently, as noted in [1], such that $d(xy) = d(x)\alpha(y) + \tau(x)d(y)$ for all $x, y \in \mathcal{N}$. A mapping $d : \mathcal{N} \to \mathcal{N}$ is said P-additive if $d(x+y) - (d(x)+d(y)) \in P$ for all $x, y \in \mathcal{N}$. A mapping $d : \mathcal{N} \to \mathcal{N}$ is P-trivial if $d(\mathcal{N}) \subseteq P$. An element x of $\mathcal{N}$ for which $d(x) \in P$ is called P-constants. A mapping $d : \mathcal{N} \to \mathcal{N}$ will be called (α, τ)-P-commuting if $[d(x), x]_{\alpha,\tau} = d(x)\alpha(x) - \tau(x)d(x) \in P$ for all $x \in \mathcal{N}$.

Many results in the literature indicate how the global structure of a near-ring $\mathcal{N}$ is often tightly connected to the behavior of derivations defined on $\mathcal{N}$. Recently, a series of more general concepts of derivations have been introduced and studied on near-rings (see, for example, [3], [5], and [6]). In the following, we define the notion of (α, τ)-P-derivation in near-rings, which generalizes the notion of (α, τ)-derivation, and we enrich this definition by an example, which justifies the existence of this type of application:

Definition 1.1. Let $\mathcal{N}$ be a near-ring and P be a subgroup of $(\mathcal{N}, +)$. A P-additive mapping $d : \mathcal{N} \to \mathcal{N}$ is a (α, τ)-P-derivation of $\mathcal{N}$, if there exist maps $\alpha, \tau : \mathcal{N} \to \mathcal{N}$ such that $d(xy) - (\tau(x)d(y) + d(x)\alpha(y)) \in P$ for all $x, y \in \mathcal{N}$.

Definition 1.2. Let $\mathcal{N}$ be a near-ring and P be a subgroup of $(\mathcal{N}, +)$. A P-additive mapping $d : \mathcal{N} \to \mathcal{N}$ is a (α, τ)-P^+-derivation of $\mathcal{N}$, if d is a (α, τ)-P-derivation such that

(**a**) $d\left(d(xy) - (\tau(x)d(y) + d(x)\alpha(y))\right) \in P$ for all $x, y \in \mathcal{N}$.
(**b**) $d\left(d(xy) - (d(x)\alpha(y) + \tau(x)d(y))\right) \in P$ for all $x, y \in \mathcal{N}$.

Definition 1.3. Let $\mathcal{N}$ be a near-ring. A normal subgroup P of $(\mathcal{N}, +)$ is called a symmetric ideal if

(**a**) P is an ideal of $\mathcal{N}$.
(**b**) $\mathcal{N}P \subseteq P$.

When $P = \{0\}$ is the symmetric ideal of a near-ring $\mathcal{N}$, we get the concept of a zero-symmetric near-ring $\mathcal{N}$.

Definition 1.4. A near-ring $\mathcal{N}$ is called symmetric if each ideal of $\mathcal{N}$ is symmetric.

It is easy to see that any (α, τ)-derivation on $\mathcal{N}$ is a (α, τ)-P-derivation on $\mathcal{N}$. The following example justifies the existence of a (α, τ)-P-derivation, which is not a (α, τ)-derivation:

Example 1.5. Let S be a left near-ring. Define $\mathcal{N}$, P by

$$\mathcal{N}=\left\{\begin{pmatrix} 0 & a & b\\ 0 & 0 & 0\\ 0 & c & 0\end{pmatrix} \mid a,b,c,0\in S\right\},\ P=\left\{\begin{pmatrix} 0 & u & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \mid 0,u\in S\right\},$$

then $\mathcal{N}$ is a left near-ring, and P is an ideal of $\mathcal{N}$.
Let us define d,α, and $\tau:\mathcal{N}\longrightarrow\mathcal{N}$ as follow:

$$d\begin{pmatrix} 0 & a & b\\ 0 & 0 & 0\\ 0 & c & 0\end{pmatrix}=\begin{pmatrix} 0 & a & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\alpha\begin{pmatrix} 0 & a & b\\ 0 & 0 & 0\\ 0 & c & 0\end{pmatrix}=\begin{pmatrix} 0 & a & 0\\ 0 & 0 & 0\\ 0 & c & 0\end{pmatrix}$$

and

$$\tau\begin{pmatrix} 0 & a & b\\ 0 & 0 & 0\\ 0 & c & 0\end{pmatrix}=\begin{pmatrix} 0 & a & b\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

It is clear to see that d is a (α,τ)-P^{+}-derivation, but not a (α,τ)-derivation on $\mathcal{N}$.

With these definitions, by using (α,τ)-P-derivations, where $\alpha,\tau:\mathcal{N}\to\mathcal{N}$ are two automorphisms and P is an ideal of a near-ring $\mathcal{N}$, we will investigate properties of the near-ring $\mathcal{N}/P$. The originality in this work is that we use a (α,τ)-P-derivation on $\mathcal{N}$ (and not on $\mathcal{N}/P$), which satisfies some algebraic identities on $\mathcal{N}$ and on P, without the primeness (semi-primeness) assumption on the considered near-ring.

2. Some preliminaries

Lemma 2.1. *Let $\mathcal{N}$ be a near-ring and P be an ideal of $\mathcal{N}$. If*

$$d:\mathcal{N}\to\mathcal{N}$$

is a (α,τ)-P-derivation of $\mathcal{N}$ such that $d(P)\subseteq P,\alpha(P)\subseteq P,\tau(P)\subseteq P$, then the mapping $\widetilde{d}:\mathcal{N}/P\to\mathcal{N}/P$ defined by $\widetilde{d}(\overline{x})=\overline{d(x)}$ is a $(\widetilde{\alpha},\widetilde{\tau})$-derivation on $\mathcal{N}/P$, where $\widetilde{\alpha}(\overline{x})=\overline{\alpha(x)}$ and $\widetilde{\tau}(\overline{x})=\overline{\tau(x)}$.

Proof. $\widetilde{d}$ is well defined. Indeed, let $y\in\overline{x}$, then $y-x=p$ for some $p\in P$, thus $d(y)=d(x)+d(p)$, hence $\widetilde{d}(\overline{y})=\overline{d(y)}=\overline{d(x)}=\widetilde{d}(\overline{x})$.

