Journal of Algebraic Systems
Vol. 12, No. 2, (2025), pp 283-299

REGULAR FILTERS OF DISTRIBUTIVE LATTICES

M. SAMBASIVA RAO* AND A. P. PHANEENDRA KUMAR

Abstract. The concepts of regular filters and π-filters are introduced in distributive lattices. A set of equivalent conditions is given for a D-filter to become a regular filter. For every D-filter, it is proved that there exists a homomorphism whose dense kernel is a regular filter. π-filters are characterized in terms of regular filters and congruences. Some equivalent conditions are given for the space of all prime π-filters to become a Hausdorff space.

Introduction

Many authors introduced the concept of annihilators in the structures of rings as well as lattices and characterized several algebraic structures in terms of annihilators. In their research, T. P. Speed [11] and W. H. Cornish [4] made an extensive study of annihilators in distributive lattices. In [5], some properties of minimal prime filters are studied in distributive lattice and the properties of dense elements and D-filters are studied in MS-algebras [9]. In [2], the notion of D-filters was introduced in pseudo-complemented semi-lattices. Later it was generalized by the author [9] in MS-algebras.

The main aim of this paper is to study some further properties of dense elements and D-filters in the form of regular filters and π-filters of distributive lattices. Some equivalent conditions are established for

DOI: 10.22044/JAS.2023.12585.1674.
MSC(2010): 06D30.
Keywords: Regular filter; Minimal prime D-filter; π-filter; Dense element; Relatively complemented lattice; Hausdorff space.
Received: 10 January 2023, Accepted: 3 March 2023.
*Corresponding author.

a D-filter to become a regular filter. For a D-filter of a distributive lattice, it is proved that there exists a homomorphism whose dense kernel is a regular filter. Finally, a sufficient condition is derived, in terms of regular filters, for every distributive lattice to become relatively complemented. A set of equivalent conditions are derived for a distributive lattice to become a Boolean algebra. Some topological properties of the space of all prime π-filters of distributive lattices are also studied.

1. Preliminaries

The reader is referred to [1] and [3] for the elementary notions and notations of distributive lattices. Some of the preliminary definitions and results of [9] and [8] are presented for the ready reference.

Definition 1.1. [1] An algebra $(L, \wedge, \vee)$ of type $(2, 2)$ is called a distributive lattice if for all $x, y, z \in L$, it satisfies the following properties $(1), (2), (3)$ and (4) along with (5) or $(5')$

(1) $x \wedge x = x$, $x \vee x = x$,
(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$,
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$,
(4) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$,
(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
$(5')$ $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

A non-empty subset A of a lattice L is called an *ideal* (*filter*) of L if $a \vee b \in A$ $(a \wedge b \in A)$ and $a \wedge x \in A$ $(a \vee x \in A)$ whenever $a, b \in A$ and $x \in L$. The set $\mathcal{I}(L)$ of all ideals of the lattice $(L, \vee, \wedge, 0)$ forms a complete distributive lattice as well as the set $\mathcal{F}(L)$ of all filters of the lattice $(L, \vee, \wedge, 1)$ forms a complete distributive lattice. A proper ideal (filter) M of a lattice is called *maximal* if there exists no proper ideal (filter) N such that $M \subset N$.

Definition 1.2. [3] Let $(L, \wedge, \vee)$ be a lattice. A partial ordering relation $\leq$ is defined on L by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$. In this case, the pair $(L, \leq)$ is called a partially ordered set. If $x \leq y$ or $y \leq x$ for all $x, y \in L$, then $(L, \leq)$ is called a totally ordered set.

The set $(a] = \{x \in L \mid x \leq a\}$ is called a *principal ideal* generated by the element a and the set of all principal ideals is a sublattice of $\mathcal{I}(L)$. Dually the set $[a) = \{x \in L \mid a \leq x\}$ is called a *principal filter* generated by the element a and the set of all principal filters is a sublattice of $\mathcal{F}(L)$. A proper ideal (proper filter) P of a lattice L is called *prime* if for all $a, b \in L$, $a \wedge b \in P$ ($a \vee b \in P$) then $a \in P$ or $b \in P$. Every maximal filter is a prime filter.

Theorem 1.3. [1] *Let F be a filter and I an ideal of a distributive lattice L such that $F \cap I = \emptyset$, then there exists a prime filter P of L such that $F \subseteq P$ and $P \cap I = \emptyset$.*

For any element a of a distributive lattice L, the *annihilator* of a is defined as the set $(a)^* = \{\ x \in L \mid x \wedge a = 0\ \}$. An element a of a lattice L is called a *dense element* if $(a)^* = \{0\}$. The set D of all dense elements of a lattice L forms a filter of L.

Definition 1.4. [9] A filter F of a lattice L is called a D-filter if $D \subseteq F$.

The set D of all dense elements of a distributive lattice is the smallest D-filter of the lattice. For any subset A of a lattice L, define

$$A^\circ = \{x \in L \mid a \vee x \in D \text{ for all } a \in A\}.$$

Clearly $L^\circ = D$ and $D^\circ = L$. It can also be observed that $D \subseteq A^\circ$ for any subset A of a lattice L. For any $a \in L$, we simply represent $(\{a\})^\circ$ by $(a)^\circ$. Then it is obvious that $(1)^\circ = L$. For any subset A of L, A° is a D-filter of L.

Lemma 1.5. [8] *For any subsets A, B of a distributive lattice L,*

(1) $A \subseteq B$ *implies* $B^\circ \subseteq A^\circ$,
(2) $A \subseteq A^{\circ\circ}$,
(3) $A^{\circ\circ\circ} = A^\circ$,
(4) $A^\circ = L \Leftrightarrow A \subseteq D$.

Proposition 1.6. [8] *For any filters F, G, H of a distributive lattice L,*

(1) $F^\circ \cap F^{\circ\circ} = D$,
(2) $F \cap G \subseteq D$ *implies* $F \subseteq G^\circ$,
(3) $(F \vee G)^\circ = F^\circ \cap G^\circ$,
(4) $(F \cap G)^{\circ\circ} = F^{\circ\circ} \cap G^{\circ\circ}$.

It is clear that $([x))^\circ = (x)^\circ$. Then clearly $(0)^\circ = D$. The following corollary is a direct consequence of the above results.

