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ON CLOSED HOMOTYPICAL VARIETIES OF SEMIGROUPS-2

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Abstract. In this paper we extended the results of paper "On Closed Homotypical Varieties of Semigroups" and have shown that the homotypical varieties of semigroups defined by the identities $axy = x^n ayx$, $axy = xa^n ya[axy = yay^n x]$, $axy = xaya^n[axy = y^n ayx]$ and $axy = xayx^n$ are closed in itself, where $(n \in \mathbb{N})$.

1. Introduction and Preliminaries

A semigroup identity $u = v$ is the formal equality of two words u and v formed by the letters over an alphabet set X. For any word u, the content of u (necessarily finite) is the set of all distinct variables appearing in u and is denoted by $C(u)$. The identity $u = v$ is said to be homotypical if $C(u) = C(v)$. The class of semigroups, in which a finite or an infinite collection $u_1 = v_1$, $u_2 = v_2 \cdots$ of identical relations is satisfied, is called the variety of semigroups determined by these identical relations. A variety $\mathcal{V}$ of semigroups is said to be homotypical if it admits a homotypical identity. Let U be a subsemigroup of a semigroup S. Isbell [10] defined the dominion of U in S as

$$Dom(U,S) = \{d \in S : \forall\ \alpha, \beta : S \longrightarrow T, \text{ if } \alpha|_U = \beta|_U \implies d\alpha = d\beta\}.$$

It is well known that $Dom(U,S)$ is a subsemigroup of S containing U. If $Dom(U,S) = U$, then U is called closed in S; and absolutely closed if $Dom(U,S) = U$ in every containing semigroup S. If each member

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of a variety $\mathcal{V}$ is closed, then the variety $\mathcal{V}$ of semigroups is said to be closed. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be any varieties of semigroups such that $\mathcal{V}_1 \subseteq \mathcal{V}_2$. Then the variety $\mathcal{V}_1$ is said to be closed in the variety $\mathcal{V}_2$ if whenever a semigroup $U \in \mathcal{V}_1$ is a subsemigroup of a member S of $\mathcal{V}_2$, then U is closed in S. Obviously, if $\mathcal{V}_1$ is closed in $\mathcal{V}_2$, then all subvarieties of $\mathcal{V}_1$ are closed in containing subvarieties of $\mathcal{V}_2$.

The following result provided by Isbell [10], known as Isbell's zigzag theorem, is a most useful characterization of semigroup dominions and is of basic importance to our investigations.

Theorem 1.1. *([10], Theorem 2.3) Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:*

$$d = a_0t_1 = y_1a_1t_1 = y_1a_2t_2 = y_2a_3t_2 = \cdots = y_ma_{2m-1}t_m = y_ma_{2m} \quad (1.1)$$

where $m \geq 1$, $a_i \in U$ $(i = 0, 1, \ldots, 2m)$, $y_i, t_i \in S$ $(i = 1, 2, \ldots, m)$, and

$$a_0 = y_1a_1, \qquad a_{2m-1}t_m = a_{2m},$$
$$a_{2i-1}t_i = a_{2i}t_{i+1}, \qquad y_ia_{2i} = y_{i+1}a_{2i+1} \qquad (1 \leq i \leq m-1).$$

Such a series of factorization is called a zigzag in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$.

The following result is also necessary for our investigations.

Theorem 1.2. *([11], Result 3) Let U and S be semigroups with U as a subsemigroup of S. Take any $d \in S\backslash U$ such that $d \in Dom(U, S)$. If (1.1) is a zigzag of minimal length m over U with value d, then $t_j, y_j \in S\backslash U$ for all $j = 1, 2, \ldots, m$.*

Semigroup theoretic notations and conventions of Clifford and Preston [7] and Howie [9] will be used throughout without explicit mention.

2. Closedness and Varieties of semigroups

In general varieties of bands containing the varieties of rectangular and normal bands are not absolutely closed as Higgins [8, Chapter 4] had given examples of a rectangular band and a normal band that were not absolutely closed. Therefore, for the varieties of semigroups, it is worthwhile to find largest subvarieties of the variety of all semigroups in which these varieties are closed. As a first step in this direction, one attempts to find those varieties of semigroups that are closed in itself. Encouraged by the fact that Scheiblich [12] had shown that the variety of all normal bands was closed, Alam and Khan in [4, 5, 6] had shown that the variety of left [right] regular bands, left [right] quasi-normal bands and left [right] semi-normal bands were closed. In [3], Ahanger and Shah had proved a stronger fact that the variety of left [right] regular bands was closed in the variety of all bands and, recently, Abbas and Ashraf [1] had shown that a variety of left [right] normal bands was closed in some containing homotypical varieties of semigroups.