Now, let $\overline{x},\overline{y}\in\mathcal{N}/P$ we have

$$\begin{aligned}\widetilde{d}(\overline{xy})=\overline{d(xy)}&=\overline{(\tau(x)d(y)+d(x)\alpha(y))}\\&=\overline{\tau(x)d(y)}+\overline{d(x)\alpha(y)}\\&=\widetilde{\tau}(\overline{x})\widetilde{d}(\overline{y})+\widetilde{d}(\overline{x})\widetilde{\alpha}(\overline{y}),\end{aligned}$$

which completes the proof. ∎

Lemma 2.2. *Let $\mathcal{N}$ be a near-ring and P be an ideal of $\mathcal{N}$. A P-additive endomorphism d on a near-ring $\mathcal{N}$ is a (α,τ)-P-derivation if and only if $d(xy)-(d(x)\alpha(y)+\tau(x)d(y)) \in P$ for all $x, y \in \mathcal{N}$.*

Proof. Suppose that $d(xy)-(\tau(x)d(y)+d(x)\alpha(y)) \in P$, for all $x, y \in \mathcal{N}$. Since $x(y+y)=xy+xy$, it follows that

$$\overline{d(x(y+y))} = \overline{\tau(x)d(y+y)} + \overline{d(x)}(\overline{\alpha(y)}+\overline{\alpha(y)}) \tag{2.1}$$
$$= \overline{\tau(x)d(y)} + \overline{\tau(x)d(y)} + \overline{d(x)\alpha(y)} + \overline{d(x)\alpha(y)} \tag{2.2}$$

for all $x, y \in \mathcal{N}$. Also,

$$\overline{d(xy+xy)} = \overline{d(xy)} + \overline{d(xy)}$$
$$= \overline{\tau(x)d(y)} + \overline{d(x)\alpha(y)} + \overline{\tau(x)d(y)} + \overline{d(x)\alpha(y)} \tag{2.3}$$

for all $x, y \in \mathcal{N}$. Combining (2.2) and (2.3), we get

$$\overline{\tau(x)d(y)} + \overline{d(x)\alpha(y)} = \overline{d(x)\alpha(y)} + \overline{\tau(x)d(y)}$$

for all $x, y \in \mathcal{N}$. Hence, $d(xy)-(d(x)\alpha(y)+\tau(x)d(y)) \in P$ for all $x, y \in \mathcal{N}$.
For the converse suppose that $d(xy)-(d(x)\alpha(y)+\tau(x)d(y)) \in P$ for all $x, y \in \mathcal{N}$. We have

$$\overline{d(x(y+y))} = \overline{d(x)\alpha(y+y)} + \overline{\tau(x)d(y+y)}$$
$$= \overline{d(x)\alpha(y)} + \overline{d(x)\alpha(y)} + \overline{\tau(x)d(y)} + \overline{\tau(x)d(y)} \tag{2.4}$$

for all $x, y \in \mathcal{N}$. On the other hand,

$$\overline{d(xy+xy)} = \overline{d(xy)} + \overline{d(xy)}$$
$$= \overline{d(x)\alpha(y)} + \overline{\tau(x)d(y)} + \overline{d(x)\alpha(y)} + \overline{\tau(x)d(y)} \tag{2.5}$$

for all $x, y \in \mathcal{N}$. From (2.4) and (2.5), we get

$$\overline{d(x)\alpha(y)} + \overline{\tau(x)d(y)} = \overline{\tau(x)d(y)} + \overline{d(x)\alpha(y)}$$

for all $x, y \in \mathcal{N}$. Thus, $d(xy)-(\tau(x)d(y)+d(x)\alpha(y)) \in P$ for all $x, y \in \mathcal{N}$. Hence, d is a (α,τ)-P-derivation. □

Lemma 2.3. *Let $\mathcal{N}$ be a near-ring, P an ideal of $\mathcal{N}$ and d be an arbitrary (α,τ)-P-derivation of $\mathcal{N}$. Then, $\mathcal{N}/P$ satisfies the following partial distributive laws:*

(a) $\left(\overline{\tau(x)d(y)} + \overline{d(x)\alpha(y)}\right)\overline{\alpha(z)} = \overline{\tau(x)d(y)\alpha(z)} + \overline{d(x)\alpha(y)\alpha(z)}$ *for all $x, y, z \in \mathcal{N}$.*

(b) $\left(\overline{d(x)\alpha(y)} + \overline{\tau(x)d(y)}\right)\overline{\alpha(z)} = \overline{d(x)\alpha(y)\alpha(z)} + \overline{\tau(x)d(y)\alpha(z)}$ *for all $x, y, z \in \mathcal{N}$.*

Proof. (**a**) We know that $\overline{d(xy)} = \overline{\tau(x)d(y)} + \overline{d(x)\alpha(y)}$ for all $x, y \in \mathcal{N}$. So,

$$\begin{aligned} \overline{d((xy)z)} &= \overline{\tau(x)\tau(y)d(z)} + \overline{d(xy)\alpha(z)} \\ &= \overline{\tau(x)\tau(y)d(z)} + \left(\overline{\tau(x)d(y)} + \overline{d(x)\alpha(y)}\right)\overline{\alpha(z)} \end{aligned} \tag{2.6}$$

for all $x, y, z \in \mathcal{N}$. Also,

$$\begin{aligned} \overline{d(x(yz))} &= \overline{\tau(x)d(yz)} + \overline{d(x)\alpha(y)\alpha(z)} \\ &= \overline{\tau(x)}\left(\overline{\tau(y)d(z)} + \overline{d(y)\alpha(z)}\right) + \overline{d(x)\alpha(y)\alpha(z)} \\ &= \overline{\tau(x)\tau(y)d(z)} + \overline{\tau(x)d(y)\alpha(z)} + \overline{d(x)\alpha(y)\alpha(z)} \end{aligned} \tag{2.7}$$

for all $x, y, z \in \mathcal{N}$. From (2.6) and (2.7), we get

$$\begin{aligned} &\overline{\tau(x)\tau(y)d(z)} + \left(\overline{\tau(x)d(y)} + \overline{d(x)\alpha(y)}\right)\overline{\alpha(z)} \\ &= \overline{\tau(x)\tau(y)d(z)} + \overline{\tau(x)d(y)\alpha(z)} + \overline{d(x)\alpha(y)\alpha(z)} \end{aligned}$$

for all $x, y, z \in \mathcal{N}$, i.e.,

$$\left(\overline{\tau(x)d(y)} + \overline{d(x)\alpha(y)}\right)\overline{\alpha(z)} = \overline{\tau(x)d(y)\alpha(z)} + \overline{d(x)\alpha(y)\alpha(z)}$$

for all $x, y, z \in \mathcal{N}$.

