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GENERALIZED π-BAER $*$-RINGS

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ABSTRACT. A $*$-ring R is called a generalized π-Baer $*$-ring, if for any projection invariant left ideal Y of R, the right annihilator of Y^n is generated, as a right ideal, by a projection, for some positive integer n, depending on Y. In this paper, we study some properties of generalized π-Baer $*$-rings. We show that this notion is well-behaved with respect to polynomial extensions, full matrix rings, and several classes of triangular matrix rings. We indicate interrelationships between the generalized π-Baer $*$-rings and related classes of rings such as generalized π-Baer rings, generalized Baer $*$-rings, generalized quasi-Baer $*$-rings, and π-Baer $*$-rings. We obtain algebraic examples which are generalized π-Baer $*$-rings but are not π-Baer $*$-rings. We show that for pre-C*-algebras these two notions are equivalent. We obtain classes of Banach $*$-algebras which are generalized π-Baer $*$-rings but are not π-Baer $*$-rings. We finish the paper by showing that for a locally compact abelian group G, the group algebra $L^1(G)$ is a generalized π-Baer $*$-ring, if and only if so is the group C*-algebra $C^*(G)$, if and only if G is finite.

1. Introduction

Throughout this paper R denotes an associative ring with unity. Let us recall that a $*$-*ring* (or an *involutive ring*) R is a ring with a map $* : R \to R$, called *involution*, such that $(x+y)^* = x^*+y^*$, $(xy)^* = y^*x^*$, and $(x^*)^* = x$, for all $x, y \in R$. An idempotent p of a $*$-ring R is called

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a *projection* if p is self-adjoint (i.e., $p^* = p$). An idempotent $e \in R$ is called *right* (resp., *left*) *semicentral* if $ex = exe$ (resp., $xe = exe$), for all $x \in R$ [9]. We denote by $\mathbf{S}_r(R)$ (resp., $\mathbf{S}_\ell(R)$) the set of all right (resp., left) semicentral idempotents of R. If X is a nonempty subset of R, then $r_R(X)$ (resp., $\ell_R(X)$) is used for the right (resp., left) annihilator of X over R. We use $\mathrm{M}_n(R)$, $R[x]$, and $R[[x]]$ for the n by n full matrix ring over R, the ring of polynomials, and the ring of formal power series, respectively. The ring of integers and the ring of integers modulo n are denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively.

Recall from [24] that, a ring R is called a *Baer ring* if the right annihilator of every nonempty subset of R is generated, as a right ideal, by an idempotent. If R is a $*$-ring, then R is called a *Baer $*$-ring* if the right annihilator of every nonempty subset is generated, as a right ideal, by a projection. In [24] Kaplansky has shown that the definitions of a Baer ring and a Baer $*$-ring are left-right symmetric. The subject of Baer $*$-rings is essentially pure algebra, with historic roots in operator algebras and lattice theory. Baer $*$-rings are a common generalization of AW*-algebras and complete $*$-regular rings. The AW*-algebras are precisely the Baer $*$-rings that happen to be C*-algebras, the complete $*$-regular rings are the Baer $*$-rings that happen to be regular in the sense of von Neumann.

Various weaker versions of Baer and Baer $*$-rings have been studied. From [20], a ring R is *quasi-Baer* if the right annihilator of every right ideal is generated, as a right ideal, by an idempotent. This is a nontrivial generalization of the class of Baer rings. For example, prime rings with nonzero right singular ideal are quasi-Baer and not Baer, since Baer rings are nonsingular. The quasi-Baer ring property is left-right symmetric.

Recall from [27] that, a ring R is called a *generalized quasi-Baer ring* if for every ideal I of R, the right annihilator of I^n is generated by an idempotent for some positive integer n depending on I. In [15], Birkenmeier and Park introduced a *quasi-Baer $*$-ring* as a $*$-ring R in which the right annihilator of every ideal is generated by a projection. As in the case of Baer $*$-rings, the involution can be used to show that this notion is left-right symmetric. If R is a commutative non-Prüfer domain then $\mathrm{M}_n(R)$, with the transpose involution, is a quasi-Baer $*$-ring which is not a Baer $*$-ring.