In [2], Abbas and Ashraf had shown that some closed homotypical varieties of semigroups determined by the identities $axy = x^2ayx$, $axy = xa^2ya$, $axy = yay^2x$, $axy = xaya^2$, $axy = y^2ayx$ and $axy = xayx^2$. In the same direction, we have shown that the homotypical varieties of semigroups defined by the identities $axy = x^nayx$, $axy = xa^nya[axy = yay^nx]$, $axy = xaya^n[axy = y^nayx]$ and $axy = xayx^n$ are closed in itself.

Finally, the results in this section raise open problem of whether the varieties considered in the paper are absolutely closed; if not, of finding out largest varieties of semigroups in which these varieties are closed.

Lemma 2.1. *Let U be a subsemigroup of semigroup S such that S satisfies an identity $axy = x^nayx$ $[axy = xa^nya]$ and let $d \in Dom(U,S)\backslash U$ has a zigzag of type (1.1) in S over U with value d of shortest possible length m. Then*

$$d = (\prod_{i=1}^{k} a_{2i-1}^{2n-1})y_k a_{2k-1} t_k (\prod_{i=1}^{k} a_{2k-(2i-1)}).$$

for each $k = 1, 2, ..., m$.

Proof. Let $\mathcal{V}_1 = [axy = x^nayx]$ and $\mathcal{V}_2 = [axy = xa^nya]$ be the varieties of semigroups. First we show that in both cases whether $S \in \mathcal{V}_1$ or $S \in \mathcal{V}_2$, S satisfies $xyz = y^{2n-1}xyzy$.

Case (i): When $S \in \mathcal{V}_1$, then for any $x, y, z \in S$, we have

$$\begin{aligned} xyz &= y^nxzy \text{ (as } S \in \mathcal{V}_1 \text{)} \\ &= (y^{n-1}yx)zy \text{ (for } n = 1\text{, we treat } y^{n-1}y \text{ as } y) \\ &= y^ny^{n-1}xyzy \text{ (as } S \in \mathcal{V}_1 \text{)} \\ &= y^{2n-1}xyzy. \end{aligned} \tag{2.1}$$

Case (ii): When $S \in \mathcal{V}_2$, then for any $x, y, z \in S$, we have

$$\begin{aligned} xyz &= (yx^nz)x \text{ (as } S \in \mathcal{V}_2 \text{)} \\ &= x^ny^nzyx \text{ (as } S \in \mathcal{V}_2 \text{)} \end{aligned}$$

$$\begin{aligned}
&= x^n(yy^{n-1}z)yx \text{ (for } n=1\text{, we treat } yy^{n-1} \text{ as } y)\\
&= x^n y^{n-1} y^n zyyx \text{ (as } S \in \mathcal{V}_2 \text{)}\\
&= (x^n(y^{2n-1}z)y)yx\\
&= y^{2n-1}(z(x^n)^n yx^n)yx \text{ (as } S \in \mathcal{V}_2 \text{)}\\
&= y^{2n-1}((x^n z)(yy)x) \text{ (as } S \in \mathcal{V}_2 \text{)}\\
&= y^{2n-1}y(y(x^n z)^n x(x^n z)) \text{ (as } S \in \mathcal{V}_2 \text{)}\\
&= y^{2n-1}(yx^n(zy)x) \text{ (as } S \in \mathcal{V}_2 \text{)}\\
&= y^{2n-1}xyzy \text{ (as } S \in \mathcal{V}_2 \text{)}
\end{aligned}$$

Thus the claim is proved.

Now, we shall prove the lemma by using induction on k. Let U be a subsemigroup of semigroup S such that S belongs to either $S \in \mathcal{V}_1$ or $S \in \mathcal{V}_2$ and let $d \in Dom(U,S)\backslash U$ has a zigzag of type (1.1) in S over U with value d of shortest possible length m. Now, for $k=1$, we have

$$\begin{aligned}
d &= y_1 a_1 t_1 \text{ (by zigzag equations)}\\
&= a_1^{2n-1} y_1 a_1 t_1 a_1 \text{ (by equation (2.1)).}
\end{aligned}$$