(**b**) We have $\overline{d(xy)} = \overline{d(x)\alpha(y)} + \overline{\tau(x)d(y)}$ for all $x, y \in \mathcal{N}$. Then,

$$\begin{aligned} \overline{d(x(yz))} &= \overline{d(x)\alpha(y)\alpha(z)} + \overline{\tau(x)d(yz)} \\ &= \overline{d(x)\alpha(y)\alpha(z)} + \overline{\tau(x)}\left(\overline{d(y)\alpha(z)} + \overline{\tau(y)d(z)}\right) \\ &= \overline{d(x)\alpha(y)\alpha(z)} + \overline{\tau(x)d(y)\alpha(z)} + \overline{\tau(x)\tau(y)d(z)} \end{aligned} \tag{2.8}$$

for all $x, y, z \in \mathcal{N}$. On the other side,

$$\begin{aligned} \overline{d((xy)z)} &= \overline{d(xy)\alpha(z)} + \overline{\tau(x)\tau(y)d(z)} \\ &= \left(\overline{d(x)\alpha(y)} + \overline{\tau(x)d(y)}\right)\overline{\alpha(z)} + \overline{\tau(x)\tau(y)d(z)} \\ &= \left(\overline{d(x)\alpha(y)} + \overline{\tau(x)d(y)}\right)\overline{\alpha(z)} + \overline{\tau(x)\tau(y)d(z)} \end{aligned} \tag{2.9}$$

for all $x, y, z \in \mathcal{N}$. Combining (2.8) and (2.9), we obtain

$$\begin{aligned} &\overline{d(x)\alpha(y)\alpha(z)} + \overline{\tau(x)d(y)\alpha(z)} + \overline{\tau(x)\tau(y)d(z)} \\ &= \left(\overline{d(x)\alpha(y)} + \overline{\tau(x)d(y)}\right)\overline{\alpha(z)} + \overline{\tau(x)\tau(y)d(z)} \end{aligned}$$

for all $x, y, z \in \mathcal{N}$. Thus, we have

$$\left(\overline{d(x)\alpha(y)} + \overline{\tau(x)d(y)}\right)\overline{\alpha(z)} = \overline{d(x)\alpha(y)\alpha(z)} + \overline{\tau(x)d(y)\alpha(z)}$$

for all $x, y, z \in \mathcal{N}$. ∎

Lemma 2.4. *Let P be a 3-prime ideal of a near-ring $\mathcal{N}$.*

(**a**) *If $\overline{z} \in Z(\mathcal{N}/P) \setminus \{\overline{0}\}$, then $\overline{z}$ is not a zero divisor.*
(**b**) *If $Z(\mathcal{N}/P)$ contains a nonzero element $\overline{z}$ for which $\overline{z} + \overline{z} \in Z(\mathcal{N}/P)$, then $(\mathcal{N}/P, +)$ is abelian.*
(**c**) *If $\overline{z} \in Z(\mathcal{N}/P) \setminus \{\overline{0}\}$ and $\overline{x} \in \mathcal{N}/P$ such that $\overline{xz} \in Z(\mathcal{N}/P)$, then $\overline{x} \in Z(\mathcal{N}/P)$.*

Proof. By hypothesis we have P is a 3-prime ideal of $\mathcal{N}$. Thus $\{\overline{0}\}$ is a 3-prime ideal of $\mathcal{N}/P$. Therefore, (**a**), (**b**) and (**c**) are consequences of [2, Lemma 1.2 & Lemma 1.3]. □

Lemma 2.5. *Let P be a 3-prime ideal of a near-ring $\mathcal{N}$ and d be a non P-trivial (α, τ)-P-derivation on $\mathcal{N}$. Then*

(**a**) $\overline{x}\overline{d(\mathcal{N})} = \{\overline{0}\}$ *implies* $\overline{x} = \overline{0}$,
(**b**) $\overline{d(\mathcal{N})\alpha(x)} = \{\overline{0}\}$ *implies* $\overline{\alpha(x)} = \overline{0}$.

Proof. (**a**) Suppose that $\overline{x}\overline{d(\mathcal{N})} = \{\overline{0}\}$. Then,

$$\overline{0} = \overline{x}\overline{d(yz)} = \overline{x}\overline{d(y)\alpha(z)} + \overline{x}\overline{\tau(y)d(z)} = \overline{x}\overline{\tau(y)d(z)}$$

for all $y, z \in \mathcal{N}$. That is, $x\mathcal{N}d(z) \subseteq P$ for all $z \in \mathcal{N}$. By the 3-primeness of P, we have $\overline{0} = \overline{x}$ or $\overline{0} = \overline{d(z)}$ for all $z \in \mathcal{N}$. Since $d(\mathcal{N}) \nsubseteq P$, we conclude that $\overline{0} = \overline{x}$.
(**b**) A similar argument works if $\overline{d(\mathcal{N})\alpha(x)} = \{\overline{0}\}$. □

Lemma 2.6. *Let P be a 3-prime ideal of a near-ring $\mathcal{N}$ and d be a (α, τ)-P^+-derivation of $\mathcal{N}$ such that $\overline{\alpha d} = \overline{d\alpha}$ and $\overline{\tau d} = \overline{d\tau}$. If $d^2(\mathcal{N}) \subseteq P$, then, $d(\mathcal{N}) \subseteq P$ or $2(\mathcal{N}/P) = \{\overline{0}\}$.*

Proof. Assume that $d(\mathcal{N}) \nsubseteq P$. By the hypothesis, we have

$$\begin{aligned}
\overline{0} &= \overline{d^2(xy)} \\
&= \overline{d(d(x)\alpha(y) + \tau(x)d(y))} \\
&= \overline{d^2(x)\alpha^2(y)} + \overline{\tau(d(x))d(\alpha(y))} + \overline{d(\tau(x))\alpha(d(y))} + \overline{\tau^2(x)d^2(y)} \\
&= \overline{d(\tau(x))(2d(\alpha(y)))}.
\end{aligned}$$

This implies that $\overline{d(\mathcal{N})\alpha(d(2y))} = \{\overline{0}\}$ for all $y \in \mathcal{N}$. By using Lemma 2.5 (**b**), we get $\overline{d(2y)} = \overline{0}$ for all $y \in \mathcal{N}$. Hence,

$$\begin{aligned}
\overline{0} &= \overline{d(2xy)} \\
&= \overline{d(xy)} + \overline{d(xy)} \\
&= \overline{d(x)\alpha(y)} + \overline{\tau(x)d(y)} + \overline{\tau(x)d(y)} + \overline{d(x)\alpha(y)} \\
&= \overline{d(x)\alpha(y)} + \overline{\tau(x)d(2\alpha(y))} + \overline{d(x)\alpha(y)}
\end{aligned}$$

$$= \overline{d(x)\alpha(y)} + \overline{d(x)\alpha(y)}$$
$$= \overline{d(x)\alpha(2y)} \text{ for all } x, y \in \mathcal{N},$$

that is, $\overline{d(\mathcal{N})\alpha(2y)} = \{\overline{0}\}$ for all $y \in \mathcal{N}$. By Lemma 2.5 (**b**), we get $2(\mathcal{N}/P) = \{\overline{0}\}$. □

3. Commutativity of $\mathcal{N}/P$

Lemma 3.1. *Let P be a 3-prime ideal of a near-ring $\mathcal{N}$ and d be a (α,τ)-P-derivation of $\mathcal{N}$. Suppose that $\overline{\tau(u)}$ is not a left zero divisor on $\mathcal{N}/P$. If $[d(u), u]_{\alpha,\tau} \in P$, then $d((x, u)) \in P$ for all $x \in \mathcal{N}$.*