Corollary 1.7. [8] *Let L be a distributive lattice. For any $a, b, c \in L$,*

(1) $a \leq b$ *implies* $(a)^\circ \subseteq (b)^\circ$,
(2) $(a \wedge b)^\circ = (a)^\circ \cap (b)^\circ$,
(3) $(a \vee b)^{\circ\circ} = (a)^{\circ\circ} \cap (b)^{\circ\circ}$,
(4) $(a)^\circ = L$ *if and only if a is dense.*

Proposition 1.8. [8] *For any D-filter F of a distributive lattice L,*

(1) $F \cap F^{\circ} = D$
(2) $F^{\circ} = \bigcap\{P \mid P$ *is a prime D-filter such that* $F \nsubseteq P\}$
(3) *For any ideal I of L such that $F \cap I = \emptyset$, there exists a prime D-filter P of L such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Let F be a D-filter and $x \notin F$. Then there exists a prime D-filter P such that $F \subseteq P$ and $x \notin P$. A prime D-filter P of a lattice L is called minimal if there exists no prime D-filter Q such that $Q \subset P$.

Theorem 1.9. [8] *A prime D-filter P of a lattice L is minimal if and only if to each $x \in P$, there exists $y \notin P$ such that $x \vee y \in D$.*

Throughout this article, all lattices are bounded distributive lattices unless otherwise mentioned.

2. Regular filters of lattices

In this section, the concept of regular filters is introduced in lattices. Some properties of D-filters are observed. A set of equivalent conditions is derived for a D-filter of a lattice to become a regular filter.

Definition 2.1. A filter F of a lattice L is called *regular* if $F = F^{\circ\circ}$.

Obviously, A° is a regular filter for each subset A of a lattice L. It is also clear that D and L are both regular filters. Since $D \subseteq F^{\circ\circ} = F$ for any filter F, it is obvious that every regular filter is a D-filter. Moreover, for any $E \subseteq D$, it is clear that $L = D^{\circ} \subseteq E^{\circ}$. Hence $E^{\circ\circ} = L^{\circ} = D$. Therefore D is the smallest regular filter in the lattice L. Let $\mathcal{RF}(L)$ be the set of all regular filters of L.

Theorem 2.2. *For any lattice L, the class $\mathcal{RF}(L)$ of all regular filters of L forms a complete Boolean algebra.*

Proof. Clearly $(\mathcal{RF}(L), \subseteq)$ is a partially ordered set, where $\subseteq$ is the set-inclusion. Let $F, G \in \mathcal{RF}(L)$. Then clearly $F^{\circ\circ} \cap G^{\circ\circ} = (F \cap G)^{\circ\circ}$ is the infimum of both F and G in $\mathcal{RF}(L)$. Again, define the binary operation $\sqcup$ on $\mathcal{RF}(L)$ as follows:

$$F \sqcup G = (F^{\circ} \cap G^{\circ})^{\circ}$$

It can be easily observed that $(F^{\circ} \cap G^{\circ})^{\circ}$ is the supremum for F and G in $\mathcal{RF}(L)$. Clearly D and L are the least and greatest elements in $\mathcal{RF}(L)$. Hence $(\mathcal{RF}(L), \cap, \sqcup, D, L)$ is a bounded distributive lattice. Now, for any $F \in \mathcal{RF}(L)$, $F \cap F^{\circ} = F^{\circ\circ} \cap F^{\circ} = D$ and

$$F \sqcup F^{\circ} = (F^{\circ} \cap F^{\circ\circ})^{\circ} = D^{\circ} = L.$$

Hence F° is the unique complement of F in $\mathcal{RF}(L)$. Therefore $(\mathcal{RF}(L), \cap, \sqcup, D, L, ^\circ)$ is a complete Boolean algebra. $\square$

For any $x \in L$, $(x)^\circ$ is a regular filter and hence for any two $(x)^\circ$ and $(y)^\circ$ their supremum in $\mathcal{RF}(L)$ is

$$(x)^\circ \sqcup (y)^\circ = ((x)^{\circ\circ} \cap (y)^{\circ\circ})^\circ = (x \vee y)^{\circ\circ\circ} = (x \vee y)^\circ$$

Also their infimum in $\mathcal{RF}(L)$ is $(x)^\circ \cap (y)^\circ = (x \wedge y)^\circ$. We are thus lead to the following result, which is a direct consequence of the above observation.

Theorem 2.3. *Let L be a lattice. Then the class $\mathcal{RF}_\circ(L)$ of all regular filters of the form $(x)^\circ, x \in L$ is a lattice $\langle \mathcal{RF}_\circ(L), \cap, \sqcup \rangle$ and sublattice of the distributive lattice $\langle \mathcal{RF}(L), \cap, \sqcup \rangle$ of all regular filters of L. Moreover, $\mathcal{RF}_\circ(L)$ has the greatest element $L = (d)^\circ$ for arbitrary $d \in D$ while $\mathcal{RF}_\circ(L)$ has the smallest element $(0)^\circ = D$.*

Theorem 2.4. *Let F be a D-filter of a lattice L. Then $F \vee F^\circ = L$ if and only if F is regular and $F^\circ \vee F^{\circ\circ} = L$.*

Proof. Assume that $F \vee F^\circ = L$ for any D-filter F of L. Then

$$\begin{aligned} F^{\circ\circ} &= F^{\circ\circ} \cap L \\ &= F^{\circ\circ} \cap (F \vee F^\circ) \\ &= (F^{\circ\circ} \cap F) \vee (F^{\circ\circ} \cap F^\circ) \\ &= F \vee D \\ &= F. \end{aligned}$$

Hence F is regular. Also $F^\circ \vee F^{\circ\circ} = F^\circ \vee F = L$. The converse is clear. $\square$

In the following result, a set of equivalent conditions is derived for a prime D-filter of a lattice to become a minimal prime D-filter.

Theorem 2.5. *The following assertions are equivalent in a lattice L:*

(1) *every prime D-filter is minimal;*
(2) *for each $x \in L, [x) \vee (x)^\circ = L$;*
(3) *for each $x \in L, [x) = (x)^{\circ\circ}$ and $(x)^\circ \vee (x)^{\circ\circ} = L$.*

Proof. (1) $\Rightarrow$ (2) : Assume condition (1). Let $x \in L$. Suppose $[x) \vee (x)^\circ \neq L$. Then there exists a prime filter P such that

$$[x) \vee (x)^\circ \subseteq P.$$

Since $(x)^\circ$ is a D-filter, we get that P is a D-filter. Hence by the hypothesis, P is minimal. Since $(x)^\circ \subseteq P$, by Theorem 1.8(2), we get that $x \notin P$, which is a contradiction. Therefore $[x) \vee (x)^\circ = L$.

(2) $\Rightarrow$ (3) : From Theorem 2.4, it is clear.