Thus the result holds for $k=1$. Assume inductively that the result holds for $k=j<m$. Then we shall show that it also holds for $k=j+1$. Now

$$\begin{aligned}
d &= (\prod_{i=1}^{j} a_{2i-1}^{2n-1}) y_j a_{2j-1} t_j (\prod_{i=1}^{j} a_{2j-(2i-1)})\\
&\quad \text{(by inductive hypothesis)}\\
&= (\prod_{i=1}^{j} a_{2i-1}^{2n-1}) y_{j+1} a_{2j+1} t_{j+1} (\prod_{i=1}^{j} a_{2j-(2i-1)})\\
&\quad \text{(by zigzag equations)}\\
&= (\prod_{i=1}^{j} a_{2i-1}^{2n-1}) a_{2j+1}^{2n-1} y_{j+1} a_{2j+1} t_{j+1} a_{2j+1} (\prod_{i=1}^{j} a_{2j-(2i-1)})\\
&\quad \text{(by equation (2.1))}\\
&= (\prod_{i=1}^{j+1} a_{2i-1}^{2n-1}) y_{j+1} a_{2j+1} t_{j+1} (\prod_{i=1}^{j+1} a_{2(j+1)-(2i-1)}),
\end{aligned}$$

as required and, by induction, the lemma is established. $\square$

Theorem 2.2. *The variety $\mathcal{V} = [axy = x^n ayx]$ of semigroups, i.e. the class of all semigroups satisfying the identity $axy = x^n ayx$, is closed.*

Proof. Take any U, $S \in \mathcal{V}$ with U as a subsemigroup of S such that $d \in Dom(U,S)\setminus U$. Let d has zigzag of type (1.1) in S over U of shortest possible length m. Now

$$\begin{aligned}
d &= (\prod_{i=1}^{m} a_{2i-1}^{2n-1}) y_m a_{2m-1} t_m (\prod_{i=1}^{m} a_{2m-(2i-1)}) \\
&\quad \text{(by Lemma (2.1))} \\
&= (\prod_{i=1}^{m-1} a_{2i-1}^{2n-1}) a_{2m-1}^{n-1} (a_{2m-1}^{n} (y_m a_{2m-1}) t_m a_{2m-1}) (\prod_{i=2}^{m} a_{2m-(2i-1)}) \\
&\quad \text{(for } n=1\text{, we treat } a_{2m-1}^{n-1} a_{2m-1}^{n} \text{ as } a_{2m-1}^{n}) \\
&= (\prod_{i=1}^{m-1} a_{2i-1}^{2n-1}) a_{2m-1}^{n-1} (y_m a_{2m-1})(a_{2m-1} t_m)(\prod_{i=2}^{m} a_{2m-(2i-1)}) \\
&\quad \text{(as } S \in \mathcal{V}\text{)} \\
&= (\prod_{i=1}^{m-1} a_{2i-1}^{2n-1}) a_{2m-1}^{n-1} y_{m-1} a_{2m-2} a_{2m} (\prod_{i=2}^{m} a_{2m-(2i-1)}) \\
&\quad \text{(by zigzag equations)} \\
&= (\prod_{i=1}^{m-2} a_{2i-1}^{2n-1}) a_{2m-3}^{n-1} (a_{2m-3}^{n} (a_{2m-1}^{n-1} y_{m-1})(a_{2m-2} a_{2m}) a_{2m-3}) \\
&\quad (\prod_{i=3}^{m} a_{2m-(2i-1)}) \\
&= (\prod_{i=1}^{m-2} a_{2i-1}^{2n-1}) a_{2m-3}^{n-1} a_{2m-1}^{n-1} (y_{m-1} a_{2m-3}) a_{2m-2} a_{2m} (\prod_{i=3}^{m} a_{2m-(2i-1)}) \\
&\quad \text{(as } S \in \mathcal{V}\text{)} \\
&= (\prod_{i=1}^{m-2} a_{2i-1}^{2n-1}) a_{2m-3}^{n-1} a_{2m-1}^{n-1} y_{m-2} a_{2m-4} a_{2m-2} a_{2m} (\prod_{i=3}^{m} a_{2m-(2i-1)}) \\
&\quad \text{(by zigzag equations)} \\
&\;\vdots \\
&= a_1^{2n-1} a_3^{n-1} \cdots a_{2m-3}^{n-1} a_{2m-1}^{n-1} y_1 a_2 a_4 \cdots a_{2m-2} a_{2m} a_1 \\
&= a_1^{n-1} (a_1^{n} (a_3^{n-1} \cdots a_{2m-3}^{n-1} a_{2m-1}^{n-1} y_1)(a_2 a_4 \cdots a_{2m-2} a_{2m}) a_1)
\end{aligned}$$

$$= a_1^{n-1}a_3^{n-1}\cdots a_{2m-3}^{n-1}a_{2m-1}^{n-1}(y_1a_1)a_2a_4\cdots a_{2m-2}a_{2m}$$
(as $S \in \mathcal{V}$)
$$= a_1^{n-1}a_3^{n-1}\cdots a_{2m-3}^{n-1}a_{2m-1}^{n-1}a_0a_2a_4a_6\cdots a_{2m-2}a_{2m}$$
(by zigzag equations)
$$\in U$$

$\Rightarrow Dom(U,S) = U$.