Proof. Given $x \in \mathcal{N}$. From $u(u + x) = u^2 + ux$, we have

$$\overline{\tau(u)d(u+x)} + \overline{d(u)}(\overline{\alpha(u)} + \overline{\alpha(x)})$$
$$= \overline{\tau(u)d(u)} + \overline{d(u)\alpha(u)} + \overline{\tau(u)d(x)} + \overline{d(u)\alpha(x)},$$

which reduces to $\overline{\tau(u)d(x)} + \overline{d(u)\alpha(u)} = \overline{d(u)\alpha(u)} + \overline{\tau(u)d(x)}$. Since $\overline{d(u)\alpha(u)} = \overline{\tau(u)d(u)}$, the above equation can be expressed as

$$\overline{0} = \overline{\tau(u)(d(x) + d(u) - d(x) - d(u))} = \overline{\tau(u)d((x, u))}.$$

Thus, $\overline{d((x, u))} = \overline{0}$ for all $x \in \mathcal{N}$. □

Theorem 3.2. *Let P be a 3-prime ideal of a near-ring $\mathcal{N}$. Suppose that $\mathcal{N}/P$ has no nonzero divisors of zero. If $\mathcal{N}$ admits a non P-trivial (α,τ)-P-commuting P-(α,τ)-derivation d, then $(\mathcal{N}/P, +)$ is abelian.*

Proof. Given $x, y \in \mathcal{N}$. Let $c = \alpha^{-1}((x, y))$. Then, by Lemma 3.1, $\overline{d(c)} = \overline{0}$. Moreover, wc is an additive commutator for any $w \in \mathcal{N}$, thus, also a P-constant. i.e.,

$$\overline{0} = \overline{d(wc)} = \overline{\tau(w)d(c) + d(w)\alpha(c)} = \overline{d(w)\alpha(c)}$$

for all $w \in \mathcal{N}$. In view of Lemma 2.5 (**b**), we conclude that

$$\overline{0} = \overline{\alpha(c)} = \overline{(x, y)}$$

for all $x, y \in \mathcal{N}$. The proof is now complete. □

Theorem 3.3. *Let P be a 3-prime ideal of a near-ring $\mathcal{N}$ and d be a non P-trivial (α,τ)-P-derivation such that $\alpha(d(\mathcal{N})) - d(\alpha(\mathcal{N})) \subseteq P$ and $\tau(d(\mathcal{N})) - d(\tau(\mathcal{N})) \subseteq P$. If $\overline{d(N)} \subseteq Z(\mathcal{N}/P)$, then $(\mathcal{N}/P, +)$ is abelian. Moreover, if $d^2(\mathcal{N}) \nsubseteq P$, then $\mathcal{N}/P$ is a commutative ring.*

Proof. Suppose that $u \in \mathcal{N}$ such that $\overline{d(u)} \neq \overline{0}$. Then,

$$\overline{d(u)} \in Z(\mathcal{N}/P) \setminus \{\overline{0}\}$$

and $\overline{d(u)} + \overline{d(u)} \in Z(\mathcal{N}/P)$. Let $x, y \in \mathcal{N}$, we have

$$(\overline{d(u)} + \overline{d(u)})(\overline{x} + \overline{y}) = (\overline{x} + \overline{y})(\overline{d(u)} + \overline{d(u)}),$$

i.e.,

$$\overline{x}\overline{d(u)} + \overline{x}\overline{d(u)} + \overline{y}\overline{d(u)} + \overline{y}\overline{d(u)} = \overline{x}\overline{d(u)} + \overline{y}\overline{d(u)} + \overline{x}\overline{d(u)} + \overline{y}\overline{d(u)}$$

and we find that

$$\overline{x}\overline{d(u)} + \overline{y}\overline{d(u)} = \overline{y}\overline{d(u)} + \overline{x}\overline{d(u)}.$$

Hence, $\overline{d(u)}(\overline{x}, \overline{y}) = \overline{0}$ for all $\overline{x}, \overline{y} \in \mathcal{N}/P$. Since $\overline{d(u)} \in Z(\mathcal{N}/P) \setminus \{\overline{0}\}$ and $\mathcal{N}/P$ is a 3-prime near-ring, we see that $(\overline{x}, \overline{y}) = \overline{0}$ for all $\overline{x}, \overline{y} \in \mathcal{N}/P$. Thus, $(\mathcal{N}/P, +)$ is abelian. Using the hypothesis, we have $\overline{\alpha(z)d(xy)} = \overline{d(xy)\alpha(z)}$ for all $x, y, z \in \mathcal{N}$. An application of Lemma 2.3 yields

$$\overline{\alpha(z)d(x)\alpha(y)} + \overline{\alpha(z)\tau(x)d(y)} = \overline{d(x)\alpha(y)\alpha(z)} + \overline{\tau(x)d(y)\alpha(z)}$$

for all $x, y, z \in \mathcal{N}$. Since $\overline{d(N)} \subseteq Z(\mathcal{N}/P)$ and $(\mathcal{N}/P, +)$ is abelian, we get

$$\overline{d(x)\alpha(z)\alpha(y)} + \overline{d(y)\alpha(z)\tau(x)} = \overline{d(x)\alpha(y)\alpha(z)} + \overline{d(y)\tau(x)\alpha(z)}$$

for all $x, y, z \in \mathcal{N}$. Hence,

$$\overline{d(x)}[\overline{\alpha(z)}, \overline{\alpha(y)}] = \overline{d(y)}[\overline{\tau(x)}, \overline{\alpha(z)}]$$

for all $x, y, z \in \mathcal{N}$. Suppose now that $\mathcal{N}/P$ is not commutative. Choosing $x, z \in \mathcal{N}$ such that $[\overline{\tau(x)}, \overline{\alpha(z)}] \neq \overline{0}$ and replacing y by $d(y)$, we infer that $\overline{d^2(y)}[\overline{\tau(x)}, \overline{\alpha(z)}] = \overline{0}$ for all $y \in \mathcal{N}$. Since the central element $\overline{d^2(y)}$ cannot be a nonzero divisor of zero, we conclude that $\overline{d^2(y)} = \overline{0}$ for all $y \in \mathcal{N}$; a contradiction. □

Corollary 3.4. *Let P be a 3-prime ideal of a near-ring $\mathcal{N}$ and d be a non P-trivial (α, τ)-P^+-derivation such that $\alpha(d(\mathcal{N})) - d(\alpha(\mathcal{N})) \subseteq P$ and $\tau(d(\mathcal{N})) - d(\tau(\mathcal{N})) \subseteq P$. If $\overline{d(\mathcal{N})} \subseteq Z(\mathcal{N}/P)$, then $(\mathcal{N}/P, +)$ is abelian. Moreover, if $2(\mathcal{N}/P) \neq \{\overline{0}\}$, then $\mathcal{N}/P$ is a commutative ring.*