(3) $\Rightarrow$ (1) : Assume the condition (3). Let P be a prime D-filter. Suppose there exists a prime D-filter Q such that $Q \subset P$. Then, choose $x \in P - Q$. Since $x \notin Q$, we get that $(x)^\circ \subseteq Q$. Since $x \in P$, by the assumed condition we get

$$L = (x)^{\circ\circ} \vee (x)^\circ = [x) \vee (x)^\circ \subseteq P \vee Q = P$$

which is a contradiction. Thus P is a minimal prime D-filter of L. $\square$

A filter F of a lattice L is called *condensed* if $F^\circ = D$. Then clearly the set of all condensed filters of a lattice L forms a sublattice to the lattice of all filters of L. It can be easily seen that a proper condensed filter will never be a regular filter. It can also be observed that every regular filter is a D-filter. The converse of this is not true in general. i.e. not every D-filter of a lattice has to be a regular filter. However, in the following theorem, some equivalent conditions are derived for every D-filter of a lattice to become a regular filter.

Theorem 2.6. *Let L be a lattice in which every proper filter is non-condensed. Then the following assertions are equivalent:*

(1) *every D-filter is a regular filter;*
(2) *every prime D-filter is a regular filter;*
(3) *every prime D-filter is minimal;*
(4) *every prime D-filter is maximal.*

Proof. (1) $\Rightarrow$ (2): It is obvious.

(2) $\Rightarrow$ (3): Assume that every prime D-filter is a regular filter. Let P be a prime D-filter of L. Then $P^{\circ\circ} = P$. Suppose P is not a minimal prime D-filter. Then there exists a prime D-filter Q such that $Q \subset P$. Choose $x \in P - Q$. Let $a \in P^\circ$. Since $x \in P$, we get that $a \vee x \in D \subseteq Q$. Since Q is prime and $x \notin Q$, it yield that $a \in Q \subset P$. Hence $P^\circ \subseteq P \subseteq P^{\circ\circ}$. Thus $P^\circ = P^\circ \cap P^{\circ\circ} = D$. Hence $P = P^{\circ\circ} = L$, which is a contradiction. Therefore P is a minimal prime D-filter of L.

(3) $\Rightarrow$ (4): Since every maximal D-filter is prime, this is clear.

(4) $\Rightarrow$ (1): Assume that every prime D-filter is maximal. Let F be a non-dense filter. Clearly $F \subseteq F^{\circ\circ}$. Let $x \in F^{\circ\circ}$. Then we get $F^\circ \subseteq (x)^\circ$. Suppose $x \notin F$. Then there exists a prime D-filter P such that $F \subseteq P$ and $x \notin P$. By the condition (4), P is maximal. Since $x \notin P$, we get $P \vee [x) = L$. Hence $P^\circ \cap (x)^\circ = (P \vee [x))^\circ = L^\circ = D$. Thus $P^\circ = P^\circ \cap F^\circ = D$, which is a contradiction. Hence $x \in F$ and thus $F^{\circ\circ} \subseteq F$. Therefore F is a regular filter of L. $\square$

For any filter F of a bounded lattice L, denote the set of all homomorphisms defined on F by $Hom_L(F)$. It can be easily observed that $Hom_L(F)$ is a distributive lattice with respect to the point-wise operations. Then the following proposition can be routinely verified.

Proposition 2.7. *Let F be a filter of a lattice L. For any $r \in L$, consider $\phi_r : F \longrightarrow F$ by $\phi_r(x) = x \vee r$ for all $x \in F$. Then we have the following properties hold:*

(1) *ϕ_r is a homomorphism,*
(2) *$\phi_{r\wedge s} = \phi_r \wedge \phi_s$ for $r, s \in L$,*
(3) *$\phi_{r\vee s} = \phi_r \vee \phi_s$ for $r, s \in L$.*

Let F be a D-filter of a bounded lattice L. A homomorphism $\upsilon : F \to F$ is called dense-valued if $\upsilon(x)$ is dense for all $x \in F$. Now, consider the set of all $\upsilon \in Hom_L(F)$ such that υ is a dense-valued homomorphism denoted by $\mathcal{D}(F)$. It is clear that the unit element of $Hom_L(F)$ is in $\mathcal{D}(F)$. In fact, the map $\mathbf{1} : F \to F$ given by $\mathbf{1}(x) = x$ for all $x \in F$, is a dense-valued homomorphism. Hence $\mathbf{1} \in \mathcal{D}(F)$. Moreover, it can be easily observed that $\mathcal{D}(F)$ is a filter of $Hom_L(F)$. Also, $\Phi_d \in \mathcal{D}(F)$ for all $d \in D$.

Definition 2.8. Let F be a D-filter of a lattice L and

$$f : L \longrightarrow Hom_L(F)$$

a homomorphism. Then define the dense-kernel $Ker^D(f)$ of the homomorphism f as $Ker^D(f) = \{x \in L \mid f(x) \in \mathcal{D}(F)\}$. Also, define a map $\Phi_F : L \to Hom_L(F)$ given by $\Phi_F(r) = \Phi_r$ for all $r \in L$.

Lemma 2.9. *Let F be a D-filter of a lattice L and $f : L \longrightarrow Hom_L(F)$ a homomorphism. Then $Ker^D(f)$ is a filter in L.*

Theorem 2.10. *For any D-filter F of a bounded lattice L, we have*

$$Ker^D(\Phi_F) = F^\circ$$

So, F° can be considered as the dense-kernel of a homomorphism.

Proof. Assume that $r \in Ker^D(\Phi_F)$. Then $\Phi_r \in \mathcal{D}(F)$, hence $x \vee r = \Phi_r(x)$ is dense for all $x \in F$. Hence $r \in F^\circ$. Conversely, let $r \in F^\circ$. Then $x \vee r \in D$ for all $x \in F$. Thus Φ_r is dense for all $x \in F$. Thus $\Phi_F(r) = \Phi_r \in \mathcal{D}(F)$. Hence $r \in Ker^D(\Phi_F)$. $\square$

Theorem 2.11. *If every D-filter of a lattice L is a regular filter, then any two prime D-filters are incomparable.*

Proof. Assume that every D-filter is a regular filter. Suppose that there exists two distinct prime D-filters P, Q such that $P \subset Q$. Choose

$q \in Q - P$. For any $x \in Q^\circ$, we have $x \vee q \in D \subseteq P$. Hence $x \in P$ because of P is prime and $q \notin P$. Thus $Q^\circ \subseteq P \subseteq Q$. Hence $Q^\circ = Q \cap Q^\circ = D$. Since every D-filter is regular, Q is regular. So $Q = Q^{\circ\circ} = D^\circ = L$, which is a contradiction. Hence any two prime D-filters are incomparable. $\square$

In the following theorem, a sufficient condition is derived, in terms of regular filters, for a lattice to become relatively complemented.