Thus the proof of the theorem is complete. $\square$

Theorem 2.3. *The variety $\mathcal{V} = [axy = xa^nya]$ of semigroups, i.e. the class of all semigroups satisfying the identity $axy = xa^nya$, is closed.*

Proof. Take any U, $S \in \mathcal{V}$ with U as a subsemigroup of S such that $d \in Dom(U,S)\backslash U$. Let d has zigzag of type (1.1) in S over U of shortest possible length m. Now

$$d = (\prod_{i=1}^{m} a_{2i-1}^{2n-1})y_m a_{2m-1}t_m(\prod_{i=1}^{m} a_{2m-(2i-1)})$$
(by Lemma (2.1)).
$$= (\prod_{i=1}^{m-1} a_{2i-1}^{2n-1})(a_{2m-1}^{n-1}a_{2m-1}^{n}(y_m a_{2m-1}t_m)a_{2m-1})(\prod_{i=2}^{m} a_{2m-(2i-1)})$$
(for $n = 1$, we treat $a_{2m-1}^{n-1}a_{2m-1}^{n}$ as a_{2m-1}^{n})
$$= (\prod_{i=1}^{m-1} a_{2i-1}^{2n-1})(a_{2m-1}(a_{2m-1}^{n-1}y_m a_{2m-1})t_m)(\prod_{i=2}^{m} a_{2m-(2i-1)})$$
(as $S \in \mathcal{V}$)
$$= (\prod_{i=1}^{m-1} a_{2i-1}^{2n-1})a_{2m-1}^{n-1}y_m(a_{2m-1}a_{2m-1}^{n}t_m a_{2m-1})(\prod_{i=2}^{m} a_{2m-(2i-1)})$$
(as $S \in \mathcal{V}$)
$$= (\prod_{i=1}^{m-1} a_{2i-1}^{2n-1})a_{2m-1}^{n-1}(y_m a_{2m-1})(a_{2m-1}t_m)(\prod_{i=2}^{m} a_{2m-(2i-1)})$$
(as $S \in \mathcal{V}$)
$$= (\prod_{i=1}^{m-1} a_{2i-1}^{2n-1})a_{2m-1}^{n-1}y_{m-1}a_{2m-2}a_{2m}(\prod_{i=2}^{m} a_{2m-(2i-1)})$$
(by zigzag equations)

$$= (\prod_{i=1}^{m-2} a_{2i-1}^{2n-1})(a_{2m-3}^{n-1}a_{2m-3}^{n}(a_{2m-1}^{n-1}y_{m-1}a_{2m-2}a_{2m})a_{2m-3})$$
$$(\prod_{i=3}^{m} a_{2m-(2i-1)})$$
$$= (\prod_{i=1}^{m-2} a_{2i-1}^{2n-1})(a_{2m-3}(a_{2m-3}^{n-1}a_{2m-1}^{n-1}y_{m-1}a_{2m-2})a_{2m})(\prod_{i=3}^{m} a_{2m-(2i-1)})$$
(as $S \in \mathcal{V}$)
$$= (\prod_{i=1}^{m-2} a_{2i-1}^{2n-1})a_{2m-3}^{n-1}a_{2m-1}^{n-1}y_{m-1}(a_{2m-2}a_{2m-3}^{n}a_{2m}a_{2m-3})(\prod_{i=3}^{m} a_{2m-(2i-1)})$$
(as $S \in \mathcal{V}$)
$$= (\prod_{i=1}^{m-2} a_{2i-1}^{2n-1})a_{2m-3}^{n-1}a_{2m-1}^{n-1}(y_{m-1}a_{2m-3})a_{2m-2}a_{2m}(\prod_{i=3}^{m} a_{2m-(2i-1)})$$
(as $U \in \mathcal{V}$)
$$= (\prod_{i=1}^{m-2} a_{2i-1}^{2n-1})a_{2m-3}^{n-1}a_{2m-1}^{n-1}y_{m-2}a_{2m-4}a_{2m-2}a_{2m}(\prod_{i=3}^{m} a_{2m-(2i-1)})$$
(by zigzag equations)