Proof. The proof is given by Lemma 2.6 and Theorem 3.3. □

Theorem 3.5. *Let P be a 3-prime ideal of a near-ring $\mathcal{N}$ and d be a non P-trivial (α, τ)-P-derivation such that $\alpha(d(\mathcal{N})) - d(\alpha(\mathcal{N})) \subseteq P$ and $\tau(d(\mathcal{N})) - d(\tau(\mathcal{N})) \subseteq P$. If $[\overline{d(x)}, \overline{d(y)}] = \overline{0}$ for all $x, y \in \mathcal{N}$, then one of the following assertions holds:*

(a) *$d^2(\mathcal{N}) \subseteq P$.*
(b) *$\mathcal{N}/P$ is a commutative ring.*

Proof. Suppose that $d^2(\mathcal{N}) \nsubseteq P$. Let $u \in \mathcal{N}$ such that $\overline{d(u)} \neq \overline{0}$. Then, $\overline{d(u)}$ and $\overline{d(u)} + \overline{d(u)}$ commute with $\overline{d(\mathcal{N})}$. Given $x, y \in \mathcal{N}$, we have

$$(\overline{d(u)} + \overline{d(u)})(\overline{d(x)} + \overline{d(y)}) = (\overline{d(x)} + \overline{d(y)})(\overline{d(u)} + \overline{d(u)}).$$

Using our hypothesis, we obtain

$$\begin{aligned}&\overline{d(u)d(x)} + \overline{d(u)d(x)} + \overline{d(u)d(y)} + \overline{d(u)d(y)}\\ &= \overline{d(u)d(x)} + \overline{d(u)d(y)} + \overline{d(u)d(x)} + \overline{d(u)d(y)}.\end{aligned}$$

Thus,

$$\overline{d(u)d(x)} + \overline{d(u)d(y)} = \overline{d(u)d(y)} + \overline{d(u)d(x)}.$$

Hence, $\overline{d(u)d((x,y))} = \overline{0}$ for all $u, x, y \in \mathcal{N}$. Since α is an automorphism of $\mathcal{N}$, we have

$$\overline{d(\mathcal{N})\alpha(d((x,y)))} = \{\overline{0}\} \text{ for all } x, y \in \mathcal{N}. \tag{3.1}$$

In view of Lemma 2.5 (**b**), we conclude that $\overline{\alpha(d((x,y)))} = \overline{0}$ for all $x, y \in \mathcal{N}$. Thus, $\overline{d((x,y))} = \overline{0}$ for all $x, y \in \mathcal{N}$. Which implies that

$$\overline{0} = \overline{d((wx, wy))} = \overline{d(w(x,y))} = \overline{d(w)\alpha((x,y))}$$

for all $w \in \mathcal{N}$. In view of Lemma 2.5 (**b**), we have $(\overline{x}, \overline{y}) = \overline{0}$ for all $x, y \in \mathcal{N}$, implying that $(\mathcal{N}/P, +)$ is abelian.
Using our hypothesis, we have

$$\overline{d(\alpha(z))d(d(x)y)} = \overline{d(d(x)y)d(\alpha(z))}.$$

for all $x, y, z \in \mathcal{N}$. Thus,

$$\overline{d(\alpha(z))}\left(\overline{d^2(x)\alpha(y)} + \overline{\tau(d(x))d(y)}\right) = \left(\overline{d^2(x)\alpha(y)} + \overline{\tau(d(x))d(y)}\right)\overline{d(\alpha(z))}$$

for all $x, y, z \in \mathcal{N}$. According to Lemma 2.3,

$$\begin{aligned}&\overline{d(\alpha(z))d^2(x)\alpha(y)} + \overline{d(\alpha(z))d(\tau(x))d(y)}\\ &= \overline{d^2(x)\alpha(y)d(\alpha(z))} + \overline{d(\tau(x))d(y)d(\alpha(z))}\end{aligned}$$

for all $x, y, z \in \mathcal{N}$. Using the fact that $(\mathcal{N}/P, +)$ is abelian and $\overline{d(\alpha(z))d(\tau(x))d(y)} = \overline{d(\tau(x))d(y)d(\alpha(z))}$, the last relation yields

$$\overline{d^2(x)d(\alpha(z))\alpha(y)} = \overline{d^2(x)\alpha(y)d(\alpha(z))}.$$

Replacing y by yt, where $t \in \mathcal{N}$, in the last equation and using it again, we infer that

$$\begin{aligned}\overline{d^2(x)d(\alpha(z))\alpha(y)\alpha(t)} &= \overline{d^2(x)\alpha(y)\alpha(t)d(\alpha(z))}\\ &= \overline{d^2(x)\alpha(y)d(\alpha(z))\alpha(t)}\end{aligned}$$

for all $t, x, y, z \in \mathcal{N}$. Hence,

$$\overline{d^2(x)}(\mathcal{N}/P)[\overline{d(\alpha(z))}, \overline{\alpha(t)}] = \{\overline{0}\}$$

for all $t, x, z \in \mathcal{N}$. The 3-primeness of $\mathcal{N}/P$, forces that

$$\overline{d(\mathcal{N})} \subseteq Z(\mathcal{N}/P).$$

Consequently, $\mathcal{N}/P$ is a commutative ring by Theorem 3.3. □

Corollary 3.6. *Let P be a symmetric 3-prime ideal of a near-ring $\mathcal{N}$ and d be a (α, τ)-P^+-derivation of $\mathcal{N}$ such that $\alpha(d(\mathcal{N})) - d(\alpha(\mathcal{N})) \subseteq P$ and $\tau(d(\mathcal{N})) - d(\tau(\mathcal{N})) \subseteq P$. If $[d(x), d(y)] \in P$ for all $x, y \in \mathcal{N}$, then one of the following assertions holds:*

(**a**) $2(\mathcal{N}/P) = \{\overline{0}\}$.
(**b**) $d(\mathcal{N}) \subseteq P$.
(**c**) *$\mathcal{N}/P$ is a commutative ring.*

Proof. The proof is obtained by Lemma 2.6 and Theorem 3.5. □

Theorem 3.7. *Let P be a 3-prime ideal of a near-ring $\mathcal{N}$ with a non P-trivial (α, τ)-P-derivation d and $a \in \mathcal{N}$. If $[d(\mathcal{N}), a]_{(\alpha,\tau)} \subseteq P$, then $\overline{\alpha(a)} \in Z(\mathcal{N}/P)$ or $\overline{d(a)} = \overline{0}$.*

Proof. By the hypothesis, we have

$$\overline{d(ax)\alpha(a)} = \overline{\tau(a)d(ax)} \text{ for all } x \in \mathcal{N}.$$

