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WEAKLY PRIME AND SUPER-MAX FILTERS IN BL-ALGEBRAS

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ABSTRACT. In this paper, the concepts of weakly prime filters and super-max filters in BL-algebras are introduced, and the relationships between them are discussed. Also, some properties and relations between these filters and other types of filters in BL-algebras are given. With some examples, it is shown that these filters have differences. After that, the notions of weakly linear BL-algebras and weak top BL-algebras are defined and investigated. Finally, using the notion of a weakly prime filter, a new topology on BL-algebras is defined and studied.

Basic fuzzy logic, BL for short, and its corresponding BL-algebras were introduced by Hájek (see [8] and the references given there) with the purpose of formalizing the many-valued semantics induced by the continuous t-norms on the real unit interval $[0, 1]$. BL-algebras are the algebraic structures for Hájek's Basic logic [8]. BL-algebras rise as Lindenbaum algebras from certain logical axioms in a similar manner that Boolean algebras or MV-algebras [7] do from classical logic or Lukasiewicz logic, respectively. Turunen [14] studied BL-algebras by deductive systems. Deductive systems correspond to subsets closed with respect to Modus Ponens, and they are called filters, too. In [15], Boolean deductive systems and implicative deductive systems were introduced. Moreover, it was proved that these deductive systems coincide. MV-algebras, product algebras, and Godel algebras are the most common classes of BL-algebras. Filters theory plays an important

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role in studying these algebras. From a logical point of view, various filters correspond to various sets of provable formulas. Hájek [8] introduced the concepts of (prime) filters of BL-algebras. Using prime filters of BL-algebras, he proved the completeness of basic logic. Turunen [14] studied some properties of the prime filters of BL-algebras. This is the motivation for the researchers of this study to introduce some new filters in BL-algebras. The study of algebras motivated by logic is interesting and very useful, especially when these structures are not isomorphic. BL-algebras are important classes of algebras inspired by logic. In fact, the objective of this paper is to develop and define new concepts for investigating BL-algebras. This paper is motivated by the previous researches on filters in BL-algebras. The aim of this paper is to introduce a new filter in BL-algebras, which is weaker than the prime filter (weakly prime filter) and super-max filter in BL-algebras. In this paper, we introduce the notions of weakly prime filters and super-max filters in BL-algebras and study some properties of them. Also, we define the notion of weakly linear BL-algebra ($Wl-BL$-algebras) and a new topology on BL-algebras and characterize them. BL-algebras, MV-algebras, and lattice implication algebras are closely related. Thus, all results in this paper will contribute much to studying MV-algebras and lattice implication algebras.

1. Preliminaries

Definition 1.1 ([8])**.** A BL-algebra is an algebra $(A, \wedge, \vee, *, \to, 0, 1)$ with four binary operations $\wedge, \vee, *, \to$ and two constants $0, 1$ satisfying the following conditions:

(BL_1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice $L(A)$,

(BL_2) $(A, *, 1)$ is a commutative monoid,

(BL_3) $*$ and $\to$ form an adjoint pair; that is, $c \leq a \to b$ if and only if $a * c \leq b$, for all $a, b, c \in A$,

(BL_4) $a \wedge b = a * (a \to b)$,

(BL_5) $(a \to b) \vee (b \to a) = 1$.

It is easy to prove that if A is a BL-algebra and $x, y, z \in A$, then we have the following rules of calculus (for more details, see [5, 6, 8, 16]):

(BL_6) $x \leq y$ if and only if $x \to y = 1$,

(BL_7) $1 \to x = x$ and $x \leq y \to x$,

(BL_8) $x \to (y \to z) = (x * y) \to z = y \to (x \to z)$,

(BL_9) If $x \leq y$, then $y \to z \leq x \to z$, $z \to x \leq z \to y$, $x * z \leq y * z$ and $y^- \leq x^-$, where $x^- = x \to 0$,

(BL_{10}) $y \leq (y \to x) \to x$, so $y \leq y^{--}$,

$$x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x),$$

(BL_{11}) $x \vee (y * z) \geq (x \vee y) * (x \vee z)$, $x * (y \vee z) = (x * y) \vee (x * z)$,
(BL_{12}) $x^m \vee y^n \geqslant (x \vee y)^{mn}$, for all $m, n \in \mathbb{N}$.

Let A be a BL-algebra.
• A is said to be an MV-algebra, if for all $x \in A$, $x^{--} = x$, where $x^- = x \to 0$; see [7].
• A is said to be an integral, if for all $x, y \in A$, $x * y = 0$, implies $x = 0$ or $y = 0$; see [4].

Throughout this paper, it is assumed that $(A, \wedge, \vee, *, \to, 0, 1)$, in short A, is a BL-algebra (unless we write otherwise).

Definition 1.2 ([8, 14])**.** Let F be a nonempty subset of a BL-algebra A and let $a, b \in A$. We say that F is a filter of A if one of the following equivalent conditions is satisfied:
(i) Let $a, b \in F$ with $a \leq b$. If $a * b \in F$, then $b \in F$;
(ii) $1 \in F$ and $x, x \to y \in F$ implies $y \in F$;

A filter F of a BL-algebra A is proper if $F \neq A$; that is, $0 \notin F$.

Theorem 1.3 ([8, 17])**.** *Let F be a proper filter of a* BL*-algebra A. Then the following conditions are equivalent:*
(i) *$x \vee y \in F$ implies $x \in F$ or $y \in F$, for all $x, y \in A$;*
(ii) *$x \to y \in F$ or $y \to x \in F$, for all $x, y \in A$;*
(iii) *F is a prime filter of A.*

Definition 1.4 ([8, 17, 3])**.** Consider a proper filter F of a BL-algebra A.
(i) It is maximal if it is not contained in any other proper filter.
(ii) It is a primary filter, if for any $x, y \in A$, $(x * y)^- \in F$ implies $(x^n)^- \in F$ or $(y^n)^- \in F$, for some $n \in N$.
(iii) It is an obstinate filter, if $x, y \notin F$ implies $x \to y \in F$ and $y \to x \in F$.

Definition 1.5 ([13])**.** Let F be a proper filter of a BL-algebra A. The intersection of all maximal filters of A that contain F is called the radical of F, and it is denoted by $Rad(F)$. Clearly, $F \subseteq Rad(F)$.

Definition 1.6 ([8])**.** Let F be a proper filter of a BL-algebra A. The relation $\sim_F$ defined on a BL-algebra A by $(x, y) \in \sim_F$ if and only if $x \to y \in F$ and $y \to x \in F$, is a congruence relation on A. The quotient algebra $A/\sim_F$ denoted by A/F becomes a BL-algebra in a natural way, with the operations induced from those of A. So, the order relation on A/F is given by $x/F \leq y/F$ if and only if $x \to y \in F$.

Hence $x/F = 1/F$ if and only if $x \in F$ and $x/F = 0/F$ if and only if $x^- \in F$.

2. Weakly Prime Filters in BL-Algebras

In this section, the concept of a new filter in BL-algebras is introduced and also characterized.

Definition 2.1. A proper filter F of a BL-algebra A is called a weakly prime filter, if for any $x, y \in A$, $x \vee y \in F$ implies $x \in F$ or $y^{--} \in F$.

Proposition 2.2. *In any* BL-*algebra, every prime filter is a weakly prime filter.*

Proof. Based on Definition 2.1 and the condition (BL_{10}), the proof is clear. □

In the following example, we show that the converse of Proposition 2.2 is not true, in general.