$\vdots$

$$= a_1^{2n-1}a_3^{n-1}\cdots a_{2m-3}^{n-1}a_{2m-1}^{n-1}y_1a_2a_4\cdots a_{2m-2}a_{2m}a_1$$
$$= (a_1^{n-1}a_1^{n}(a_3^{n-1}\cdots a_{2m-3}^{n-1}a_{2m-1}^{n-1}y_1a_2a_4\cdots a_{2m-2}a_{2m})a_1)$$
$$= (a_1(a_1^{n-1}a_3^{n-1}\cdots a_{2m-3}^{n-1}a_{2m-1}^{n-1}y_1a_2)(a_4\cdots a_{2m-2}a_{2m}))$$
(as $S \in \mathcal{V}$)
$$= a_1^{n-1}a_3^{n-1}\cdots a_{2m-3}^{n-1}a_{2m-1}^{n-1}y_1(a_2a_1^{n}(a_4\cdots a_{2m-2}a_{2m})a_1)$$
(as $S \in \mathcal{V}$)
$$= a_1^{n-1}a_3^{n-1}\cdots a_{2m-3}^{n-1}a_{2m-1}^{n-1}(y_1a_1)a_2a_4\cdots a_{2m-2}a_{2m}$$
(as $U \in \mathcal{V}$)
$$= a_1^{n-1}a_3^{n-1}\cdots a_{2m-3}^{n-1}a_{2m-1}^{n-1}a_0a_2a_4\cdots a_{2m-2}a_{2m}$$
(by zigzag equations)
$$= a_1^{n-1}a_3^{n-1}\cdots a_{2m-3}^{n-1}a_{2m-1}^{n-1}(\prod_{i=0}^{m} a_{2i}).$$
$$\in U$$

$\Rightarrow Dom(U,S) = U$.

Thus the proof of the theorem is complete. $\square$

Dually, we may prove the following result.

Theorem 2.4. *The variety $\mathcal{V} = [axy = yay^n x]$ of semigroups, i.e. the class of all semigroups satisfying the identity $axy = yay^n x$, is closed.*

Lemma 2.5. *Let U be a subsemigroup of semigroup S such that S satisfies an identity $axy = xaya^n$ $[axy = xayx^n]$ and let $d \in Dom(U,S)\backslash U$ has a zigzag of type (1.1) in S over U with value d of shortest possible length m. Then*

$$d = (\prod_{i=1}^{k} a_{2i-1}) y_k a_{2k-1} t_k (\prod_{i=1}^{k} a_{2k-(2i-1)}^{2n-1}).$$

for each $k = 1, 2, ..., m$.

Proof. Let $\mathcal{V}_1 = [axy = xaya^n]$ and $\mathcal{V}_2 = [axy = xayx^n]$ be the varieties of semigroups. First we show that in both cases whether $S \in \mathcal{V}_1$ or $S \in \mathcal{V}_2$, S satisfies $xyz = yxyzy^{2n-1}$.

Case (i): When $S \in \mathcal{V}_1$, then for any $x, y, z \in S$, we have

$$\begin{aligned}
xyz &= (yxz)x^n \text{ (as } S \in \mathcal{V}_1 \text{)} \\
&= xyzy^n x^n \text{ (as } S \in \mathcal{V}_1 \text{)} \\
&= x(yzy)y^{n-1}x^n \text{ (for } n = 1, \text{ we treat } yy^{n-1} \text{ as } y) \\
&= (x(zyy)y^n)y^{n-1}x^n \text{ (as } S \in \mathcal{V}_1 \text{)} \\
&= zy(yxy^n x^n)y^{n-1}x^n \text{ (as } S \in \mathcal{V}_1 \text{)} \\
&= ((zy)x(yy^n y^{n-1})x^n) \text{ (as } S \in \mathcal{V}_1 \text{)} \\
&= x(z(yy)y^n)y^{n-1} \text{ (as } S \in \mathcal{V}_1 \text{)} \\
&= xy(yzy^n z^n)y^{n-1} \text{ (as } S \in \mathcal{V}_1 \text{)} \\
&= ((xyz)(yy^n)y^{n-1}) \text{ (as } S \in \mathcal{V}_1 \text{)} \\
&= y(y^n(xyz)y^{n-1}(xyz)^n) \text{ (as } S \in \mathcal{V}_1 \text{)} \\
&= yxyzy^n y^{n-1} \text{ (as } S \in \mathcal{V}_1 \text{)} \\
&= yxyzy^{2n-1}. \qquad (2.2)
\end{aligned}$$