Hence,

$$\left(\overline{d(a)\alpha(x)} + \overline{\tau(a)d(x)}\right) \overline{\alpha(a)} = \overline{\tau(a)} \left(\overline{d(a)\alpha(x)} + \overline{\tau(a)d(x)}\right)$$

for all $x \in \mathcal{N}$. Since $\mathcal{N}$ satisfies the partial distributive law by Lemma 2.3, we have

$$\overline{d(a)\alpha(x)\alpha(a)} + \overline{\tau(a)d(x)\alpha(a)} = \overline{\tau(a)d(a)\alpha(x)} + \overline{\tau(a)\tau(a)d(x)}$$

for all $x \in \mathcal{N}$. Using our hypothesis again

$$\overline{d(a)\alpha(x)\alpha(a)} + \overline{\tau(a)\tau(a)d(x)} = \overline{\tau(a)d(a)\alpha(x)} + \overline{\tau(a)\tau(a)d(x)}$$

for all $x \in \mathcal{N}$, i.e.,

$$\overline{d(a)\alpha(x)\alpha(a)} = \overline{\tau(a)d(a)\alpha(x)} \text{ for all } x \in \mathcal{N}.$$

Substituting xy, where $y \in \mathcal{N}$, for x in the above equation and using it again, we get

$$\overline{d(a)\alpha(x)\alpha(y)\alpha(a)} = \overline{\tau(a)d(a)\alpha(x)\alpha(y)} = \overline{d(a)\alpha(x)\alpha(a)\alpha(y)}$$

for all $x, y \in \mathcal{N}$. The latter relation can be rewritten as

$$\overline{d(a)}(\mathcal{N}/P)[\overline{\alpha(a)}, \overline{\alpha(y)}] = \{\overline{0}\}$$

for all $y \in \mathcal{N}$. Using the 3-primeness of $\mathcal{N}/P$, we find that $\overline{d(a)} = \overline{0}$ or $\overline{\alpha(a)} \in Z(\mathcal{N}/P)$. The proof is now complete. □

Theorem 3.8. *Let P be a 3-prime ideal of a near-ring $\mathcal{N}$ with a non P-trivial (α,τ)-P-derivation d such that $\alpha(d(\mathcal{N})) - d(\alpha(\mathcal{N})) \subseteq P$ and $\tau(d(\mathcal{N})) - d(\tau(\mathcal{N})) \subseteq P$. If $[d(\mathcal{N}), d(\mathcal{N})]_{\alpha,\tau} \subseteq P$, then one of the following assertions holds:*

(**a**) *$d^2(\mathcal{N}) \subseteq P$.*
(**b**) *$\mathcal{N}/P$ is a commutative ring.*

Proof. According to Theorem 3.7, $\overline{d^2(x)} = \overline{0}$ or $\overline{\alpha(d(x))} \in Z(\mathcal{N}/P)$ for all $x \in \mathcal{N}$. Put $K = \{x \in \mathcal{N} \mid \overline{d^2(x)} = \overline{0}\}$ and

$$L = \{x \in \mathcal{N} \mid \overline{\alpha(d(x))} \in Z(\mathcal{N}/P)\}.$$

Both K and L are additive subgroups of $\mathcal{N}$. Moreover, $\mathcal{N} = K \cup L$. But a group cannot be the set-theoretic union of any two of its proper subgroups. Hence, $K = \mathcal{N}$ or $L = \mathcal{N}$.

If $K = \mathcal{N}$ then $d^2(\mathcal{N}) \subseteq P$. If $L = \mathcal{N}$ then $d(\mathcal{N}) \subseteq Z(\mathcal{N}/P)$. As a result, the Theorem 3.3 completes the proof. □

Corollary 3.9. *Let P be a symmetric 3-prime ideal of a near-ring $\mathcal{N}$ and d be a (α,τ)-P^+-derivation of $\mathcal{N}$ such that $\alpha(d(\mathcal{N})) - d(\alpha(\mathcal{N})) \subseteq P$ and $\tau(d(\mathcal{N})) - d(\tau(\mathcal{N})) \subseteq P$. If $[d(\mathcal{N}), d(\mathcal{N})]_{\alpha,\tau} \subseteq P$, then one of the following assertions holds:*

(**a**) *$2(\mathcal{N}/P) = \{\overline{0}\}$.*
(**b**) *$d(\mathcal{N}) \subseteq P$.*
(**c**) *$\mathcal{N}/P$ is a commutative ring.*

Proof. We get the proof by using Lemma 2.6 and Theorem 3.8. □

Theorem 3.10. *Let P be a symmetric 3-prime ideal of a near-ring $\mathcal{N}$ and d be a (α,τ)-P-derivation of $\mathcal{N}$ such that $\alpha(d(\mathcal{N})) - d(\alpha(\mathcal{N})) \subseteq P$ and $\tau(d(\mathcal{N})) - d(\tau(\mathcal{N})) \subseteq P$. If $d(xy) - d(x)d(y) \in P$ for all $x, y \in \mathcal{N}$, then $d(\mathcal{N}) \subseteq P$.*

Proof. Assume that

$$\overline{d(xy)} = \overline{d(x)d(y)} \tag{3.2}$$

for all $x, y \in \mathcal{N}$. First we have

$$\begin{aligned}\overline{d(xy\alpha(x))} &= \overline{d(xy)\alpha^2(x)} + \overline{\tau(x)\tau(y)d(\alpha(x))} \\ &= \overline{d(x)\alpha(y)\alpha^2(x)} + \overline{\tau(x)d(y)\alpha^2(x)} + \overline{\tau(x)\tau(y)d(\alpha(x))}\end{aligned}$$

for all $x, y \in \mathcal{N}$. Secondly,

$$\overline{d(xy\alpha(x))} = \overline{d(xy)d(\alpha(x))}$$

$$
\begin{aligned}
&= \overline{d(xy)\alpha(d(x))}\\
&= \overline{d(x)\alpha(y)\alpha(d(x))} + \overline{\tau(x)d(y)\alpha(d(x))}\\
&= \overline{d(x)\alpha(y)\alpha(d(x))} + \overline{\tau(x)d(y\alpha(x))}\\
&= \overline{d(x)\alpha(y)\alpha(d(x))} + \overline{\tau(x)d(y)\alpha^2(x)} + \overline{\tau(x)\tau(y)d(\alpha(x))}
\end{aligned}
$$

for all $x, y \in \mathcal{N}$. Comparing the last relations, we get

$$\overline{d(x)\alpha(y)\alpha^2(x)} = \overline{d(x)\alpha(y)\alpha(d(x))}$$

for all $x, y \in \mathcal{N}$. Equivalently,

$$\overline{d(x)}(\mathcal{N}/P)(\overline{\alpha(d(x))} - \overline{\alpha^2(x)}) = \{\overline{0}\}$$