Theorem 2.12. *Let L be a lattice in which every principal filter is a D-filter. If every D-filter is a regular filter, then L is relatively complemented.*

Proof. Let L be a lattice in which every principal filter is a D-filter. Assume that every D-filter of L is a regular filter. Suppose L is not relatively complemented. Then there exists three elements $a, b, c \in L$ such that $b < c < a$ and c has no complement in the interval $[b, a]$. Consider the set $I = \{x \in L \mid c \wedge x \leq b\}$. It is easy to check that I is an ideal in L. Now consider the ideal $E = I \vee (c]$. Suppose $a \in E$. Then, we can write $a = c \vee i$ for some $i \in I$. Now

$$\begin{aligned} a &= a \vee b \\ &= (c \vee i) \vee b \\ &= c \vee (i \vee b) \qquad\qquad \longrightarrow (1) \end{aligned}$$

Again we get the following:

$$\begin{aligned} (i \vee b) \wedge c &= (i \wedge c) \vee (b \wedge c) \\ &= (c \wedge i) \vee b \\ &= b \qquad \text{since } i \in I \qquad\qquad \longrightarrow (2) \end{aligned}$$

From(1) and (2), we get that $i \vee b$ is a relative complement of c in $[b, a]$, which is a contradiction. Hence $a \notin E$. Therefore $[a) \cap E = \emptyset$. Since $[a)$ is a D-filter, by Proposition 1.8(3), there exists a prime D-filter P of L such that $[a) \subseteq P$ and $P \cap E = \emptyset$. Hence $P \cap I = \emptyset$. Now

$$\begin{aligned} P \cap E = \emptyset \;&\Rightarrow\; P \cap \{I \vee (c]\} = \emptyset \\ &\Rightarrow\; P \cap I = \emptyset \text{ and } P \cap (c] = \emptyset. \end{aligned}$$

Now consider $F = [c) \vee P$. Clearly F is a D-filter in L. Suppose $b \in F$. Then

$$\begin{aligned} b \in [c) \vee P \;&\Rightarrow\; b = c \wedge p \text{ for some } p \in P \\ &\Rightarrow\; p \in I. \end{aligned}$$

Hence $p \in P \cap I$, which is a contradiction. Hence $b \notin F$. Therefore $F \cap (b] = \emptyset$. Thus, by Proposition 1.8(3), there exists a prime D-filter

Q such that $F \subseteq Q$ and $(b] \cap Q = \emptyset$. Thus $P \subset F \subseteq Q$. Hence P and Q are distinct prime D-filters such that $P \subset Q$. Therefore there exists two prime D-filters which are comparable. Hence by the above theorem, we conclude that L is relatively complemented. □

3. π-FILTERS OF LATTICES

In this section, the concept of π-filters is introduced in a lattices and then these class of filters is characterized by using the regular filters and congruences. Some equivalent conditions are derived for a lattice to become a Boolean algebra.

Definition 3.1. A filter F of a lattice L is called a *π-filter* if $x \in F$ implies $(x)^{\circ\circ} \subseteq F$ for all $x \in L$.

For any $x \in D$, it is clear that $(x)^{\circ} = L$ and hence $(x)^{\circ\circ} = D$. Therefore D is a π-filter and also it is the smallest π-filter in the lattice L. Every regular filter is a π-filter. For, consider a regular filter F. Let $x \in F$. Then $(x)^{\circ\circ} \subseteq F^{\circ\circ} = F$. Therefore F is a π-filter. However, the converse is not true. A π-filter F satisfying the property $F^{\circ} = D$ is not a regular filter because of $F^{\circ\circ} = D^{\circ} = L \neq F$.

Proposition 3.2. *Every minimal prime D-filter is a π-filter.*

Proof. Let P be a minimal prime D-filter of a lattice L. Let $x \in P$. Since P is minimal, we get that $x \vee y \in D$ for some $y \notin P$. Let $t \in (x)^{\circ\circ}$. Then $(x)^{\circ} \subseteq (t)^{\circ}$. Hence $y \in (t)^{\circ}$. Thus $t \in (t)^{\circ\circ} \subseteq (y)^{\circ} \subseteq P$ because of $y \notin P$. Hence $(x)^{\circ\circ} \subseteq P$. Therefore P is a π-filter of L. □

Definition 3.3. For any filter F of a lattice L, define an extension to F as given by $F^{e} = \{x \in L \mid (a)^{\circ} \subseteq (x)^{\circ} \text{ for some } a \in F\}$.

It can be easily observed that $D \subseteq F^{e}$ and $D^{e} = D$. Moreover, the following result is a direct consequence from the above definition.

Lemma 3.4. *For any two filters F, G of a lattice L, the following properties hold:*

(1) *$F \subseteq G$ implies $F^{e} \subseteq G^{e}$,*
(2) *$(F \cap G)^{e} = F^{e} \cap G^{e}$,*
(3) *$(F^{e})^{e} = F^{e}$.*

Proposition 3.5. *For any filter F of lattice L, F^{e} is the smallest π-filter containing F.*

Proof. Obviously $D \subseteq F^{e}$. Let $x, y \in F^{e}$. Then there exist $a, b \in F$ such that $(a)^{\circ} \subseteq (x)^{\circ}$ and $(b)^{\circ} \subseteq (y)^{\circ}$. Thus

$$(a \wedge b)^{\circ} = (a)^{\circ} \cap (b)^{\circ} \subseteq (x)^{\circ} \cap (y)^{\circ} = (x \wedge y)^{\circ}.$$

Hence $x \wedge y \in F^e$. Now, let $x \in F^e$ and $x \leq y$. Then

$$(a)^\circ \subseteq (x)^\circ \subseteq (y)^\circ$$

for some $a \in F$. Therefore F^e is a filter of L. Clearly $F \subseteq F^e$. Let $x \in F^e$ and $t \in (x)^{\circ\circ}$. Then there exists $a \in F$ such that $(a)^\circ \subseteq (x)^\circ \subseteq (t)^\circ$. Hence $(x)^{\circ\circ} \subseteq F^e$ and so F^e is a π-filter of L. Let G be a π-filter of L such that $F \subseteq G$. Let $x \in F^e$. Then we get $(a)^\circ \subseteq (x)^\circ$ for some $a \in F \subseteq G$. Since G is a π-filter of L, we get that $x \in (x)^{\circ\circ} \subseteq (a)^{\circ\circ} \subseteq G$. Hence $F^e \subseteq G$. Therefore F^e is the smallest π-filter of L such that $F \subseteq F^e$. $\Box$

From Lemma 3.4, it can be easily seen that a filter F is a π-filter if and only if $F = F^e$. Hence D is the smallest π-filter in L. In view of the above two results, it can be easily observed that the class $\mathcal{F}^\pi(L)$ of all π-filters of a lattice L forms a complete distributive lattice with respect the operations given by

$$F \wedge G = F \cap G \quad \text{and} \quad F \sqcup G = (F \vee G)^e$$

for any $F, G \in \mathcal{F}^\pi(L)$ in which the smallest element is D.