Case (ii): When $S \in \mathcal{V}_2$, then for any $x, y, z \in S$, we have

$$\begin{aligned}
xyz &= yxzy^n \text{ (as } S \in \mathcal{V}_2 \text{)} \\
&= yx(zyy^{n-1}) \text{ (for } n = 1, \text{ we treat } yy^{n-1} \text{ as } y) \\
&= yxyzy^{n-1}y^n \text{ (as } S \in \mathcal{V}_2 \text{)}
\end{aligned}$$

$$= yxyzy^{2n-1} \text{ (as } S \in \mathcal{V}_2 \text{).}$$

Thus the claim is proved.

Now, we shall prove the lemma by using induction on k. Let U be a subsemigroup of semigroup S such that S belongs to either $S \in \mathcal{V}_1$ or $S \in \mathcal{V}_2$ and let $d \in Dom(U,S)\backslash U$ has a zigzag of type (1.1) in S over U with value d of shortest possible length m. Now, for $k = 1$, we have

$$\begin{aligned} d &= y_1 a_1 t_1 \text{ (by zigzag equations)} \\ &= a_1 y_1 a_1 t_1 a_1^{2n-1} \text{ (by equation (2.2)).} \end{aligned}$$

Thus the result holds for $k = 1$. Assume inductively that the result holds for $k = j < m$. Then we shall show that it also holds for $k = j+1$. Now

$$\begin{aligned} d &= (\prod_{i=1}^{j} a_{2i-1}) y_j a_{2j-1} t_j (\prod_{i=1}^{j} a_{2j-(2i-1)}^{2n-1}) \\ &\quad \text{(by inductive hypothesis)} \\ &= (\prod_{i=1}^{j} a_{2i-1}) y_{j+1} a_{2j+1} t_{j+1} (\prod_{i=1}^{j} a_{2j-(2i-1)}^{2n-1}) \\ &\quad \text{(by zigzag equations)} \\ &= (\prod_{i=1}^{j} a_{2i-1}) a_{2j+1} y_{j+1} a_{2j+1} t_{j+1} a_{2j+1}^{2n-1} (\prod_{i=1}^{j} a_{2j-(2i-1)}^{2n-1}) \\ &\quad \text{(by equation (2.2))} \\ &= (\prod_{i=1}^{j+1} a_{2i-1}) y_{j+1} a_{2j+1} t_{j+1} (\prod_{i=1}^{j+1} a_{2(j+1)-(2i-1)}^{2n-1}), \end{aligned}$$

as required and, by induction, the lemma is established. □

Theorem 2.6. *The variety $\mathcal{V} = [axy = xaya^n]$ of semigroups, i.e. the class of all semigroups satisfying the identity $axy = xaya^n$, is closed.*

Proof. Take any U, $S \in \mathcal{V}$ with U as a subsemigroup of S such that $d \in Dom(U,S)\backslash U$. Let d has zigzag of type (1.1) in S over U of shortest possible length m. Now