Example 2.3. (i) Let $A = \{0, a, b, c, d, e, f, g, 1\}$, where

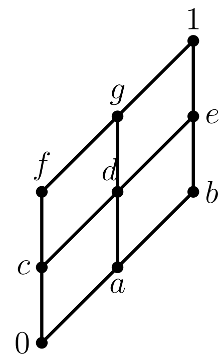
$$0 < a < b, d, e, g < 1, \qquad 0 < d < e, g < 1,$$
$$0 < f < g < 1, \qquad 0 < b < e < 1,$$

and $0 < c < d, e, f, g < 1$. Operations $*$ and $\to$ are defined as follows:

$*$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	a	a	0	a	a	0	a	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	c	c	c	c	c	c
d	0	a	a	c	d	d	c	d	d
e	0	a	b	c	d	e	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	a	a	c	d	d	f	g	g
1	0	a	b	c	d	e	f	g	1

$\to$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	f	1	1	f	1	1	f	1	1
b	f	g	1	f	g	1	f	g	1
c	b	b	b	1	1	1	1	1	1
d	0	b	b	f	1	1	f	1	1
e	0	a	b	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	0	b	b	c	e	e	f	1	1
1	0	a	b	c	d	e	f	g	1

Its Hasse diagram is as follows:

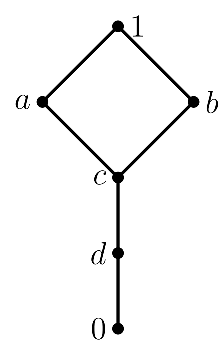


Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra. Moreover, $F = \{g, 1\}$ is not a weakly prime filter, since $a \vee f = g \in F$, while $a, f, a^{--}, f^{--} \notin F$.
(ii) Let $B = \{0, a, b, c, d, 1\}$, where $0 < d < c < a, b < 1$. Operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	d	a
b	0	c	b	c	d	b
c	0	c	c	c	d	c
d	0	d	d	d	0	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	b	d	1
b	0	a	1	a	d	1
c	0	1	1	1	d	1
d	d	1	1	1	1	1
1	0	a	b	c	d	1

Its Hasse diagram is as follows:



Then $(B, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra. Clearly, $\{1\}$ is a weakly prime filter, which is not prime.

Note. In MV-algebras, the notions of prime filters and weakly prime filters are equal.

Remark 2.4. Let F be a weakly prime filter in a BL-algebra A. Then $F \cap B(A)$ is a prime filter in $B(A)$.

Theorem 2.5. *Let F be a proper filter of a* BL*-algebra A. Then F is a weakly prime filter if and only if for all $x, y \in A$, $x \to y \in F$ or $y^{--} \to x^{--} \in F$.*

Proof. Let F be a weakly prime filter and let $x, y \in A$. Then as $(x \to y) \vee (y \to x) = 1 \in F$, we have $x \to y \in F$ or

$$y^{--} \to x^{--} = (y \to x)^{--} \in F.$$

Now assume that $x \to y \in F$ or $y^{--} \to x^{--} \in F$ for all $x, y \in A$ and that $x \vee y \in F$ for $x, y \in A$. If $x \to y \in F$, then as $x \vee y \leq (x \to y) \to y$ we have $y \in F$. If $y^{--} \to x^{--} \in F$, then as $x \vee y \in F$ and

$$x \vee y \leq (y \to x) \to x \leq (y^{--} \to x^{--}) \to x^{--},$$

we have $(y^{--} \to x^{--}) \to x^{--} \in F$ and so $x^{--} \in F$. Therefore F is a weakly prime filter. □

The following theorem is a straightforward consequence of Theorem 2.5.

Theorem 2.6. *Let F be a weakly prime filter of a* BL*-algebra A and let G be a proper filter of A containing F. Then G is a weakly prime filter.*

For a filter G of a BL-algebra A, consider $G^{--} = \{x \in A : x^{--} \in G\}$. It is clear that G^{--} is a filter of A containing G and $0 \in G$ if and only if $0 \in G^{--}$.

Proposition 2.7. *Let F be a weakly prime filter of a* BL*-algebra A. Then F^{--} and $Rad(F^{--})$ are prime, primary, and weakly prime filters.*

Proof. Let F be a weakly prime filter of A. Then F is a proper filter of A, and according to the definition of F^{--}, F^{--} is a proper filter of A. Using Theorem 2.5, for all $x, y \in A$, $x \to y \in F \subseteq F^{--}$ or $y^{--} \to x^{--} \in F$, which implies $y \to x \in F^{--}$. So based on the definition of prime filter, F^{--} is a prime filter of A. Therefore F^{--} is a primary and weakly prime filter of A, too. Also $Rad(F^{--})$ is a prime, primary, and weakly prime filter. □

Proposition 2.8. *Let F be a weakly prime filter of a* BL*-algebra A. Then $T = \{G^{--} : \; G \;$ is a filter of A and $F \subseteq G\}$ is a totally order set (by inclusion).*

Proof. Let $G_1^{--}, G_2^{--} \in T$. Then G_1 and G_2 are filters of A such that $F \subseteq G_1 \cap G_2$. Assume that $G_1^{--} \nsubseteq G_2^{--}$ and that $G_2^{--} \nsubseteq G_1^{--}$. Thus there exist $x \in G_1^{--} - G_2^{--}$ and $y \in G_2^{--} - G_1^{--}$. So $x^{--} \in G_1 - G_2$ and $y^{--} \in G_2 - G_1$. Based on Theorem 2.5, $x \to y \in F$ implies

$x^{--} \to y^{--} \in G_1$ and so $y^{--} \in G_1$, which is a contradiction. Also, $y^{--} \to x^{--} \in F$ implies $y^{--} \to x^{--} \in G_2$, and so $x^{--} \in G_2$, which is a contradiction. □

According to [14, Proposition 7], we obtain the following result.

Corollary 2.9. *Any proper filter of a* BL*-algebra can be extended to a weakly prime one.*

Proposition 2.10. *Any proper filter of a* BL*-algebra can be extended to a maximal weakly prime filter, with respect to inclusion.*

Proof. Let F be a proper filter of A. Based on Corollary 2.9, there exists a weakly prime filter P containing F. Put

$$T = \{G : G \text{ is a proper filter of } A,\ P \subseteq G\}.$$

Moreover, $P \in T$, and by Zorn's Lemma, T has a maximal element like G_0. So $F \subseteq P \subseteq G_0$ and using Theorem 2.6, G_0 is a maximal weakly prime filter of A that contains F. □

Proposition 2.11. *Let F_1 and F_2 be weakly prime filters of an integral* BL*-algebra A such that F_2 is a maximal filter. Then $F_1 \cap F_2$ is a weakly prime filter of A.*

Proof. Let $x \vee y \in F_1 \cap F_2$, for $x, y \in A$ and let $x \notin F_1 \cap F_2$. If $x \in F_1 - F_2$, then $F_2 \vee < x >= A$. So $0 \in F_2 \vee < x >$; hence $f_2 * x^n \leq 0$, for some $f_2 \in F_2$ and $n \geq 1$. Thus as A is an integral BL-algebra, $f_2 = 0$ or $x^n = 0$, that is, $F_1 = A$ or $F_2 = A$, which is a contradiction. If $x \in F_2 - F_1$, then as F_1 is a weakly prime filter and $x \notin F_1$, so $y^{--} \in F_1$. Hence $y^{--} \in F_1 \cap F_2$ (if $y^{--} \notin F_2$, then similarly we get $F_1 = A$ or $F_2 = A$, which is a contradiction). If $x \notin F_1 \cup F_2$, then $y^{--} \in F_1 \cap F_2$. Therefore $F_1 \cap F_2$ is a weakly prime filter. □

Proposition 2.12. *Let $\{P_i : i \in I\}$ be a nonempty totally ordered subset of weakly prime filters. Then $\cap_{i \in I} P_i$ and $\cup_{i \in I} P_i$ are weakly prime filters.*

Proof. Suppose that $x \vee y \in \cap_{i \in I} P_i$, for $x, y \in A$. Assume that there exists $j \in I$ such that $x \notin P_j$, so $y^{--} \in P_j$. Assume that $t \in I$. If $P_t \subseteq P_j$, then $x \notin P_t$, and so $y^{--} \in P_t$. If $P_j \subseteq P_t$, then $y^{--} \in P_t$. So for all $t \in I$, $y^{--} \in P_t$, and therefore $y^{--} \in \cap_{i \in I} P_i$. Clearly $\cup_{i \in I} P_i$ is a weakly prime filter, according to Theorem 2.6. □