In [10], Birkenmeier et al. introduced another generalization of Baer rings. Recall that a ring R is said to be a *π-Baer ring* if the right annihilator of every projection invariant left ideal Y (i.e., $Ye \subseteq Y$ for all $e = e^2 \in R$) is generated by an idempotent.

Recall from [31] that, a $*$-ring R is said to be a *π-Baer $*$-ring* if the right annihilator of every projection invariant left ideal Y is generated by a projection.

Like the Baer and Baer $*$ properties, the π-Baer and π-Baer $*$ properties are left-right symmetric. The π-Baer condition is strictly between the Baer and quasi-Baer conditions, and the π-Baer $*$ condition is strictly between the Baer $*$ and quasi-Baer-$*$ conditions.

In [31], the authors generalized the notion of π-Baer rings. Recall that a ring R is said to be a *generalized π-Baer ring* if for every projection invariant left ideal Y of R, the right annihilator of Y^n is generated by an idempotent for some positive integer n depending on Y.

From [4], a $*$-ring R is a *generalized Baer $*$-ring* if for any nonempty subset S of R, the right annihilator of S^n is generated, as a right ideal, by a projection for some positive integer n depending on S, where $S^n = \{s_1 s_2 \cdots s_n |\ s_1, s_2, \ldots, s_n \in S\}$.

Recall from [3] that, a $*$-ring R is said to be a *generalized quasi-Baer $*$-ring* if for any ideal I of R, the right annihilator of I^n is generated, as a right ideal, by a projection for some positive integer n depending on I.

To transfer the generalized quasi-Baer $*$-condition from a $*$-ring R to various extensions (e.g., $R[x]$ or $R[[x]]$ or full matrix rings over R) one needs no additional conditions which is certainly not the case for the generalized Baer $*$-condition (see [3, Theorem 3.17] and [4, Example 2.24]). So, it is natural to ask: is there a condition strictly between the generalized Baer $*$ and generalized quasi-Baer $*$-conditions, which is able to combine some of the notable features of the generalized Baer $*$ and generalized quasi-Baer $*$-conditions?

On the other hand, in the presence of an involution, the projections are "vastly easier to work with than idempotents". In this paper, we introduce a *generalized π-Baer $*$-ring* as a $*$-ring R in which the right annihilator of every projection invariant left ideal of R is generated by a projection. These $*$-rings are generalizations of π-Baer $*$-rings, and there are examples distinguishing these classes.

The organization of our paper is as follows. In Section 2, we introduce the notion of generalized π-Baer $*$-rings, and we study its properties and relations with other Baer-type rings such as generalized π-Baer rings, generalized Baer $*$-rings, generalized quasi-Baer $*$-rings, and π-Baer $*$-rings.

Section 3 is devoted to the study of extensions of generalized π-Baer $*$-rings. We prove that the n by n full matrix rings over generalized π-Baer $*$-rings are generalized π-Baer $*$-rings. It is shown that being a generalized π-Baer $*$-ring is preserved by polynomial extensions. Also, it is proved that the essential overrings of generalized π-Baer $*$-rings are generalized π-Baer $*$-rings. The analytic part of the paper is presented in Section 4. In Section 4 we show that for pre-C*-algebras the notions of a generalized π-Baer $*$-ring and a π-Baer $*$-ring are equivalent. We give examples of Banach $*$-algebras which are generalized π-Baer $*$-rings but are not π-Baer $*$-rings. Also, we characterize locally compact abelian groups G where the group algebra $L^1(G)$ and the group C*-algebra $C^*(G)$ are (generalized) π-Baer $*$-rings.