for all $x \in \mathcal{N}$. Using the 3-primeness of $\mathcal{N}/P$, we find

$$\overline{d(x)} = \overline{0} \text{ or } \overline{d(x)} = \overline{\alpha(x)}$$

for all $x \in \mathcal{N}$. The subsets $A = \{x \in \mathcal{N} \mid \overline{d(x)} = \overline{0}\}$ and

$$B = \{x \in \mathcal{N} \mid \overline{d(x)} = \overline{\alpha(x)}\}$$

are additive subgroups of $\mathcal{N}$ and their union is equal to $\mathcal{N}$. Hence, either $\mathcal{N} = A$ or $\mathcal{N} = B$. If $\mathcal{N} = A$, then $d(\mathcal{N}) \subseteq P$. If $\mathcal{N} = B$, then $\overline{d(x)} = \overline{\alpha(x)}$ for all $x \in \mathcal{N}$. Putting xy instead of x in the last equation, we find

$$\overline{d(xy)} = \overline{\alpha(xy)} = \overline{\alpha(x)\alpha(y)} = \overline{d(x)\alpha(y)}$$

for all $x, y \in \mathcal{N}$. So,

$$\overline{d(x)\alpha(y)} + \overline{\tau(x)d(y)} = \overline{d(x)\alpha(y)}$$

for all $x, y \in \mathcal{N}$. In view of Lemma 2.5 (**a**), we conclude that $d(\mathcal{N}) \subseteq P$ and the proof is complete. □

Theorem 3.11. *Let P be a symmetric 3-prime ideal of a near-ring $\mathcal{N}$ and d be a (α, τ)-P-derivation of $\mathcal{N}$ such that $\alpha(d(\mathcal{N})) - d(\alpha(\mathcal{N})) \subseteq P$ and $\tau(d(\mathcal{N})) - d(\tau(\mathcal{N})) \subseteq P$. If $d(xy) - d(y)d(x) \in P$ for all $x, y \in \mathcal{N}$, then $d(\mathcal{N}) \subseteq P$.*

Proof. Assume that $\overline{d(xy)} = \overline{d(y)d(x)}$ for all $x, y \in \mathcal{N}$. It follows that

$$
\begin{aligned}
\overline{d(\alpha(x)\alpha(x)y)} &= \overline{d(\alpha(x))\alpha^2(x)\alpha(y)} + \overline{\tau(\alpha(x))d(\alpha(x)y)}\\
&= \overline{d(\alpha(x))\alpha^2(x)\alpha(y)} + \overline{\tau(\alpha(x))d(\alpha(x))d(y)}
\end{aligned}
$$

for all $x, y \in \mathcal{N}$. On the other hand, we have

$$
\begin{aligned}
\overline{d(\alpha(x)\alpha(x)y)} &= \overline{d(\alpha(x)y)d(\alpha(x))}\\
&= \overline{d(\alpha(x)y)\alpha(d(x))}\\
&= \overline{d(\alpha(x))\alpha(y)\alpha(d(x))} + \overline{\tau(\alpha(x))d(\alpha(y))\alpha(d(x))}
\end{aligned}
$$

$$= \overline{d(\alpha(x))\alpha(y)d(\alpha(x))} + \overline{\tau(\alpha(x))d(\alpha(y))d(\alpha(x))}$$

for all $x, y \in \mathcal{N}$. When we combine the last two expressions, we get

$$\overline{d(\alpha(x))\alpha^2(x)\alpha(y)} = \overline{d(\alpha(x))\alpha(y)d(\alpha(x))}$$

for all $x, y \in \mathcal{N}$. Substituting yz, where $z \in \mathcal{N}$, for y in the last equation yields

$$\begin{aligned}\overline{d(\alpha(x))\alpha^2(x)\alpha(y)\alpha(z)} &= \overline{d(\alpha(x))\alpha(y)d(\alpha(x))\alpha(z)} \\ &= \overline{d(\alpha(x))\alpha(y)\alpha(z)d(\alpha(x))}\end{aligned}$$

for all $x, y, z \in \mathcal{N}$, which implies that

$$\overline{d(\alpha(x))\alpha(y)}[\overline{\alpha(z)}, \overline{d(\alpha(x))}] = \overline{0}$$

for all $x, y, z \in \mathcal{N}$, that is, $\overline{d(\alpha(x))}(\mathcal{N}/P)[\overline{\alpha(z)}, \overline{d(\alpha(x))}] = \{\overline{0}\}$ for all $x, z \in \mathcal{N}$. Using the 3-primeness of $\mathcal{N}/P$, we obtain

$$\overline{d(\mathcal{N})} \subseteq Z(\mathcal{N}/P).$$

Thus, using our hypothesis, we obtain $\overline{d(xy)} = \overline{d(x)d(y)}$ for all $x, y \in \mathcal{N}$. Hence, by Theorem 3.10, we conclude that $d(\mathcal{N}) \subseteq P$. □

4. Semiprime ideal and (α,τ)-derivation

Theorem 4.1. *Let P be a semiprime ideal of a symmetric near-ring $\mathcal{N}$, where $\mathcal{N}/P$ is 2-torsion free, d be a (α,τ)-derivation such that $\alpha(d(\mathcal{N})) - d(\alpha(\mathcal{N})) \subseteq P$ and $\tau(d(\mathcal{N})) - d(\tau(\mathcal{N})) \subseteq P$. If*

$$\overline{d(N)} \subseteq Z(\mathcal{N}/P),$$

then one of the following assertions holds:

- (**a**) *There exists a prime ideal $P_\lambda \supseteq P$ such that $d(\mathcal{N}) \subseteq P_\lambda$.*
- (**b**) *$\mathcal{N}/P$ is a commutative ring.*

Proof. The semiprimeness of P implies that there exists a family $\mathcal{P}$ of 3-prime ideals P_λ such that $\cap P_\lambda = P$. Therefore,

$$\overline{d(N)} \subseteq Z(\mathcal{N}/P_\lambda) \tag{4.1}$$

for all $P_\lambda \in \mathcal{P}$. Since d is a (α,τ)-derivation, we get d is a (α,τ)-P_λ-derivation on $\mathcal{N}$ for all $P_\lambda \in \mathcal{P}$. By using (4.1) and the fact that $2(\mathcal{N}/P_\lambda) \neq \{\overline{0}\}$, then the corollary 3.4 gives

$$d(\mathcal{N}) \subseteq P_\lambda \text{ or } \mathcal{N}/P_\lambda \text{ is a commutative ring for all } P_\lambda \in \mathcal{P}. \tag{4.2}$$

Suppose that $d(\mathcal{N}) \nsubseteq P_\lambda$ for all $P_\lambda \in \mathcal{P}$. Thus, (4.2) implies that $\mathcal{N}/P = \mathcal{N}/\cap P_\lambda$ is a commutative ring. □