Theorem 2.13. *Let F be a proper filter of A. Then for any $a \in A \setminus F$, there exists a weakly prime filter P such that $F \subseteq P$ and $a \notin P$.*

Proof. Consider $\sum = \{G : G \text{ is a filter of } A,\ F \subseteq G \text{ and } a \notin G\}$. It is clear that $F \in \sum$ and so by Zorn's Lemma, $\sum$ has a maximal element like P. We show that P is a weakly prime filter. Suppose on contrary that P is not a weakly prime filter. Thus there exist $x, y \in A$ such that $x \vee y \in P$; while $x \notin P$ and $y^{--} \notin P$. So $y \notin P$, $P\vee < x >$, and $P\vee < y > \notin \sum$, since P is a maximal element of $\sum$. So as $F \subseteq P\vee < x >$ and $F \subseteq P\vee < y >$, therefore $a \in P\vee < x >$ and $a \in P\vee < y >$. Hence $\alpha * x^n \leq a$ and $\beta * y^m \leq a$, for some $\alpha, \beta \in P$ and $n, m \in \mathbb{N}$. By the conditions (BL_{11}) and (BL_{12}), we have

$$\begin{aligned} a &\geq (\alpha * x^n) \vee (\beta * y^m) \geq (\alpha \vee (\beta * y^m)) * (x^n \vee (\beta * y^m)), \\ &\geq (\alpha \vee \beta) * (\alpha \vee y^m) * (x^n \vee \beta) * (x^n \vee y^m), \\ &\geq (\alpha \vee \beta) * (\alpha \vee y^m) * (x^n \vee \beta) * (x \vee y)^{mn}. \end{aligned}$$

Thus we get $a \in P$, which is a contradiction. So P is a weakly prime filter of A. □

According to Theorem 2.13, we get the following result.

Corollary 2.14. *Every proper filter is the intersection of all weakly prime filters containing it.*

Theorem 2.15. *Let P be a proper filter of A. Then P is a weakly prime filter if and only if for any filters F and G of A, $F \cap G \subseteq P$ implies $F \subseteq P$ or $G \subseteq P^{--}$.*

Proof. Assume that P is a weakly prime filter, and suppose that for filters F and G of A, $F \cap G \subseteq P$, but $F \nsubseteq P$. So there exists $a \in F \smallsetminus P$. Let $b \in G$. Then $a \vee b \in F \cap G$. Thus $a \vee b \in P$ and since $a \notin P$ and P is a weakly prime filter, $b^{--} \in P$, that is, $G \subseteq P^{--}$. Now assume that for any filters F and G of A, $F \cap G \subseteq P$ implies $F \subseteq P$ or $G \subseteq P^{--}$, and for $x, y \in A$, $x \vee y \in P$. So $< x > \cap < y > \subseteq < x \vee y > \subseteq P$. Hence $< x > \subseteq P$, which implies $x \in P$ or $< y > \subseteq P^{--}$, which implies $y \in P^{--}$, that is, $y^{--} \in P$. Therefore, the proof is completed. □

Corollary 2.16. *Let P be a weakly prime filter and let F_i ($i \in I = \{1, \ldots, n\}$) be filters of A such that $\cap_{i \in I} F_i \subseteq P$. Then there exists $j \in I$ such that $F_j \subseteq P$ or there exists $t \in I$ such that $F_t \subseteq P^{--}$.*

Proof. Let $\cap_{i \in I} F_i \subseteq P$. Then $\cap_{i \in I \smallsetminus \{1\}} F_i \cap F_1 \subseteq P$. Therefore by Theorem 2.15, we have two cases:

(1) $\cap_{i \in I \smallsetminus \{1\}} F_i \subseteq P$ or $F_1 \subseteq P^{--}$. If $F_1 \subseteq P^{--}$, then the proof is completed. Now if $\cap_{i \in I \smallsetminus \{1\}} F_i \subseteq P$, then by continuing this process, the proof is completed.

(2) $\cap_{i\in I\smallsetminus\{1\}}F_i \subseteq P^{--}$ or $F_1 \subseteq P$. If $F_1 \subseteq P$, then the proof is completed. Now let $\cap_{i\in I\smallsetminus\{1\}}F_i \subseteq P^{--}$. As $P \subseteq P^{--}$, so P^{--} is a weakly prime filter. Hence by continuing this process and using Theorem 2.15, the proof is completed. □

Corollary 2.17. *Let P be a weakly prime filter and let F and G be filters of A such that $P = F \cap G$. Then $F = P$ or $G^{--} = P^{--}$.*

Proof. According to Theorem 2.15, we have $F \subseteq P$ or $G \subseteq P^{--}$. If $F \subseteq P$, then $F = P$. Suppose that $G \subseteq P^{--}$ and that $x \in G^{--}$. Then $x^{--} \in G \subseteq P^{--}$, so $x^{--} = (x^{--})^{--} \in P$. Hence $x \in P^{--}$, that is, $G^{--} \subseteq P^{--}$. On the other hand, $P^{--} \subseteq G^{--}$, since $P \subseteq G$. Therefore $G^{--} = P^{--}$. □

Using Corollary 2.17, we have the following result.

Corollary 2.18. *Let P be a weakly prime filter and let F_i ($i \in I = \{1,\dots,n\}$) be filters of A such that $P = \cap_{i\in I}F_i$. Then there exists $j \in I$ such that $P = F_j$ or there exists $t \in I$ such that $P^{--} = F_t^{--}$.*

Proposition 2.19. *Let $f : A \longrightarrow B$ be a* BL*-homomorphism. If P is a weakly prime filter of B, then $f^{-1}(P)$ is a weakly prime filter of A.*

Proof. It is clear that $f^{-1}(P)$ is a proper filter of A. Now let $x \vee y \in f^{-1}(P)$, for $x, y \in A$. Thus $f(x) \vee f(y) \in P$ and so $x \in f^{-1}(P)$ or $y^{--} \in f^{-1}(P)$. Therefore $f^{-1}(P)$ is a weakly prime filter of A. □

Recall that for a BL-homomorphism $f : A \longrightarrow B$, we set

$$\ker(f) = \{x \in A : \ f(x) = 1\}.$$

Proposition 2.20. *Let $f : A \longrightarrow B$ be a* BL*-epimorphism. If P is a weakly prime filter of A such that* $\ker(f) \subseteq P$*, then $f(P)$ is a weakly prime filter of B.*

Proof. As f is onto, it is clear that $f(P)$ is a proper filter of B. Now let $x \vee y \in f(P)$, for $x, y \in B$. Then there exist $a, b \in A$ such that $x = f(a)$ and $y = f(b)$. Hence $f(a \vee b) \in f(P)$ and so there exists $c \in P$ such that $f(a \vee b) = f(c)$. So $c \to (a \vee b) \in \ker(f) \subseteq P$. Then $a \vee b \in P$. Thus $a \in P$ or $b^{--} \in P$. Hence $x \in f(P)$ or $y^{--} \in f(P)$. Therefore $f(P)$ is a weakly prime filter of B. □

Proposition 2.21. *Let F and G be proper filters of A such that $F \subseteq G$. Then G is a weakly prime filter of A if and only if G/F is a weakly prime filter of A/F.*

Proof. By the property of quotient BL-algebras (Definition 1.6), the proof is clear. □

Proposition 2.22. *Let F be a proper filter of A. Then F is a weakly prime filter of A if and only if every proper filter of A/F is a weakly prime filter.*

Proof. By the property of quotient BL-algebras (Definition 1.6), the proof is clear. □

From [12], for a filter F of A and $x \in A$, set
$$(F : x) = \{a \in A : \ a \vee x \in F\},$$
which is a filter of A containing F.