2. Basic Results

In this section, the generalized π-Baer $*$-rings and their basic properties are introduced. Furthermore, the relations between the notion of a generalized π-Baer $*$-ring and other Baer-type notions are verified.

Definition 2.1. A $*$-ring R is called a generalized π-Baer $*$-ring if for any projection invariant left ideal Y of R, the right annihilator of Y^n is generated, as a right ideal, by a projection for some positive integer n, depending on Y; i.e., there is a projection $p \in R$ such that $r_R(Y^n) = pR$.

Remark 2.2. Let R be a $*$-ring.

(1) Taking $Y = 0$ in Definition 2.1 yields that R has a unity element.
(2) The definition of a generalized π-Baer $*$-ring is left-right symmetric. For this, let Y be a projection invariant right ideal of R. It is not hard to see that Y^* is a projection invariant left ideal of R. Then $r_R((Y^*)^n) = pR$, for some projection $p \in R$ and some positive integer n. Hence

$$\ell_R(Y^n) = (r_R((Y^n)^*))^* = (r_R((Y^*)^n))^* = Rp.$$

The following example demonstrates that there exists a $*$-ring which is a generalized π-Baer $*$-ring, but it is not a π-Baer $*$-ring.

Example 2.3. Let $R = \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$. Define $* : R \to R$ by

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^* = \begin{pmatrix} \overline{c} & \overline{b} \\ 0 & \overline{a} \end{pmatrix},$$

where $\overline{a}$ (resp., $\overline{b}$, and $\overline{c}$) is the conjugate of a (resp., b, and c). Then all the projection invariant left ideals of R are

$$0, R, \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}.$$

Since $r_R(\begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix})$ is not generated by a projection of R, R is not a π-Baer $*$-ring. But, it is a generalized π-Baer $*$-ring, since

$$r_R(\begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix}^2) = r_R(0) = R.$$

In the next example we see that there exists a $*$-ring which is not a generalized π-Baer $*$-ring, but it is a generalized π-Baer ring.

Example 2.4. Let n be an integer and $R = T_{2^n}(\mathbb{Z} \oplus \mathbb{Z})$ be the ring of $2^n \times 2^n$ upper triangular matrices over $\mathbb{Z} \oplus \mathbb{Z}$. Take $S = R \oplus R^{op}$, where R^{op} denote the opposite ring of R. Then by [31, Corollary 3.4] S is a generalized π-Baer ring. Let $* : S \to S$ be the exchange involution; i.e., $(A, B)^* = (B^{\circledast}, A^{\circledast})$ (see Definition 3.5 for the definition of $\circledast$). We will now show that S is not a generalized π-Baer $*$-ring. Note that R is a projection invariant left ideal of S, and the set of all projections of S is

$$\{(A, A) \mid A \in S, A = A^2 = A^{\circledast}\}.$$

One can show that $r_S(R^n) = r_S(R) = (0, \mathrm{I}_{2^n})S$, for each integer $n \geq 1$. Thus $r_S(R^n)$ does not contain a nonzero projection of S. Hence S is not a generalized π-Baer $*$-ring.

A $*$-ring R is called *symmetric* if, for every $x \in R$, $1 + x^*x$ is an invertible element of R (see [7, Exercise 7c, p. 9]).

Proposition 2.5. *Let R be a symmetric $*$-ring. Then the following statements are equivalent:*

(1) *R is a generalized π-Baer $*$-ring;*
(2) *R is a generalized π-Baer ring.*

Proof. Clearly (1) implies (2). Assume that (2) holds and let Y be a projection invariant left ideal of R. Then $r_R(Y^n) = eR$, for some idempotent $e \in R$ and some positive integer n. Since R is symmetric, [24, Theorem 26, p. 34] yields that there is a projection $p \in R$ such that $eR = pR$. Hence $r_R(Y^n) = pR$ and so R is a generalized π-Baer $*$-ring. $\square$

We need the following proposition in the sequel

Lemma 2.6 ([30], Lemma 1.4)**.** *Let R be a $*$-ring.*

(1) *If* $p \in R$ *is a projection and* pR *is a projection invariant right ideal, then* p *is central.*
(2) *If* $p \in R$ *is a projection and* Rp *is a projection invariant left ideal, then* p *is central.*

The following result will be used many times in the sequel.