Theorem 3.6. *Let F be any filter of a lattice L. For any $x, y \in L$, define a binary relation $\Theta(F)$ on L as follows:*

$$(x, y) \in \Theta(F) \text{ if and only if } \{D \vee [x)\} \cap (a)^\circ = \{D \vee [y)\} \cap (a)^\circ$$

for some $a \in F$. Then $\Theta(F)$ is a congruence on L.

Proof. Clearly $\Theta(F)$ is an equivalence relation on L. Let $(x, y) \in \Theta(F)$. Then $\{D \vee [x)\} \cap (a)^\circ = \{D \vee [y)\} \cap (a)^\circ$ for some $a \in F$. For any $c \in L$, we have

$$\begin{aligned} \{D \vee [x \vee c)\} \cap (a)^\circ &= \{D \vee [x)\} \cap \{D \vee [c)\} \cap (a)^\circ \\ &= \{D \vee [y)\} \cap \{D \vee [c)\} \cap (a)^\circ \\ &= \{D \vee [y \vee c)\} \cap (a)^\circ \end{aligned}$$

Therefore $(x \vee c, y \vee c) \in \Theta(F)$. Again,

$$\begin{aligned} \{D \vee [x \wedge c)\} \cap (a)^\circ &= \{D \vee [x) \vee [c)\} \cap (a)^\circ \\ &= \{\{D \vee [x)\} \cap (a)^\circ\} \vee \{[c) \cap (a)^\circ\} \\ &= \{\{D \vee [y)\} \cap (a)^\circ\} \vee \{[c) \cap (a)^\circ\} \\ &= \{D \vee [y \wedge c)\} \cap (a)^\circ \end{aligned}$$

Hence $(x \wedge c, y \wedge c) \in \Theta(F)$. Therefore $\Theta(F)$ is a congruence on L. $\Box$

Lemma 3.7. *For any element x of a lattice L, the following hold:*

(1) $\{D \vee [x)\}^{\circ\circ} = (x)^{\circ\circ}$,

(2) $\{D \vee [x)\} \cap (x)^\circ = D$.

Proof. (1) $\{D \vee [x)\}^{\circ\circ} = \{D^{\circ} \cap (x)^{\circ}\}^{\circ} = \{L \cap (x)^{\circ}\}^{\circ} = (x)^{\circ\circ}$.
(2)

$$\begin{aligned}\{D \vee [x)\} \cap (x)^{\circ} &= \{D \vee [x)\} \cap \{L \cap (x)^{\circ}\} \\ &= \{D \vee [x)\} \cap \{D^{\circ} \cap (x)^{\circ}\} \\ &= \{D \vee [x)\} \cap (D \vee [x))^{\circ} \\ &= D\end{aligned}$$

as $D \vee [x)$ is a D-filter (by Proposition 2.9). □

Proposition 3.8. *For any filter F of a lattice L, define the dense-kernel $Ker^{D}\Theta(F)$ of the congruence $\Theta(F)$ as follows:*

$$Ker^{D}\Theta(F) = \{x \in L \mid \{D \vee [x)\} \cap (a)^{\circ} = D \text{ for some } a \in F\}$$

Then $Ker^{D}\Theta(F)$ is a filter in L such that $F \subseteq Ker^{D}\Theta(F)$.

Proof. Clearly $D \subseteq Ker^{D}\Theta(F)$. Let $x, y \in Ker^{D}\Theta(F)$. Then

$$\{D \vee [x)\} \cap (a)^{\circ} = \{D \vee [y)\} \cap (b)^{\circ} = D$$

for some $a, b \in F$. Now

$$\begin{aligned}\{D \vee [x \wedge y)\} \cap (a \wedge b)^{\circ} &= \{D \vee [x) \vee D \vee [y)\} \cap (a)^{\circ} \cap (b)^{\circ} \\ &= \{(D \vee [x)) \cap (a)^{\circ} \cap (b)^{\circ}\} \\ &\quad \vee \{(D \vee [y)) \cap (a)^{\circ} \cap (b)^{\circ}\} \\ &= \{D \cap (b)^{\circ}\} \vee \{D \cap (a)^{\circ}\} \\ &= D\end{aligned}$$

Hence $x \wedge y \in Ker^{D}\Theta(F)$. Again, let $x \in Ker^{D}\Theta(F)$ and $x \leq y$. Then there exists $a \in F$ such that $\{D \vee [y)\} \cap (a)^{\circ} \subseteq \{D \vee [x)\} \cap (a)^{\circ} = D$. Hence $y \in Ker^{D}\Theta(F)$. Therefore $Ker^{D}\Theta(F)$ is a filter of L. Now, let $x \in F$. From Lemma 3.7, we get that $x \in Ker^{D}\Theta(F)$. Therefore $F \subseteq Ker^{D}\Theta(F)$. □

In the following, the π-filters are characterized.

Theorem 3.9. *For any filter F of a lattice L, the following are equivalent:*

(1) *F is a π-filter,*
(2) *$F = Ker^{D}\Theta(F)$,*
(3) *for $x, y \in L, (x)^{\circ} = (y)^{\circ}$ and $x \in F$ imply that $y \in F$,*
(4) *$x \in F$ if and only if $x \in (a)^{\circ\circ}$ for some $a \in F$.*

Proof. (1) ⇒ (2): Assume that F is a π-filter. Clearly $F \subseteq Ker^{D}\Theta(F)$. Let $x \in Ker^{D}\Theta(F)$. Then $\{D \vee [x)\} \cap (a)^{\circ} = D$ for some $a \in F$.

Hence, $x \in D \vee [x) \subseteq (a)^{\circ\circ} \subseteq F($ because of F is a π-filter). Hence $Ker^D\Theta(F) \subseteq F$. Therefore $F = Ker^D\Theta(F)$.

(2) $\Rightarrow$ (3): Assume condition (2). Let $a, b \in L$ such that $(a)^\circ = (b)^\circ$. Suppose $a \in F$. Then $\{D \vee [a)\} \cap (t)^\circ = D$ for some $t \in F$. Then

$$\begin{aligned}\{D \vee [a)\} \cap (t)^\circ = D &\Rightarrow \{D \vee [a)\}^{\circ\circ} \cap (t)^\circ = D^{\circ\circ} = D\\ &\Rightarrow (a)^{\circ\circ} \cap (t)^\circ = D\\ &\Rightarrow (b)^{\circ\circ} \cap (t)^\circ = D\\ &\Rightarrow \{D \vee [b)\} \cap (t)^\circ \subseteq \{D \vee [b)\}^{\circ\circ} \cap (t)^\circ = D\\ &\Rightarrow b \in Ker^D\Theta(F) = F\end{aligned}$$

(3) $\Rightarrow$ (4): Assume condition (3). Let $x \in F$. Then clearly $x \in (x)^{\circ\circ}$. Again, let $x \in (a)^{\circ\circ}$ for some $a \in F$. Hence $(x)^{\circ\circ} \subseteq (a)^{\circ\circ}$, which yields $(x)^{\circ\circ} = (x)^{\circ\circ} \cap (a)^{\circ\circ} = (x \vee a)^{\circ\circ}$. Thus $(x)^\circ = (x \vee a)^\circ$ and $x \vee a \in F$. By the condition (3), we get that $x \in F$.