Proposition 2.23. *Let F be a weakly prime filter of A. Then the following properties hold:*

(i) *$Rad(F)$, $(F : x)$, and $Rad((F : x))$ are weakly prime filters of A, for $x \in A \setminus F$.*

(ii) *$(Rad(F) : x)$ is a weakly prime filter of A, for $x \in A \setminus Rad(F)$.*

(iii) *$(F : x) = F$, for $x \in A \setminus F^{--}$.*

Proof. By [12, Proposition 4.2(1)], we have $F \subseteq (F : x)$.

(i) It is clear from Theorem 2.6.

(ii) It is clear from Theorem 2.6.

(iii) Let $y \in (F : x)$, for $x \in A \setminus F^{--}$. Then $x \vee y \in F$ and $x^{--} \notin F$. So as F is weakly prime, $y \in F$, and thus $(F : x) = F$.

□

Theorem 2.24. *In any* BL*-algebra, every maximal weakly prime filter is a prime filter.*

Proof. Let P be a maximal weakly prime filter of A and let $x \vee y \in P$, for $x, y \in A$. If $x \notin P$, then by Proposition 2.23(i), $(P : x)$ is a weakly prime filter of A that contains P. So by the maximality of P, we have $P = (P : x)$. On the other hand, $x \vee y \in P$ implies $y \in (P : x)$ and so $y \in P$. Therefore P is a prime filter. □

Remark 2.25. The converse of Theorem 2.24 is not true, in general. For example, consider A of Example 2.3(ii). Clearly, $\{a, 1\}$ is a prime filter while is not maximal weakly prime, since $\{a, b, c, 1\}$ is a weakly prime filter.

Theorem 2.26. *Let P be a proper filter of A. Then P is weakly prime if and only if for all $x, y \in A$, $x \vee y \in P$ implies $(P : x) = A$ or $(P : y^{--}) = A$.*

Proof. Let P be a weakly prime filter and let $x \vee y \in P$, for $x, y \in A$. Then $x \in P$, which implies $(P : x) = A$ or $y^{--} \in P$, equivalently $(P : y^{--}) = A$. Now assume that for all $x, y \in A$, $x \vee y \in P$ implies

$(P : x) = A$ or $(P : y^{--}) = A$ and $a \vee b \in P$, for $a, b \in A$. If $(P : a) = A$, then $a \in P$ and if $(P : b^{--}) = A$, then $b^{--} \in P$. So P is a weakly prime filter. $\square$

Proposition 2.27. *A proper filter F of A is a weakly prime filter if and only if $(F : x) \subseteq F^{--}$, for any $x \in A - F$.*

Proof. Let F be a weakly prime filter and let $y \in (F : x)$, when $x \in A - F$. Then $x \vee y \in F$, and so $y^{--} \in F$, as $x \notin F$. Now let $(F : x) \subseteq F^{--}$, for any $x \in A - F$ and let $a \vee b \in F$, for $a, b \in A$. If $a \notin F$, then as $b \in (F : a)$ we have $b \in F^{--}$, that is, $b^{--} \in F$. Therefore F is a weakly prime filter. $\square$

Proposition 2.28. *A proper filter F of A is a weakly prime filter if and only if $F^{--} = (F : x)^{--}$, for any $x \in A \smallsetminus F$.*

Proof. Let F be a weakly prime filter. Since $F \subseteq (F : x)$, we have $F^{--} \subseteq (F : x)^{--}$. On the other hand, by Proposition 2.27,

$$(F : x) \subseteq F^{--}$$

and so $(F : x)^{--} \subseteq F^{--}$. Now assume that $F^{--} = (F : x)^{--}$, for any $x \in A \smallsetminus F$ and that $a \vee b \in F$, for $a, b \in A$. If $a \notin F$, then $F^{--} = (F : a)^{--}$. As $b \in (F : a) \subseteq (F : a)^{--}$, we have $b \in F^{--}$, that is, $b^{--} \in F$. Therefore, F is a weakly prime filter. $\square$

Proposition 2.29. *Let F be a weakly prime filter of A and let*

$$(y \to z) \to y \in F,$$

for $y, z \in A$. Then $y^{--} \in F$.

Proof. Let $(y \to z) \to y \in F$, for $y, z \in A$. Then

$$(y^{--} \to z^{--}) \to y^{--} \in F.$$

As F is weakly prime, $y \to z \in F$ or $y^{--} \to z^{--} \in F$. Hence $y \in F$ or $y^{--} \in F$. Therefore $y^{--} \in F$. $\square$

Recall that a nonempty subset F of A is called a positive implicative filter of A if $1 \in F$ and the conditions $x \to ((y \to z) \to y) \in F$, for $x, y, z \in A$, and $x \in F$ imply $y \in F$; see [9].

By the definition of a positive implicative filter and Proposition 2.29, we get that the following corollary.

Corollary 2.30. *Let A be an MV-algebra. Then any weakly prime filter is a positive implicative filter.*

Open Problem: What is the relationship between weakly prime filters and positive implicative filters?

Proposition 2.31. *Let F be a weakly prime filter of A. Then for any $x, y \in A$, $(x * y)^- \in F$ implies $(x^2)^- \in F$ or $((y^{--})^2)^- \in F$.*

Proof. Let for $x, y \in A$, $(x*y)^- \in F$. As F is weakly prime, $x \to y \in F$ or $y^{--} \to x^{--} \in F$. Hence $(x^2)^- \in F$, since

$$(x \to y) * (x * y)^- = (x \to y) * (y \to x^-) \leq (x \to x^-) = (x^2)^-$$

or $((y^{--})^2)^- \in F$, by

$$\begin{aligned}(y^{--} \to x^{--}) * (x * y)^- &= (y^{--} \to x^{--}) * (x \to y^-)\\ &\leq (y^{--} \to x^{--}) * (x^{--} \to y^-)\\ &\leq y^{--} \to y^{---}\\ &= ((y^{--})^2)^-.\end{aligned}$$

□

Theorem 2.32. *In any* BL*-algebra, every weakly prime filter is a primary filter.*

Proof. Assume that F is a weakly prime filter and that $(x * y)^- \in F$, for $x, y \in A$. By Proposition 2.31, $(x^2)^- \in F$ or $((y^{--})^2)^- \in F$. So as $y \leq y^{--}$, we have $(x^2)^- \in F$ or $(y^2)^- \in F$. Therefore, $(x^n)^- \in F$ or $(y^n)^- \in F$, for some $n \in \mathbb{N}$, that is, F is a primary filter. □

According to Theorem 2.32 and [12, Theorem 3.6], we get the following corollary.

Corollary 2.33. *In any* BL*-algebra, radical of every weakly prime filter is a prime filter.*

3. Two New Classes of BL-algebras

Definition 3.1. A BL-algebra A is called a weakly linear BL-algebra (or briefly, $Wl - BL$-algebra), if for any $x, y \in A$, $x \vee y = 1$ implies $x = 1$ or $y^- = 0$.

Example 3.2. (i) Let $A = \{0, a, b, 1\}$, where $0 < a < b < 1$. Operations $*$ and $\to$ are defined as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

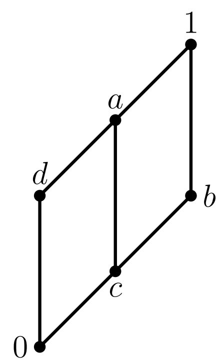
Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a $Wl - BL$-algebra.

(ii) Let $B = \{0, a, b, c, d, 1\}$, where $0 < c < a, b < 1$ and $0 < d < a < 1$. Operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	d	c	0	d	a
b	0	c	b	c	0	b
c	0	0	c	0	0	c
d	0	d	0	0	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	b	a	1
b	d	a	1	a	d	1
c	a	1	1	1	a	1
d	b	1	b	b	1	1
1	0	a	b	c	d	1

Its Hasse diagram is as follows:



Then $(B, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra while is not a $Wl - BL$-algebra. Since $a \vee b = 1$, but $a, b \neq 1$, $b^- = d \neq 0$, and $a^- = c \neq 0$.

(iii) Consider A defined in Example 2.3(ii). Clearly, A is a $Wl - BL$-algebra.