Proposition 2.7. *The following are equivalent for a* $*$*-ring* R*.*

(1) R *is a generalized* π*-Baer* $*$*-ring;*
(2) R *is a generalized* π*-Baer ring in which every left* (*right*) *semi-central idempotent is a central projection;*
(3) *For each projection invariant left* (*right*) *ideal* Y*, there exist a central projection* $p \in R$ $(q \in R)$*, and a positive integer* n (m) *such that* $r_R(Y^n) = pR$ $(\ell_R(Y^m) = Rq)$*.*

Proof. (1)⇒(2) Let R be a generalized π-Baer $*$-ring. Then obviously R is a generalized π-Baer ring. The second part of the statement follows from Proposition 2.9 and [3, Lemma 2.3(ii)].

(2)⇒(3) Let Y be a projection invariant left ideal of R. Then there is an idempotent $e \in R$ and a positive integer n such that $r_R(Y^n) = eR$. By [10, Lemma 2.1(i)] eR is a projection invariant right ideal. By [10, Lemma 2.1(iii)] e is a left semicentral idempotent. So (2) implies that e is a central projection.

(3)⇒(1) It is clear. □

Recall from [6] that, a ring R is said to satisfy the IFP *(insertion of factors property)* if $r_R(x)$ is an ideal of R for all $x \in R$. A ring R is called *abelian* if every idempotent in it is central. It is evident that any reduced ring satisfies IFP and any ring with IFP is abelian.

Proposition 2.8. *Let* R *be a* $*$*-ring satisfying IFP. Then the following conditions are equivalent.*

(1) R *is a generalized* π*-Baer* $*$*-ring.*
(2) *If* S *is a nonempty subset of* R *such that* $Se \subseteq S$ *for all idempotent* $e \in R$*, then there exist some projection* $p \in R$ *and integer* $n \geq 1$ *such that* $r_R(S^n) = pR$*.*

Proof. (1)⇒(2) Let R be a generalized π-Baer $*$-ring and S a nonempty subset of R such that $Se \subseteq S$, for all idempotent $e \in R$. Then RS is a left ideal of R and $RSe = R(Se) \subseteq RS$. Thus RS is a projection invariant left ideal of R. Hence

$$r_R(S^n) = r_R(R(S)^n) = r_R((RS)^n) = pR$$

for some projection $p \in R$ and some positive integer n, and the result follows.

(2)$\Rightarrow$(1) It is straightforward. □

Proposition 2.9. *Let R be a $*$-ring. Consider the following conditions.*

(1) *R is a generalized Baer $*$-ring;*
(2) *R is a generalized π-Baer $*$-ring;*
(3) *R is a generalized quasi-Baer $*$-ring.*

Then (1)$\Rightarrow$(2)$\Rightarrow$(3).

Proof. The fact that every projection invariant left ideal is a subset yields the implication (1)$\Rightarrow$(2). The implication (2)$\Rightarrow$(3) follows from the fact that every two-sided ideal is a projection invariant left ideal. □

We remark that when R is commutative, conditions (1), (2), and (3) of Proposition 2.9 are equivalent. The next example shows that the converse of each of the implications in Proposition 2.9 does not hold true.

Example 2.10. (i) Let $R = \mathrm{M}_2(\mathbb{C})[x]$. By [4, Example 2.24] the ring $\mathrm{M}_2(\mathbb{C})[x]$, with the conjugate transpose involution on coefficients, is not a generalized Baer $*$-ring. On the other hand, by Theorem 3.3 below, $\mathbb{C}[x]$, with the conjugate map on coefficients as the involution, is a generalized π-Baer ring. Now Proposition 3.4 below implies that R is a generalized π-Baer $*$-ring.