$$
\begin{aligned}
d &= (\prod_{i=1}^{m} a_{2i-1}) y_m a_{2m-1} t_m (\prod_{i=1}^{m} a_{2m-(2i-1)}^{2n-1}) \\
&\quad \text{(by Lemma (2.5)).} \\
&= ((\prod_{i=1}^{m-1} a_{2i-1}) a_{2m-1} (y_m a_{2m-1} t_m) a_{2m-1}^{n}) a_{2m-1}^{n-1} (\prod_{i=2}^{m} a_{2m-(2i-1)}^{2n-1}) \\
&\quad \text{(for } n=1 \text{, we treat } a_{2m-1}^{n-1} a_{2m-1}^{n} \text{ as } a_{2m-1}^{n}) \\
&= (a_{2m-1}((\prod_{i=1}^{m-1} a_{2i-1}) y_m a_{2m-1}) t_m) a_{2m-1}^{n-1} (\prod_{i=2}^{m} a_{2m-(2i-1)}^{2n-1}) \\
&\quad \text{(as } S \in \mathcal{V} \text{)} \\
&= (\prod_{i=1}^{m-1} a_{2i-1}) y_m (a_{2m-1} a_{2m-1} t_m a_{2m-1}^{n}) a_{2m-1}^{n-1}) (\prod_{i=2}^{m} a_{2m-(2i-1)}^{2n-1}) \\
&\quad \text{(as } S \in \mathcal{V} \text{)} \\
&= (\prod_{i=1}^{m-1} a_{2i-1}) (y_m a_{2m-1}) (a_{2m-1} t_m) a_{2m-1}^{n-1} (\prod_{i=2}^{m} a_{2m-(2i-1)}^{2n-1}) \\
&\quad \text{(as } S \in \mathcal{V} \text{)} \\
&= (\prod_{i=1}^{m-1} a_{2i-1}) y_{m-1} a_{2m-2} a_{2m} a_{2m-1}^{n-1} (\prod_{i=2}^{m} a_{2m-(2i-1)}^{2n-1}) \\
&\quad \text{(by zigzag equations)} \\
&= ((\prod_{i=1}^{m-2} a_{2i-1}) a_{2m-3} (y_{m-1} a_{2m-2} a_{2m} a_{2m-1}^{n-1}) a_{2m-3}^{n}) a_{2m-3}^{n-1} \\
&\quad (\prod_{i=3}^{m} a_{2m-(2i-1)}^{2n-1}) \\
&= (a_{2m-3}((\prod_{i=1}^{m-2} a_{2i-1}) y_{m-1} a_{2m-2}) a_{2m}) a_{2m-1}^{n-1} a_{2m-3}^{n-1} (\prod_{i=3}^{m} a_{2m-(2i-1)}^{2n-1}) \\
&\quad \text{(as } S \in \mathcal{V} \text{)} \\
&= (\prod_{i=1}^{m-2} a_{2i-1}) y_{m-1} (a_{2m-2} a_{2m-3} a_{2m} a_{2m-3}^{n}) a_{2m-1}^{n-1} a_{2m-3}^{n-1} (\prod_{i=3}^{m} a_{2m-(2i-1)}^{2n-1}) \\
&\quad \text{(as } S \in \mathcal{V} \text{)}
\end{aligned}
$$

$$= (\prod_{i=1}^{m-2} a_{2i-1})(y_{m-1}a_{2m-3})a_{2m-2}a_{2m}a_{2m-1}^{n-1}a_{2m-3}^{n-1}(\prod_{i=3}^{m} a_{2m-(2i-1)}^{2n-1})$$
(as $U \in \mathcal{V}$)
$$= (\prod_{i=1}^{m-2} a_{2i-1})y_{m-2}a_{2m-4}a_{2m-2}a_{2m}a_{2m-1}^{n-1}a_{2m-3}^{n-1}(\prod_{i=3}^{m} a_{2m-(2i-1)}^{2n-1})$$
(by zigzag equations)

$\vdots$

$$= a_1y_1a_2a_4\cdots a_{2m-2}a_{2m}a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1}a_1^{2n-1}$$
$$= (a_1(y_1a_2)(a_4\cdots a_{2m-2}a_{2m}))a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1}a_1^{n}a_1^{n-1}$$
$$= y_1(a_2a_1(a_4\cdots a_{2m-2}a_{2m})a_1^{n})a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1}a_1^{n}a_1^{n-1}$$
(as $S \in \mathcal{V}$)
$$= (y_1a_1(a_2a_4\cdots a_{2m-2}a_{2m}a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1})a_1^{n})a_1^{n-1}$$
(as $U \in \mathcal{V}$)
$$= (a_1(y_1a_2a_4)(a_6\cdots a_{2m-2}a_{2m}))a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1}a_1^{n-1}$$
(as $S \in \mathcal{V}$)
$$= y_1((a_2a_4)a_1(a_6\cdots a_{2m-2}a_{2m})a_1^{n})a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1}a_1^{n-1}$$
(as $S \in \mathcal{V}$)
$$= (y_1a_1)a_2a_4a_6\cdots a_{2m-2}a_{2m}a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1}a_1^{n-1}$$
(as $U \in \mathcal{V}$)
$$= a_0a_2a_4a_6\cdots a_{2m-2}a_{2m}a_{2m-1}^{n-1}a_{2m-3}^{n-1}\cdots a_3^{n-1}a_1^{n-1}$$
(by zigzag equations)
$\in U$

$\Rightarrow Dom(U,S) = U$.
Thus the proof of the theorem is complete. □

Dually, we may prove the following result.