Theorem 3.3. *A proper filter F of A is a weakly prime if and only if A/F is a $Wl - BL$-algebra.*

Proof. Let F be a weakly prime filter and let $x/F \vee y/F = 1/F$, for $x/F, y/F \in A/F$. Then $x \vee y \in F$, and so $x \in F$ or $y^{--} \in F$. Hence $x/F = 1/F$ or $y^-/F = 0/F$, that is, A/F is a $Wl - BL$-algebra. Now assume that A/F is a $Wl - BL$-algebra and that $x \vee y \in F$, for $x, y \in A$. Thus $x/F \vee y/F = 1/F$. Hence $x/F = 1/F$ or $(y/F)^- = 0/F$. Therefore, $x \in F$ or $y^{--} \in F$, that is, F is a weakly prime filter of A. $\square$

According to Theorems 3.3 and 2.6, we have the following result.

Theorem 3.4. *The following conditions are equivalent:*
(i) *Every proper filter of A is weakly prime;*

(ii) $\{1\}$ *is a weakly prime filter of* A*;*
(iii) A *is a* $Wl-BL$*-algebra.*

Recall from [11] that for a nonempty subset X of A, we denote

$$X_s = \{a \in A : \ x \to a = a \text{ and } a \to x = x, \text{ for all } x \in X\}$$

and $S_x = \{a \in A : \ x \to a = a \text{ and } a \to x = x\}$. We know that X_s and S_x are filters of A. Recall from [11, Theorem 3.3], for a subset X of A,

$$X_s = \{a \in A : \ a \vee x = 1, \text{ for all } x \in X\} = \cap_{x \in X} S_x.$$

Proposition 3.5. *Let* X *be a nonempty subset of* A*. If* X_s *is a weakly prime filter and* $a, b \in X$*, then* $X \subseteq S_{a \to b}$ *or* $X \subseteq S_{b^{--} \to a^{--}}$*.*

Proof. Let X_s be a weakly prime filter of A. Then

$$a \to b \in X_s = \cap_{x \in X} S_x \ \text{ or } \ b^{--} \to a^{--} \in X_s.$$

Hence $a \to b \in S_x$, for any $x \in X$ and so $x \in S_{a \to b}$, for any $x \in X$ or $b^{--} \to a^{--} \in \cap_{x \in X} S_x$. Thus $b^{--} \to a^{--} \in S_x$, for any $x \in X$. Hence $x \in S_{b^{--} \to a^{--}}$, for any $x \in X$. Therefore $X \subseteq S_{a \to b}$ or $X \subseteq S_{b^{--} \to a^{--}}$. □

Proposition 3.6. *Let* F *be a filter of* A *such that* F_s *is a weakly prime filter of* A*. Then for any* $a, b \in F$*,* $a \leq b$ *or* $b^{--} \leq a^{--}$*.*

Proof. Let $a, b \in F$. Then $a^{--}, b^{--} \in F$ and so $a \to b, b^{--} \to a^{--} \in F$. By Proposition 3.5, $F \subseteq S_{a \to b}$ or $F \subseteq S_{b^{--} \to a^{--}}$. If $F \subseteq S_{a \to b}$, then $a \to b \in S_{a \to b}$, and so $(a \to b) \vee (a \to b) = 1$. Hence $a \to b = 1$, and so $a \leq b$. If $F \subseteq S_{b^{--} \to a^{--}}$, then $b^{--} \to a^{--} \in S_{b^{--} \to a^{--}}$, and so $(b^{--} \to a^{--}) \vee (b^{--} \to a^{--}) = 1$. Hence $b^{--} \to a^{--} = 1$. Therefore $b^{--} \leq a^{--}$. □

Proposition 3.7. *Let* F *be a proper linear filter of* A *such that* $F \neq \{1\}$*. Then* F_s *is a weakly prime filter of* A*.*

Proof. Let $a \vee b \in F_s$, for $a, b \in A$ such that $a \notin F_s$ and $b^{--} \notin F_s$. Then $b \notin F_s$, and there exist $x, y \in F$ such that $a \vee x \neq 1$ and $b \vee y \neq 1$. Put $z = x \wedge y$. Then $z, a \vee z, b \vee z \in F$, $a \vee z \neq 1$ and $b \vee z \neq 1$. As F is linear, assume that $b \vee z \leqslant a \vee z$. Therefore as $z \in F$ and $a \vee b \in F_s$, so $1 = (a \vee b) \vee z = a \vee (b \vee z) \leqslant a \vee (a \vee z) = a \vee z$, and we have $a \vee z = 1$, which is a contradiction. So F_s is a weakly prime filter of A. □

For a subset X of A, we define

$$X_s^{--} = \{a \in A : \ a^{--} \vee x^{--} = 1, \text{ for all } x \in X\}.$$

It is clear that for a filter F of A, F_s^{--} is a filter of A. It is easy to see that $0 \in F_s^{--}$ if and only if for any $x \in F$, $x^{--} = 1$.

Proposition 3.8. *Let F be a filter of A such that F_s^{--} is a proper filter of A. Then F_s^{--} is a weakly prime filter of A if and only if for any $x, y \in F$, $x^{--} \leq y^{--}$ or $y^{--} \leq x^{--}$.*

Proof. Let F_s^{--} be a weakly prime filter. For any $x, y \in F$,

$$(x^{--} \to y^{--}) \vee (y^{--} \to x^{--}) = 1 \in F_s^{--}.$$

Hence $x^{--} \to y^{--} \in F_s^{--}$, which implies $x^{--} \to y^{--} = 1$ (since $y \in F$ so $y^{--} \in F$, and as $x^{--} \to y^{--} \in F_s^{--}$ then

$$x^{--} \to y^{--} = (x^{--} \to y^{--}) \vee y^{--} = 1),$$

or

$$y^{--} \to x^{--} = (y^{--} \to x^{--})^{--} \in F_s^{--},$$

which implies $y^{--} \to x^{--} = 1$, as $x \in F$. Therefore $x^{--} \leq y^{--}$ or $y^{--} \leq x^{--}$. Now assume that for any $x, y \in F$, $x^{--} \leq y^{--}$ or $y^{--} \leq x^{--}$ and $a \vee b \in F_s^{--}$, where $a, b^{--} \notin F_s^{--}$. So there exist $x_1, x_2 \in F$ such that $a^{--} \vee x_1^{--} \neq 1$ and $b^{--} \vee x_2^{--} \neq 1$. Put $x = x_1 \wedge x_2$. Then $a^{--} \vee x^{--} \neq 1$, $b^{--} \vee x^{--} \neq 1$, $x^{--} \in F$, and so $a^{--} \vee x^{--} \in F$ and $b^{--} \vee x^{--} \in F$. By the hypothesis, we can assume that $a^{--} \vee x^{--} \leq b^{--} \vee x^{--}$. As $a \vee b \in F_s^{--}$, we get $(a \vee b)^{--} \vee x^{--} = 1$, and so $b^{--} \vee x^{--} = 1$, which is a contradiction. Therefore F_s^{--} is a weakly prime filter. □

Now we will define weak top BL-algebras.

Let F be a filter of a BL-algebra A. We define

$$Weak - Spec(A) = WS(A) = \{P : P \text{ is a weakly prime filter of } A\}$$

and $WV(F) = \{P \in WS(A) : F \subseteq P\}$.

Lemma 3.9. *Let F, G, and F_i $(i \in I)$ be filters of a* BL-*algebra A. Then the following properties hold:*
(i) $WV(A) = \emptyset$ and $WV(\{1\}) = WS(A)$.
(ii) If $F \subseteq G$, then $WV(G) \subseteq WV(F)$.
(iii) $\cap_{i \in I} WV(F_i) = WV(\vee_{i \in I} F_i)$.

Proof. The proofs of (i) and (ii) are clear.
(iii) By (ii), the proof is clear. □

The following example shows that it is not necessary, in general, $WV(F) \cup WV(G) = WV(F \cap G)$, for filters F and G of a BL-algebra A, and so $\{WV(F) : F \text{ is a filter of } A\}$ is not closed under finite union.