(4) $\Rightarrow$ (1): Assume condition (4). Let $x \in F$. Hence $x \in (a)^{\circ\circ}$ for some $a \in F$. Let $t \in (x)^{\circ\circ}$. Then for this $a \in F$, we get that $t \in (x)^{\circ\circ} \subseteq (a)^{\circ\circ}$. Hence by condition (4), we get $t \in F$. Thus $(x)^{\circ\circ} \subseteq F$. Therefore F is a π-filter of L. □

It was already observed that every minimal prime D-filter is a prime π-filter. The converse is not true. However, a sufficient condition is derived for a prime π-filter to become a minimal prime D-filter.

Proposition 3.10. *Let L be a lattice. If each $(x)^\circ, x \in L$ is a principal filter, then every prime π-filter is a minimal prime D-filter.*

Proof. Let P be a prime π-filter of L. Let $x \in P$. By the hypothesis $(x)^\circ = [y)$ for some $y \in L$. Hence $x \vee y \in D$. Now

$$(x \wedge y)^\circ = (x)^\circ \cap (y)^\circ = (x)^\circ \cap (x)^{\circ\circ} = D.$$

Hence $x \wedge y \notin P$, which implies that $y \notin P$. Therefore P is a minimal prime D-filter, by Theorem 2.12. □

Theorem 3.11. *The following conditions are equivalent in a lattice L.*

(1) *every π-filter is a principal filter;*
(2) *each $(x)^\circ$ is a principal filter and every minimal prime D-filter is non-condensed;*
(3) *every prime π-filter is a principal filter.*

Proof. (1) $\Rightarrow$ (2): Since each $(x)^\circ$ is a π-filter, it is enough to prove that every minimal prime D-filter is non-condensed. Let P be a minimal prime D-filter. By Proposition 3.2, P is a π-filter and hence (1) implies that $P = [a)$ for some $a \in L$. Suppose $P^\circ = D$. Then $(a)^\circ = D$ and hence $L = (a)^{\circ\circ} \subseteq P$, which is a contradiction. Therefore $P^\circ \neq D$.

(2) $\Rightarrow$ (3): Assume condition (2). Let P be a prime π-filter of L. Since each $(x)^\circ$ is a principal filter, by the above proposition, we get that P is a minimal prime D-filter such that $P^\circ \neq D$. Then there exists $x \neq D$ such that $x \in P^\circ$. Hence $P \subseteq P^{\circ\circ} \subseteq (x)^\circ$. Conversely, let $t \in (x)^\circ$. Then $t \vee x \in D \subseteq P$. Since P is prime and $x \notin (x)^\circ = P$, we get $t \in P$. Hence $P = (x)^\circ$. Thus by (2), $P = (x)^\circ$ is a principal filter.

(3) $\Rightarrow$ (1): Assume condition (3). Let F be a π-filter of L. Suppose that F is not principal. Consider the collection

$$\Sigma = \{G \mid G \text{ is a } \pi\text{-filter which is not a principal filter } \}.$$

Clearly $F \in \Sigma$. Then $a \in G_i$ for some $i \in \Delta$. Hence $[a) \subseteq G_i$ for some $i \in \Delta$. On the other hand $G_i \subseteq \bigcup G_i = [a)$. Hence $G_i = [a)$ for some $i \in \Delta$, which is a contradiction. Thus $\bigcup G_i$ is an upper bound for $\{G_i\}_{i\in\Delta}$ in Σ. Let M be a maximal element of Σ containing F because of Zorn's Lemma. Choose $x \notin M$ and $y \notin M$. Then $M \subset \{M \vee [x)\}^e$ and $M \subset \{M \vee [y)\}^e$. Hence $\{M \vee [x)\}^e = [b)$ and $\{M \vee [y)\}^e = [c)$ for some $b, c \in L$. Hence we get

$$\{M \vee [x \vee y)\}^e = \{M \vee [x)\}^e \cap \{M \vee [y)\}^e = [b) \cap [c) = [b \vee c).$$

If $x \vee y \in M$, then $M = M^e = [b \vee c)$, which is a contradiction to (3). Therefore F is a principal filter. $\square$

4. The space of prime π-filters

In this section, certain topological properties of the collection of all prime π-filters of a lattice are discussed. Some equivalent conditions are established for the space of all prime π-filters of a lattice to become a Hausdorff space.

Theorem 4.1. *Let I be an ideal and F be a π-filter of a lattice L such that $F \cap I = \emptyset$. Then there exists a prime π-filter P such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Proof. Let I be an ideal and P be a π-filter of L such that $F \cap I = \emptyset$. Consider

$$\sum = \{G \mid G \text{ is a } \pi\text{-filter}, F \subseteq G \text{ and } G \cap I = \emptyset\}.$$

Clearly $F \in \sum$. Clearly $\sum$ satisfies the hypothesis of Zorn's Lemma. Let M be a maximal element of $\sum$. Then M is a π-filter of L such that $F \subseteq M$ and $M \cap I = \emptyset$. Let $x, y \in L$ be such that $x \notin M$ and $y \notin M$. Then $M \subset M \vee [x) \subseteq \{M \vee [x)\}^e$ and $M \subset M \vee [y) \subseteq \{M \vee [y)\}^e$. By the maximality of M, we get $\{M\vee[x)\}^e\cap I\neq\emptyset$ and $\{M\vee[y)\}^e\cap I\neq\emptyset$. Choose $a\in\{M\vee[x)\}^e\cap I$ and $b\in\{M\vee[y)\}^e\cap I$. Then $a\vee b\in I$ and

$$\begin{aligned} a\vee b &\in \{M\vee[x)\}^e\cap\{M\vee[y)\}^e \\ &= \{(M\vee[x))\cap(M\vee[y))\}^e \\ &= \{M\vee[x\vee y)\}^e \end{aligned}$$

Suppose $x\vee y\in M$. Then $a\vee b\in M^e=M$. Since $a\vee b\in I$, we get $a\vee b\in M\cap I$, which is a contradiction. Thus M is the required prime π-filter of L. $\square$

Corollary 4.2. *Let F be a π-filter of a lattice L and $x\in L$. If $x\notin F$, then there exists a prime π-filter P of L such that $x\notin P$ and $F\subseteq P$.*

Corollary 4.3. *For any π-filter F of a lattice L, we have*

$$F=\bigcap\{P \mid P \text{ is a prime } \pi\text{-filter of } L,\ F\subseteq P\}$$

Corollary 4.4. *The intersection of all prime π-filters of a lattice is equal to D.*

For any lattice L, denote the class of all prime π-filters of L by $Spec_F^{\pi}(L)$. For any subset A of L, take

$$K'(A)=\{P\in Spec_F^{\pi}(L)\mid A\nsubseteq P\}$$

and for any $x\in L, K'(x)=K'(\{x\})$. Then we have the following observations which can be verified directly.