Theorem 2.7. *The variety $\mathcal{V} = [axy = y^n ayx]$ of semigroups, i.e. the class of all semigroups satisfying the identity $axy = y^n ayx$, is closed.*

Theorem 2.8. *The variety $\mathcal{V} = [axy = xayx^n]$ of semigroups, i.e. the class of all semigroups satisfying the identity $axy = xayx^n$, is closed.*

Proof. Take any U, $S \in \mathcal{V}$ with U as a subsemigroup of S such that $d \in Dom(U,S)\backslash U$. Let d has zigzag of type (1.1) in S over U of shortest possible length m. Now

$$
\begin{aligned}
d &= (\prod_{i=1}^{m} a_{2i-1}) y_m a_{2m-1} t_m (\prod_{i=1}^{m} a_{2m-(2i-1)}^{2n-1}) \\
&\quad \text{(by Lemma (2.5))} \\
&= (\prod_{i=1}^{m-1} a_{2i-1})(a_{2m-1} y_m (a_{2m-1} t_m) a_{2m-1}^{n}) a_{2m-1}^{n-1} (\prod_{i=2}^{m} a_{2m-(2i-1)}^{2n-1}) \\
&\quad \text{(for } n=1\text{, we treat } a_{2m-1}^{n} a_{2m-1}^{n-1} \text{ as } a_{2m-1}^{n}) \\
&= (\prod_{i=1}^{m-1} a_{2i-1})(y_m a_{2m-1})(a_{2m-1} t_m) a_{2m-1}^{n-1} (\prod_{i=2}^{m} a_{2m-(2i-1)}^{2n-1}) \\
&\quad \text{(as } S \in \mathcal{V} \text{)} \\
&= (\prod_{i=1}^{m-1} a_{2i-1}) y_{m-1} a_{2m-2} a_{2m} a_{2m-1}^{n-1} (\prod_{i=2}^{m} a_{2m-(2i-1)}^{2n-1}) \\
&\quad \text{(by zigzag equations)} \\
&= (\prod_{i=1}^{m-2} a_{2i-1})(a_{2m-3} y_{m-1} (a_{2m-2} a_{2m} a_{2m-1}^{n-1}) a_{2m-3}^{n}) a_{2m-3}^{n-1} \\
&\quad (\prod_{i=3}^{m} a_{2m-(2i-1)}^{2n-1}) \\
&= (\prod_{i=1}^{m-2} a_{2i-1})(y_{m-1} a_{2m-3}) a_{2m-2} a_{2m} a_{2m-1}^{n-1} a_{2m-3}^{n-1} (\prod_{i=3}^{m} a_{2m-(2i-1)}^{2n-1}) \\
&\quad \text{(as } S \in \mathcal{V} \text{)} \\
&= (\prod_{i=1}^{m-2} a_{2i-1}) y_{m-2} a_{2m-4} a_{2m-2} a_{2m} a_{2m-1}^{n-1} a_{2m-3}^{n-1} (\prod_{i=3}^{m} a_{2m-(2i-1)}^{2n-1}) \\
&\quad \text{(by zigzag equations)} \\
&\ \vdots \\
&= a_1 y_1 a_2 a_4 \cdots a_{2m-2} a_{2m} a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1} a_1^{2n-1} \\
&= (a_1 y_1 (a_2 a_4 \cdots a_{2m-2} a_{2m} a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1}) a_1^{n}) a_1^{n-1} \\
&= (y_1 a_1) a_2 a_4 \cdots a_{2m-2} a_{2m} a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1} a_1^{n-1} \\
&\quad \text{(as } S \in \mathcal{V} \text{)} \\
&= a_0 a_2 a_4 \cdots a_{2m-2} a_{2m} a_{2m-1}^{n-1} a_{2m-3}^{n-1} \cdots a_3^{n-1} a_1^{n-1} \\
&\quad \text{(by zigzag equations)}
\end{aligned}
$$

$\in U$

$\Rightarrow Dom(U, S) = U$.

Thus the proof of the theorem is complete. □

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ON CLOSED HOMOTYPICAL VARIETIES OF SEMIGROUPS-2

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بررسی واریته‌های همسان از نیم‌گروه‌ها-۲

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در این مقاله ما نتایج مقاله

"On Closed Homotypical Varieties of Semigroups"

را تعمیم داده و نشان می‌دهیم که واریته‌های همسان از نیم‌گروه‌ها تحت یکسانی‌های $axy = x^nayx$، $axy = xa^nya$ $[axy = yay^nx]$، $axy = xaya^n$ $[axy = y^nayx]$ و $axy = xayx^n$ برای هر عدد طبیعی n، بسته هستند.

کلمات کلیدی: معادلات زیگزاگ، همسان، واریته، یکسانی، بسته.