Example 3.10. All (weakly prime) filters of A defined in Example 2.3(ii) are $Z = \{1\}$, $F = \{b, 1\}$, $G = \{a, 1\}$, and $H = \{a, b, c, 1\}$. Then

$$WV(F) \cup WV(G) = \{F, G, H\} \subsetneqq WV(F \cap G) = \{Z, F, G, H\}.$$

We set $\tau(A) = \{WV(F) : \ F \ \text{ is a filter of } A\}$.

Definition 3.11. A BL-algebra A is called a weak top BL-algebra, if $\tau(A)$ is closed under finite union. Then A satisfies in topology under closed sets.

Example 3.12. (i) The BL-algebra A defined in Example 3.2(i) is a weak top BL-algebra.

(ii) The BL-algebra A defined in Example 3.10 is not a weak top BL-algebra, since

$$WV(F) \cup WV(G) = \{F, G, H\} \subsetneqq WV(F \cap G) = \{Z, F, G, H\}.$$

Lemma 3.13. *Let A be a weak top* BL-*algebra and let F be a proper filter of A. Then A/F is a weak top* BL-*algebra.*

Proof. Let G/F and H/F be two filters of a BL-algebra A/F. Then G and H are filters of A. As A is a weak top BL-algebra, then $WV(G) \cup WV(H) = WV(Z)$, for some filter Z of A. We show that $WV(G/F) \cup WV(H/F) = WV(< Z \cup F > /F)$. Now, assume that

$$P/F \in WV(G/F) \cup WV(H/F).$$

So $P/F \in WV(G/F)$, that is, $P \in WS(G)$ and $G/F \subseteq P/F$. Hence $G \subseteq P$, and so $P \in WV(G) \subseteq WV(G) \cup WV(H) = WV(Z)$. Thus $Z \subseteq P$, so $< Z \cup F > \subseteq P$. Then $P/F \in WV(< Z \cup F > /F)$, that is,

$$WV(G/F) \cup WV(H/F) \subseteq WV(< Z \cup F > /F).$$

Now, let $P/F \in WV(< Z \cup F > /F)$. Hence $P \in WS(G)$ and $< Z \cup F > \subseteq P$. So we get $Z \subseteq P$ and $F \subseteq P$. Thus

$$P \in WV(Z) = WV(G) \cup WV(H).$$

Then $P \in WV(G)$, and so $G \subseteq P$, that is, $G/F \subseteq P/F$ and

$$P/F \in WV(G/F) \subseteq WV(G/F) \cup WV(H/F).$$

Therefore, $WV(< Z \cup F > /F) \subseteq WV(G/F) \cup WV(H/F)$, that is, A/F is a weak top BL-algebra. $\square$

Proposition 3.14. *Every* BL-*homomorphic image of a weak top* BL-*algebra, is a weak top* BL-*algebra.*

Proof. Let $f : A \longrightarrow B$, be a BL-epimorphism and let A be a weak top BL-algebra. So $A/\ker(f) \cong B$. Hence according to the hypothesis and Lemma 3.13, B is a weak top BL-algebra. $\square$

Definition 3.15. A filter F of a BL-algebra A is called semi-weakly prime, if $F = \cap P$, for some $P \in WS(A)$.

According to Definition 3.15 and Proposition 2.21, we have the following result.

Lemma 3.16. *Let F and G be proper filters of a* BL*-algebra A such that $F \subseteq G$ and $WS(A/F) \neq \emptyset$. Then G/F is a semi-weakly prime filter of A/F if and only if G is a semi-weakly prime filter of A.*

Definition 3.17. A weakly prime filter P of a BL-algebra A is called weakly extraordinary, if for semi-weakly prime filters F and G of A, $F \cap G \subseteq P$ implies $F \subseteq P$ or $G \subseteq P$.

Example 3.18. In Example 3.10, Z is not a weakly extraordinary filter; but H is a weakly extraordinary filter.

According to the property of the prime filters and Definition 3.17, we have the following proposition.

Proposition 3.19. *Every prime filter of a* BL*-algebra A, is a weakly extraordinary filter of A.*

Theorem 3.20. *Let A be a* BL*-algebra. Then the following conditions are equivalent:*
(i) *A is a weak top* BL*-algebra;*
(ii) *Every weakly prime filter of A is weakly extraordinary;*
(iii) *For semi-weakly prime filters F and G of A,*

$$WV(F) \cup WV(G) = WV(F \cap G).$$

Proof. If $WS(A) = \emptyset$, then there is nothing to prove. Assume that $WS(A) \neq \emptyset$.

$(i) \Rightarrow (ii)$ Let $P \in WS(A)$ and let F and G be semi-weakly prime filters such that $F \cap G \subseteq P$. Then $P \in WV(F \cap G)$. By part (i), there exists a filter H of A such that $WV(F) \cup WV(G) = WV(H)$. On the other hand, there exist $P_i \in WS(A)$ $(i \in I)$ such that $F = \cap_{i \in I} P_i$. So for all $i \in I$, $P_i \in WV(F)$. Thus for all $i \in I$, $P_i \in WV(H)$. Therefore $H \subseteq \cap_{i \in I} P_i = F$. Similarly $H \subseteq G$. Hence $H \subseteq F \cap G$. Then $WV(F) \cup WV(G) \subseteq WV(F \cap G) \subseteq WV(H) = WV(F) \cup WV(G)$. So $WV(F) \cup WV(G) = WV(F \cap G)$. Therefore $P \in WV(F)$ or $P \in WV(G)$, which implies $F \subseteq P$ or $G \subseteq P$.

$(ii) \Rightarrow (iii)$ Let F and G be semi-weakly prime filters. We have $WV(F) \cup WV(G) \subseteq WV(F \cap G)$. Let $P \in WV(F \cap G)$. Then $F \cap G \subseteq P$ and by part (ii), we have $F \subseteq P$ or $G \subseteq P$. Therefore $P \in WV(F) \cup WV(G)$.

$(iii) \Rightarrow (i)$ Let F and G be filters of A. If $WV(F) = \emptyset$ or $WV(G) = \emptyset$, then

$$WV(F) \cup WV(G) = WV(G) \text{ or } WV(F) \cup WV(G) = WV(F),$$

respectively. So assume that $WV(F) \neq \emptyset$ and $WV(G) \neq \emptyset$. According to Corollary 2.14 and part (iii), $WV(F) \cup WV(G) = WV(F \cap G)$. Hence A is a weak top BL-algebra. □

Definition 3.21. Let $Y \subseteq WS(A)$. Define the closure of Y by

$$Cl(Y) = \cap H, \text{ where } H \text{ is a closed subset containing } Y.$$

Also define $O(Y) = \cap_{P \in Y} P$.

Proposition 3.22. *Let $Y \subseteq WS(A)$. Then $Cl(Y) = WV(O(Y))$.*

Proof. Let $P \in Y$. Then $O(Y) \subseteq P$, and hence $P \in WV(O(Y))$. Thus $Y \subseteq WV(O(Y))$. Therefore $Cl(Y) \subseteq Cl(WV(O(Y))) = WV(O(Y))$. Let $P \in WV(O(Y))$. Then $O(Y) \subseteq P$. Now assume that H is a closed subset of $WS(A)$ containing Y. Then there exists a filter F such that $Y \subseteq H = WV(F)$. Hence $F \subseteq \cap_{Q \in H} Q \subseteq \cap_{Q \in Y} Q = O(Y) \subseteq P$ and so $F \subseteq P$. Then $P \in WV(F) = H$. Therefore $WV(O(Y)) \subseteq H$ and so $WV(O(Y)) \subseteq Cl(Y)$. □

Using Proposition 3.22, we obtain the next corollary.

Corollary 3.23. *Let Y be a subset of $WS(A)$. Then Y is closed if and only if $Y = WV(O(Y))$.*

In the following theorem, we answer to the question that is $\{P\}$ closed as a subset of $WS(A)$?