Lemma 4.5. *Let L be a lattice and $x,y\in L$. Then we have*

(1) $\bigcup\limits_{x\in L}K'(x)=Spec_F^{\pi}(L)$,
(2) $K'(x)\cap K'(y)=K'(x\vee y)$,
(3) $K'(x)\cup K'(y)=K'(x\wedge y)$,
(4) $K'(x)=\emptyset$ *if and only if* $x\in D$
(5) $K'(0)=Spec_F^{\pi}(L)$.

From the above set of properties, it can be obviously seen that the collection $\{K'(x)|x\in L\}$ forms a base for a topology on $Spec_F^{\pi}(L)$. This topology is called a hull-kernel topology and it produces the following topological property:

Theorem 4.6. *The set of all compact open sets of $Spec_F^{\pi}(L)$ is the base $\{K'(x)|x\in L\}$.*

Proof. Let $x\in L$. Let A be a subset of L such that

$$K'(x)\subseteq\bigcup_{y\in A}K'(y)$$

which is an open cover of $K'(x)$. Let F be the filter generated by A. Suppose $x \notin F^e$. From Corollary 4.2, there exist a prime π-filter P such that $F^e \subseteq P$ and $x \notin P$. Hence $P \in K'(x) \subseteq \bigcup_{y\in A} K'(y)$. Therefore $y \notin P$ for some $y \in A$, which is a contradiction to that $F \subseteq F^e \subseteq P$. Therefore $x \in F^e$. Then there exists $a \in F$ such that $x \in (a)^{\circ\circ}$. Since F is the filter generated by A, there exist $a_1, a_2, \ldots, a_n \in A$ such that $a = a_1 \wedge a_2 \wedge \ldots \wedge a_n$. Hence $x \in (a)^{\circ\circ} = (a_1 \wedge a_2 \wedge \ldots \wedge a_n)^{\circ\circ}$. Clearly $K'(x) \subseteq \bigcup_{i=1}^{n} K'(a_i)$, which is a finite subcover of $K'(x)$. Thus $K'(x)$ is compact in $Spec^{\pi}_{F}(L)$. It enough to prove that every compact open subset of $Spec^{\pi}_{F}(L)$ is of the form $K'(x)$ for some $x \in L$. Let C be a compact open subset of $Spec^{\pi}_{F}(L)$. Since C is open, we get $C = \bigcup_{a\in A} K'(a)$ for some $A \subseteq L$. Since C is compact, there exists $a_1, a_2, \ldots, a_n \in A$ such that

$$C = \bigcup_{i=1}^{n} K'(a_i) = K'(\bigwedge_{i=1}^{n} a_i)$$

Therefore $C = K'(x)$ for some $x \in L$. □

By a maximal π-filter, we mean a maximal element in the class of all proper π-filters of a lattice. As the class of all π-filters forms a distributive lattice, it can be easily deduced that every maximal π-filter is a prime π-filter. Now, in the following, a set of equivalent conditions is derived for every prime π-filter to become a minimal prime D-filter.

Theorem 4.7. *The following assertions are equivalent in a lattice L:*

(1) *every prime π-filter is a minimal prime D-filter;*
(2) *$Spec^{\pi}_{F}(L)$ is a T_1-space;*
(3) *every prime π-filter is maximal;*
(4) *every prime π-filter is minimal;*
(5) *for each $x \in L, (x)^{\circ} \sqcup (x)^{\circ\circ} = L$;*
(6) *$Spec^{\pi}_{F}(L)$ is a Hausdorff space;*
(7) *for any $x, y \in L$, there exists $z \in L$ such that $x \vee z \in D$ and*
$$K'(y) \cap \{Spec^{\pi}_{F}(L) - K'(x)\} = K'(y \vee z).$$

Proof. (1) ⇒ (2): Assume that every prime π-filter is a minimal prime D-filter. Let P and Q be two distinct prime π-filters of L. Since P and Q are minimal, we get $P \nsubseteq Q$ and $Q \nsubseteq P$. Let us choose $x \in P - Q$ and $y \in Q - P$. Then $Q \in K'(x) - K'(y)$ and $P \in K'(y) - K'(x)$. Hence $Spec^{\pi}_{F}(L)$ is a T_1-space.

(2) $\Rightarrow$ (3): Suppose that $Spec^{\pi}_F(L)$ is a T_1-space. Let P be a prime π-filter of L. Suppose Q is a maximal π-filter of L such that $P \subset Q$. Since $Spec^{\pi}_F(L)$ is a T_1-space, there exists two basic open sets $K'(x)$ and $K'(y)$ such that $Q \in K'(x) - K'(y)$ and $P \in K'(y) - K'(x)$. Since $x \in P \subset Q$, we get $Q \notin K'(x)$, which is a contradiction. Therefore P is a maximal π-filter.

(3) $\Rightarrow$ (4): It is clear.

(4) $\Rightarrow$ (5): Assume that every prime π-filter is minimal. Then every prime π-filter is a minimal prime D-filter. Suppose

$$(x)^{\circ} \sqcup (x)^{\circ\circ} \neq L$$

for some $x \in L$. Then there exists a prime π-filter P such that $(x)^{\circ} \sqcup (x)^{\circ\circ} \subseteq P$. Thus $x \in (x)^{\circ\circ} \subseteq P$. Since P is minimal prime D-filter and $(x)^{\circ} \subseteq P$, we get $x \notin P$, which is a contradiction. Therefore $(x)^{\circ} \sqcup (x)^{\circ\circ} = L$.