Theorem 4.2. *Let P be a semiprime ideal of a symmetric near-ring $\mathcal{N}$, where $\mathcal{N}/P$ is 2-torsion free, and d be a (α,τ)-derivation of $\mathcal{N}$ such that $\alpha(d(\mathcal{N})) - d(\alpha(\mathcal{N})) \subseteq P$ and $\tau(d(\mathcal{N})) - d(\tau(\mathcal{N})) \subseteq P$. If $[d(\mathcal{N}), d(\mathcal{N})] \subseteq P$, then one of the following assertions holds:*

(**a**) *There exists a prime ideal $P_\lambda \supseteq P$ such that $d(\mathcal{N}) \subseteq P_\lambda$.*
(**b**) *$\mathcal{N}/P$ is a commutative ring.*

Proof. Since P is semiprime, there exists a family $\mathcal{P}$ of 3-prime ideals P_λ such that $\cap P_\lambda = P$. Hence,

$$[d(\mathcal{N}), d(\mathcal{N})] \subseteq P_\lambda \text{ for all } P_\lambda \in \mathcal{P}. \tag{4.3}$$

Using the fact that d is a (α,τ)-derivation, we have d is a (α,τ)-P_λ^+-derivation on $\mathcal{N}$ for all $P_\lambda \in \mathcal{P}$. Since $2(\mathcal{N}/P_\lambda) \neq \{\overline{0}\}$, by using (4.3), then corollary 3.6 gives

$$d(\mathcal{N}) \subseteq P_\lambda \text{ or } \mathcal{N}/P_\lambda \text{ is a commutative ring for all } P_\lambda \in \mathcal{P}. \tag{4.4}$$

If $d(\mathcal{N}) \nsubseteq P_\lambda$ for all $P_\lambda \in \mathcal{P}$, then $\mathcal{N}/P = \mathcal{N}/\cap P_\lambda$ is a commutative ring by (4.4). □

Theorem 4.3. *Let P be a semiprime ideal of a symmetric near-ring $\mathcal{N}$, where $\mathcal{N}/P$ is 2-torsion free, and d be a (α,τ)-derivation of $\mathcal{N}$ such that $\alpha(d(\mathcal{N})) - d(\alpha(\mathcal{N})) \subseteq P$ and $\tau(d(\mathcal{N})) - d(\tau(\mathcal{N})) \subseteq P$. If $[d(\mathcal{N}), d(\mathcal{N})]_{\alpha,\tau} \subseteq P$, then one of the following assertions holds:*

(**a**) *There exists a prime ideal $P_\lambda \supseteq P$ such that $d(\mathcal{N}) \subseteq P_\lambda$.*
(**b**) *$\mathcal{N}/P$ is a commutative ring.*

Proof. We obtain the desired result by applying arguments similar to those used in the proof of Theorem 4.2. □

Theorem 4.4. *Let P be a semiprime ideal of a symmetric near-ring $\mathcal{N}$, and d be a (α,τ)-derivation of $\mathcal{N}$ such that $\alpha(d(\mathcal{N})) - d(\alpha(\mathcal{N})) \subseteq P$ and $\tau(d(\mathcal{N})) - d(\tau(\mathcal{N})) \subseteq P$. If $d(xy) - d(x)d(y) \in P$ for all $x, y \in \mathcal{N}$, then $d(\mathcal{N}) \subseteq P$.*

Proof. Since P is semiprime, there exists a family $\mathcal{P}$ of 3-prime ideals P_λ such that $\cap P_\lambda = P$. Hence,

$$d(xy) - d(x)d(y) \in P_\lambda \text{ for all } x, y \in \mathcal{N}, P_\lambda \in \mathcal{P}. \tag{4.5}$$

Since d is a (α,τ)-derivation, we find that d is (α,τ)-P_λ-derivations on $\mathcal{N}$ for all $P_\lambda \in \mathcal{P}$. By using (4.5), Theorem 3.10 forces that $d(\mathcal{N}) \subseteq P_\lambda$ for all $P_\lambda \in \mathcal{P}$. Hence, $d(\mathcal{N}) \subseteq P$. □

Theorem 4.5. *Let P be a semiprime ideal of a symmetric near-ring $\mathcal{N}$, and d be a (α,τ)-derivation of $\mathcal{N}$ such that $\alpha(d(\mathcal{N})) - d(\alpha(\mathcal{N})) \subseteq P$*

and $\tau(d(\mathcal{N})) - d(\tau(\mathcal{N})) \subseteq P$. *If* $d(xy) - d(y)d(x) \in P$ *for all* $x, y \in \mathcal{N}$, *then* $d(\mathcal{N}) \subseteq P$.

Proof. By using similar arguments to those used in the proof of Theorem 4.4, we get the required result. □

Acknowledgments

The authors would like to thank the referee for the valuable suggestions and comments.

References

1. M. Ashraf, A. Ali and A. Shakir, (σ,τ)-derivations on prime near-rings, *Arch. Math. (Brno)*, **40** (2004), 281–286.
2. H. E. Bell, On derivations in near-rings II. In: Nearrings, Nearfields and K-loops (Hamburg, 1995), *Math. Appl.*, **426** (1997), Kluwer Acad. Publ., Dordrecht, 191–197.
3. A. Boua and A. A. M. Kamal, Some results on 3-Prime near-rings with derivations, *Indian J. Pure Appl. Math.*, **47** (2016), 705–716.
4. N. J. Groenewald, Different prime ideals in near-rings, *Comm. Algebra*, **19**(10) (1991), 2667–2675.
5. S. Mouhssine and A. Boua, Homoderivations and semigroup ideals in 3-prime near-rings, *Algebraic Structures and their Applications*, **8**(2) (2021), 177–194.
6. S. Mouhssine and A. Boua, Right multipliers and commutativity of 3-prime near-rings, *Int. J. Appl. Math.*, **34**(1) (2021), 169–181 .

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Journal of Algebraic Systems

ON $(\alpha;\tau)$-P-DERIVATIONS OF NEAR-RINGS

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بررسی $(\alpha;\tau)$-P-مشتق‌های شبه‌حلقه‌ها

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رابطه‌ی بین مشتق‌ها و ساختارهای جبری شبه‌حلقه‌ی خارج قسمتی به یک موضوع جذاب در دهه‌های اخیر تبدیل شده است. فرض کنید $\mathcal{N}$ شبه‌حلقه و P ایده‌آل اول آن باشد. در این مقاله، مفهوم $(\alpha;\tau)$-P-مشتق در شبه‌حلقه‌ها معرفی شده است. همچنین، ساختار شبه‌حلقه‌های خارج قسمتی $\mathcal{N}/P$ که در برخی خواص جبری $(\alpha;\tau)$-P-مشتق صدق می‌کنند، بررسی شده‌اند.

کلمات کلیدی: شبه‌حلقه‌ها، ایده‌آل‌های اول، $(\alpha;\tau)$-P-مشتق‌ها، جابه‌جایی.