Theorem 3.24. *Let P be a weakly prime filter of a* BL*-algebra A. Then $\{P\}$ is a closed subset of $WS(A)$ if and only if P is a maximal filter.*

Proof. Assume that $\{P\}$ is a closed subset of $WS(A)$. Then according to Corollary 3.23, $WV(P) = \{P\}$. Let Q be a maximal filter containing P. Then $\{Q\} = WV(Q) \subseteq WV(P) = \{P\}$ and so $Q = P$. Now assume that P is a maximal filter and that Q is a weakly prime filter such that $Q \in WV(P)$. Then $P = Q$, and so $WV(P) = \{P\}$. Therefore based on Corollary 3.23, $\{P\}$ is a closed set. □

From [10], a subset Y of a topology space (X, τ) is called irreducible, if for any two closed subsets Y_1 and Y_2, $Y = Y_1 \cup Y_2$ implies $Y = Y_1$ or $Y = Y_2$.

Theorem 3.25. *Let P be a weakly prime filter of a* BL*-algebra A. Then $WV(P)$ is an irreducible subset in $(WS(A), \tau(A))$.*

Proof. Let Y_1 and Y_2 be closed subsets in $WS(A)$ such that $WV(P) = Y_1 \cup Y_2$. Then $P \in Y_1 \cup Y_2$ and so $\{P\} \subseteq Y_1$ or $\{P\} \subseteq Y_2$. Hence $Cl(\{P\}) \subseteq Y_1$ or $Cl(\{P\}) \subseteq Y_2$. So by Proposition 3.22, $WV(P) \subseteq Y_1$ or $WV(P) \subseteq Y_2$. Therefore $WV(P) = Y_1$ or $WV(P) = Y_2$. $\square$

Theorem 3.26. *Let Y be a subset of $WS(A)$ such that $Cl(Y)$ is an irreducible subset of $WS(A)$. Then Y is an irreducible subset of $WS(A)$.*

Proof. Let Y_1 and Y_2 be closed subsets in $WS(A)$ such that $Y = Y_1 \cup Y_2$. Then $Cl(Y) = Cl(Y_1 \cup Y_2) = Cl(Y_1) \cup Cl(Y_2) = Y_1 \cup Y_2$. So $Cl(Y) = Y_1$ or $Cl(Y) = Y_2$. Thus $Y \subseteq Y_1$ or $Y \subseteq Y_2$. Therefore $Y = Y_1$ or $Y = Y_2$. $\square$

Proposition 3.27. *Let Y be a subset of $WS(A)$ such that $O(Y)$ is a weakly prime filter. Then Y is irreducible.*

Proof. Using Theorem 3.25, $WV(O(Y))$ is irreducible. So according to Proposition 3.22, $Cl(Y)$ is irreducible, and thus based on Theorem 3.26, Y is an irreducible subset. $\square$

According to Propositions 2.12 and 3.27, we obtain the following result.

Corollary 3.28. *Let $Y = \{P_i : i \in I\}$ be a nonempty totally ordered subset of $WS(A)$. Then Y is irreducible.*

4. Super-Max Filters in BL-algebras

In this section, a new filter is introduced and also characterized.

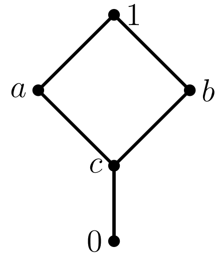
Definition 4.1. A proper filter F of A is called super-max filter, if for any $x, y \in A \setminus \{0, 1\}$, $x \vee y \in F$ implies $x * y \in F$.

Example 4.2. (i) Let $A = \{0, a, b, c, 1\}$, where $0 < c < a, b < 1$. Operations $*$ and $\to$ as follows:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	c	c	a
b	0	c	b	c	b
c	0	c	c	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	0	1	1	1	1
1	0	a	b	c	1

Its Hasse diagram is as follows:

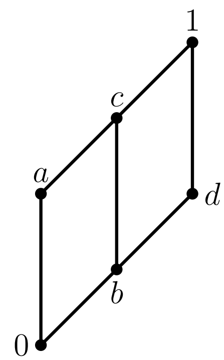


Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra and $F = \{a, b, c, 1\}$ is a super-max filter.
(ii) Let $B = \{0, a, b, c, d, 1\}$, where $0 < a < c < 1$ and $0 < b < c, d < 1$. Operations $*$ and $\rightarrow$ as follows:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

Its Hasse diagram is as follows:



Then $(B, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra and $G = \{a, c, 1\}$ is not a super-max filter.

By the definition of a super-max filter, we have the following theorem.

Theorem 4.3. *Let F be a proper filter of A. Then F is a super-max filter if and only if for any $x, y \in A \setminus \{0, 1\}$, $x \vee y \in F$ implies $x, y \in F$.*

Based on Theorem 4.3, we have the next result.

Theorem 4.4. *In any* BL-*algebra, every super-max filter is a prime filter.*

According to Proposition 2.2 and Theorem 4.4, we obtain the following proposition.

Proposition 4.5. *In any* BL-*algebras, every super-max filter is a weakly prime filter.*

Note. (i) Consider the filter G defined in Example 4.2(ii). Then in BL-algebras, every prime or even maximal filter is not super-max, in general.

(ii) In Example 4.2(ii), G is a weakly prime filter and is not a super-max filter.

Theorem 4.6. *In any* BL-*algebra, every super-max filter is* $\{1\}$ *or a maximal filter.*

Proof. Let $F \neq \{1\}$ be a super-max filter of a BL-algebra A. If there exists a proper filter G of A such that $F \subsetneqq G$, then there exists $x \in G \smallsetminus F$ such that $x \neq 0, 1$. As $F \neq \{1\}$, there exists $y \in F \smallsetminus \{0, 1\}$. So $x \vee y \in F$. Hence $x, y \in F$, which is a contradiction. Therefore, F is a maximal filter. $\square$

Theorem 4.7. $\{1\}$ *is a super-max filter of* A *if and only if there exists* $a \in A \smallsetminus \{1\}$ *such that for any* $b \in A \smallsetminus \{1\}$, $b \leq a$.

Proof. Let $\{1\}$ be a super-max filter. If there exist $a, b \in A \smallsetminus \{1\}$ such that $a \neq b$ and $a \vee b = 1$, then $a, b \neq 0$. Hence $\{1\}$ is a super-max filter. According to Theorem 4.3, $a = b = 1$, which is a contradiction. So there exists $a \in A \smallsetminus \{1\}$ such that for any $b \in A \smallsetminus \{1\}$, $b \leq a$. The converse is clear. $\square$

Using Definition 4.1, we obtain the next proposition.

Proposition 4.8. *If* $A \smallsetminus \{0\}$ *is a filter of* A*, then* $A \smallsetminus \{0\}$ *is a super-max filter.*

Corollary 4.9. $A \smallsetminus \{0\}$ *is a super-max filter, if for any* $a \in A \smallsetminus \{0\}$, $a^- = 0$.

Proof. By Proposition 4.8, it is enough to show that $A \smallsetminus \{0\}$ is a filter. Let $a, b \in A \smallsetminus \{0\}$. If $a * b = 0$, then $a^{--} * b^{--} = (a * b)^{--} = 0$, which is a contradiction. Since $a, b \in A \smallsetminus \{0\}$, then by the hypothesis, $a^{--} = 1 = b^{--}$, so $a^{--} * b^{--} = (a * b)^{--} = 1 \neq 0$. Thus $a * b \neq 0$, that is, $a * b \in A \smallsetminus \{0\}$. Therefore as $A \smallsetminus \{0\}$ is an upper set, we get that $A \smallsetminus \{0\}$ is a filter. $\square$

Proposition 4.10. *If* $A \smallsetminus \{0\}$ *is a super-max filter of* A*, then* $A \smallsetminus \{0\}$ *is the only maximal filter of* A.