(5) $\Rightarrow$ (6): Assume condition (5). Let P and Q be two distinct elements of $Spec^{\pi}_F(L)$. Choose $x \in P$ be such that $x \notin Q$. Then by hypothesis, $(x)^{\circ} \sqcup (x)^{\circ\circ} = L$. Hence $0 \in (x)^{\circ} \sqcup (x)^{\circ\circ} = \{(x)^{\circ} \vee (x)^{\circ\circ}\}^{e}$. Hence $(a)^{\circ} \subseteq (0)^{\circ} = D$ for some $a \in (x)^{\circ} \vee (x)^{\circ\circ}$. Thus $a = r \wedge s$ for some $r \in (x)^{\circ}$ and $s \in (x)^{\circ\circ}$. Hence $r \vee x \in D$. Suppose $r \in P$. Since P is a π-filter, we get $(r)^{\circ\circ} \subseteq P$. Now $(r)^{\circ} \cap (s)^{\circ} = (r \wedge s)^{\circ} = (a)^{\circ} = D$ implies that $(s)^{\circ} \subseteq (r)^{\circ\circ} \subseteq P$. Again, $s \in (x)^{\circ\circ}$ implies that $(x)^{\circ} \subseteq (s)^{\circ} \subseteq P$. Since $x \in P$, we get $L = (x)^{\circ} \sqcup (x)^{\circ\circ} \subseteq P$ which is a contradiction. Hence $r \notin P$. Thus $P \in K'(r)$. Hence $P \in K'(r)$ and $Q \in K'(x)$. Since $x \vee r \in D$, we get $K'(x) \cap K'(r) = K'(x \vee r) = \emptyset$. Therefore $Spec^{\pi}_F(L)$ is Hausdorff.

(6) $\Rightarrow$ (7): Assume that $Spec^{\pi}_F(L)$ is a Hausdorff space. Hence $K'(a)$ is a compact subset of $Spec^{\pi}_F(L)$, for each $a \in L$. Then $K'(a)$ is a clopen subset of $Spec^{\pi}_F(L)$. Let $x, y \in L$ be such that $x \neq y$. Then $K'(y) \cap \{Spec^{\pi}_F(L) - K'(x)\}$ is a compact subset of the compact space $K'(y)$. Since $K'(y)$ is open in $Spec^{\pi}_F(L)$, $K'(y) \cap \{Spec^{\pi}_F(L) - K'(x)\}$ is a compact open subset of $Spec^{\pi}_F(L)$. By Theorem 4.6, there exists $z \in L$ such that

$$K'(z) = K'(y) \cap \{Spec^{\pi}_F(L) - K'(x)\}$$

Therefore $K'(y) \cap \{Spec^{\pi}_F(L) - K'(x)\} = K'(y) \cap K'(z) = K'(y \vee z)$. Also $K'(x \vee z) = K'(x) \cap K'(z) = \emptyset$. Therefore $x \vee z \in D$.

(7) $\Rightarrow$ (1): Let P be a prime π-filter of L. Choose $x, y \in L$ such that $x \in P$ and $y \notin P$. Then by the condition (7), there exists $z \in L$ such that $x \vee z \in D$ and

$$K'(y) \cap \{Spec^{\pi}_F(L) - K'(x)\} = K'(y \vee z)$$

Then clearly $P \in K'(y) \cap \{Spec_F^{\pi}(L) - K'(x)\} = K'(y \vee z)$. If $z \in P$, then $y \vee z \in P$, which is a contradiction to $P \in K'(y \vee z)$. Hence $z \notin P$. Thus for each $x \in P$, there exists $z \notin P$ such that $x \vee z \in D$. Therefore P is a minimal prime D-filter of L. $\square$

Acknowledgments

The authors wish to thank the referees for their valuable suggestions and comments which improves the presentation.

References

1. G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloq. XXV, Providence, 1967.
2. T. S. Blyth, Ideals and filters of pseudo-complemented semilattices, *Proc. Edinb. Math. Soc.*, **23** (1980), Article ID: 301316.
3. S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer Verlag, 1981.
4. W. H. Cornish, Annulets and α-ideals in distributive lattices, *J. Aust. Math. Soc.*, **15** (1973), 70–77.
5. W. H. Cornish, Normal lattices, *J. Aust. Math. Soc.*, **14** (1972), 200–215.
6. D. Dikranjan and W. Tholen, *Categorical Structure of Closure Operators: With Applications to Topology*, Spinger Dordrecht, 1995.
7. A. P. Phaneendra Kumar, M. Sambasiva Rao and K. Sobhan Babu, Filters of distributive lattices generated by dense elements, *Palest. J. Math.*, **11**(2) (2022), 45–54.
8. A. P. Phaneendra Kumar, M. Sambasiva Rao and K. Sobhan Babu, Generalized prime D-filters of distributive lattices, *Arch. Math.*, **57**(3) (2021), 157–174.
9. M. Sambasiva Rao, e-filters of MS-algebras, *Acta Math. Sci. Ser. B*, **33**(3), (2013), 738–746.
10. M. Sambasiva Rao, Normal filters of distributive lattices, *Bull. Sec. logic*, **41**(3) (2012), 131–143.
11. T. P. Speed, Some remarks on a class of distributive lattices, *J. Aust. Math. Soc.*, **9** (1969), 289–296.

Mukkamala Sambasiva Rao

Department of Mathematics, MVGR College of Engineering, Vizianagaram, P.O. Box 535005, Andhra Pradesh, India.

Email: `mssraomaths35@rediffmail.com`

Ananthapatnayakuni Phaneendra Kumar

Department of Mathematics, MVGR College of Engineering, Vizianagaram, P.O. Box 535005, Andhra Pradesh, India.

Email: `phaneendra.arjun@gmail.com`.

Journal of Algebraic Systems

REGULAR FILTERS OF DISTRIBUTIVE LATTICES

M. SAMBASIVA RAO AND A. P. PHANEENDRA KUMAR

فیلترهای منظم مشبکه‌های توزیع‌پذیر

موکامالا سامباسیوارائو[1] و ای. پی. فنندرا کومار[2]

[1,2]گروه ریاضی، کالج‌مهندسی $MVGR$، ویزیاناگارام، آندراپرادش، هند

مفاهیم فیلترهای منظم و π-فیلترها در مشبکه‌های توزیع‌پذیر معرفی می‌شوند. مجموعه‌ای از شرایط معادل برای یک D-فیلتر ارائه شده است که تحت آن‌ها، به یک فیلتر منظم تبدیل می‌شود. نشان داده‌ایم که برای هر D-فیلتر، همریختی‌ای وجود دارد که هسته‌ی چگال آن، یک فیلتر منظم است. π-فیلترها برحسب فیلترهای منظم و تجانس‌ها رده‌بندی می‌شوند. شرایط معادلی برای فضای همه‌ی فیلترهای اول ارائه شده است که تحت آن‌ها، به یک فضای هاسدورف تبدیل می‌شود.

کلمات کلیدی: فیلتر منظم، D-فیلتر اول مینیمال، π-فیلتر، عنصر چگال، مشبکه نسبتاً کامل شده، فضای هاسدورف.