Proof. By Theorem 4.6, $A \smallsetminus \{0\}$ is a maximal filter. Then $A \smallsetminus \{0\}$ is the only maximal filter of A. $\square$

Proposition 4.11. *If $F \neq \{1\}$ is a super-max filter of A, then $F = A \smallsetminus \{0\}$.*

Proof. If there exists $x \in A \smallsetminus (F \cup \{0\})$, then $x \neq 0, 1$. On the other hand, there exists $y \in F \smallsetminus \{1\}$ and so $y \neq 0, 1$. Thus $x \vee y \in F$, and therefore $x \in F$, which is a contradiction. Therefore, $F = A \smallsetminus \{0\}$. $\square$

According to Proposition 4.11 and Theorem 4.6, we obtain the following result.

Corollary 4.12. *Let F be a super-max filter of A. Then $F = \{1\}$ or $F = A \smallsetminus \{0\}$.*

Based on Theorem 4.7 and Propositions 4.8 and 4.11, we have the next theorem.

Theorem 4.13. *Let F be a proper filter of A. The following statements are equivalent:*
(i) *F is a super-max filter.*
(ii) *$F = A \smallsetminus \{0\}$ or $F = \{1\}$, and there exists $a \in A \smallsetminus \{1\}$ such that for any $b \in A \smallsetminus \{1\}$, $b \leq a$.*

Note. Using Theorem 4.7 and Corollary 4.9, we can find that a BL-algebra A has a super-max filter, if for some $a \in A \smallsetminus \{1\}$, the table of binary operation $\rightarrow$ is as follow:

$\rightarrow$	0	$\ldots$	$\ldots$	1
0	1			
.	0			
.	0			
.	0			
.	0			
.	0			
1	0			

or

$\rightarrow$	0	$\ldots$	a	$\ldots$	1
0			1		
.			1		
.			1		
a			1		
.			1		
.			1		
1			a		

Corollary 4.14. (i) *In any* BL-*algebra, the intersection of super-max filters is a super-max filter.*
(ii) *In any* BL-*algebra, the union of a family of totally ordered super-max filters is a super-max filter.*

Proof. By the definition of a super-max filter, the proofs are easy. $\square$

According to Theorem 4.13, we have the following result.

Corollary 4.15. *Let $F \neq \{1\}$ be a super-max filter of A. Then any proper filter containing F is a super-max filter.*

Proposition 4.16. *If $\{1\}$ is an obstinate filter of A. Then $A = \{0, 1\}$.*

Proof. Assume that $x, y \in A \setminus \{1\}$. Then as $\{1\}$ is an obstinate filter, we have $x \to y = 1$ and $y \to x = 1$. So $x = y$, for any $x, y \in A \setminus \{1\}$, which implies $A = \{0, 1\}$. □

By Proposition 4.16 and the definition of super-max filters, we have the next corollary.

Corollary 4.17. *If $\{1\}$ is an obstinate filter of A, then $\{1\}$ is a super-max filter.*

Proposition 4.18. *Let $F \neq \{1\}$ be a super-max filter of A. Then F is an obstinate filter.*

Proof. By Theorem 4.13, $F = A \setminus \{0\}$. So for any $x, y \in A \setminus F$, $x = y = 0$. Thus for any $x, y \in A \setminus F$, $x \to y$, $y \to x \in F$. Thus F is an obstinate filter. □

Corollary 4.19. *In any* BL*-algebra, every super-max filter is $\{1\}$ or obstinate.*

Proof. By Corollary 4.12 and Proposition 4.18, the proof is clear. □

Using [3, Example 3.3], $F = \{c, d, 1\}$ is an obstinate filter, which is not a super-max filter, according to Theorem 4.13.

From [12], recall that for a filter F of a BL-algebra A and $x \in A$, $(F : x) = \{a \in A : \ a \vee x \in F\}$.

Proposition 4.20. *Let F be a super-max filter of A. Then for any $x \in A$, $(F : x)$ is A or a super-max filter.*

Proof. Let $x \in A$. According to Theorem 4.13, $F = \{1\}$ or $F = A \setminus \{0\}$. If $F = \{1\}$, then for $x = 1$, $(F : x) = A$ and otherwise for $x \neq 1$, according to Theorem 4.13, for any $b \neq 1$, $x \vee b \notin F$. Hence $(F : x) = \{1\} = F$. If $F = A \setminus \{0\}$, then for $x \in F$, $(F : x) = A$ and otherwise $(F : x) = A \setminus \{0\} = F$. Thus for any $x \in A$, $(F : x)$ is A or a super-max filter. □

Using Theorem 4.4 and [12, Proposition 4.2(8)], we get the following corollary.

Corollary 4.21. *Let F be a super-max filter of A and let G be a filter of A containing F. Then for any $x \in A \setminus F$, $(F : x) \subseteq G$.*

In Figure 1, we show the relationship of the filters with a diagram.

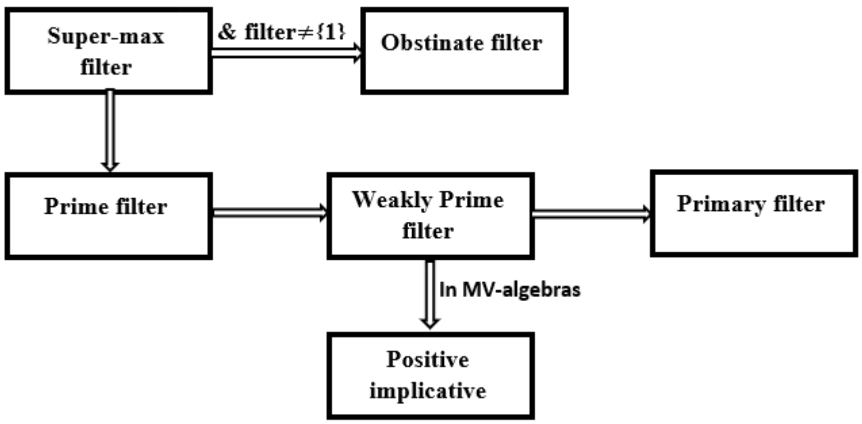


FIGURE 1. Relations of filters

5. Conclusion

BL-algebras have the most important algebraic structure among all the various logical algebras proposed as the semantic systems of nonclassical logical systems. Moreover, they include some important classes of algebras, like MV. In this paper, we introduced the notions of weakly prime filters and super-max filters in BL-algebras and studied some of their properties and also gave some of their characterizations. We showed that every super-max filter is prime, every prime filter is weakly prime, and every weakly prime is primary. Also, the concept of weakly linear BL-algebras ($Wl - BL$-algebras) and weak top BL-algebras were defined and investigated. Since BL-algebras, MV-algebras, and lattice implication algebras are closely related, all results in this paper will contribute much to studying MV-algebras and lattice implication algebras. Also, in our future research, we will compare this new filter with other filters in BL-algebras.

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WEAKLY PRIME AND SUPER-MAX FILTERS IN BL-ALGEBRAS

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فیلترهای اول ضعیف و فیلترهای فوق ماکزیمم در BL- جبرها

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در این مقاله، مفاهیم فیلترهای اول ضعیف و فیلترهای فوق ماکزیمم در BL -جبرها معرفی شده و روابط بین آنها مورد مطالعه قرار گرفته است. همچنین برخی از خصوصیات و روابط بین این فیلترها و انواع دیگر فیلترها در BL -جبرها بیان شده است. با چند مثال نشان داده می شود که این فیلترها متفاوت هستند. پس از آن مفاهیم BL -جبرهای خطی ضعیف و BL -جبرهای خطی ضعیف توپولوژی تعریف و بررسی می‌شوند. در نهایت، با استفاده از مفهوم فیلتر اول ضعیف، یک توپولوژی جدید بر روی BL -جبرها تعریف و مطالعه می‌شود.

کلمات کلیدی: فیلتر اول، فیلتر فوق ماکزیمم، فیلتر اول ضعیف، BL -جبر ضعیف توپولوژی.