(ii) Let R be a prime ring with trivial idempotents which is not domain. For example, let $R = KG$, where K is a field of characteristic $p > 0$, and $G = C_p \wr A$ be the restricted wreath product of C_p, the cyclic group of order p, and an infinite elementary abelian p-group (see [19, Example 3.4]). Let $*$ be the involution on the group ring R defined by $(\sum a_g g)^* = \sum a_g^* g^{-1}$. By [15, Corollary 1.2], R is a quasi-Baer $*$-ring. On the other hand R is not π-Baer (see [10, Theorem 2.1]). Then [31, Proposition 2.11] implies that R is not generalized π-Baer and so it is not a geeralized π-Baer $*$-ring. By Theorem 3.6 below, the ring $S_n(R)$ with the involution $\circledast$ is not a generalized π-Baer $*$-ring, for each integer $n \geq 2$ (See Definition 3.5 for the definitions of $S_n(R)$ and $\circledast$). But [3, Theorem 3.4] implies that $S_n(R)$ is a generalized quasi-Baer $*$-ring.

We include the following results to demonstrate the conditions in which the generalized π-Baer $*$-ring, π-Baer $*$-ring, and generalized quasi-Baer $*$-ring are coincide.

Proposition 2.11. *Let R be a $*$-ring, then:*

(1) *If R is semiprime, then R is a generalized π-Baer $*$-ring if and only if it is a π-Baer $*$-ring.*

(2) *If R satisfies IFP, then R is a generalized π-Baer $*$-ring if and only if it is a generalized quasi-Baer $*$-ring.*

(3) *If R is generated by its idempotents, then R is a generalized π-Baer $*$-ring if and only if it is a generalized quasi-Baer $*$-ring.*

Proof. (1) Let R be a generalized π-Baer $*$-ring and Y a projection invariant left ideal of R. Then $r_R(Y^n) = pR$, for some positive integer n, and some central projection $p \in R$. So $Y^n p = (Yp)^n = 0$. Since R is semiprime, $Yp = 0$. Thus $pR \subseteq r_R(Y) \subseteq r_R(Y^n) = pR$. Hence $r_R(Y) = pR$ and that R is a π-Baer $*$-ring. The converse is obvious.

(2) Let Y be a projection invariant left ideal of R. Since R satisfies IFP, $r_R((YR)^n) = r_R(Y^n)$ for each positive integer n. Thus if R is a generalized quasi-Baer $*$-ring, then it is a generalized π-Baer $*$-ring. The converse follows from Proposition 2.7.

(3) Note that every projection invariant one-sided ideal of R is an ideal of R by [10, Corollary 2.2(iii)]. Then Proposition 2.7 yields the result. □

The following is an example of a $*$-ring which is not a generalized π-Baer $*$-ring.

Example 2.12. Let R be the ring constructed in [12, Example 1.6]. More precisely, let $\mathbb{C}_n = \mathbb{C}$ for $n = 1, 2, \ldots$, and let

$$R = \begin{pmatrix} \prod_{n=1}^{\infty} \mathbb{C}_n & \bigoplus_{n=1}^{\infty} \mathbb{C}_n \\ \bigoplus_{n=1}^{\infty} \mathbb{C}_n & \langle \bigoplus_{n=1}^{\infty} \mathbb{C}_n, 1 \rangle \end{pmatrix}$$

which is considered as a subring of $\mathrm{M}_2(\prod_{n=1}^{\infty} \mathbb{C}_n)$, where $\langle \bigoplus_{n=1}^{\infty} \mathbb{C}_n, 1 \rangle$ denotes the $\mathbb{C}$-algebra generated by $\bigoplus_{n=1}^{\infty} \mathbb{C}_n$ and $1_{\prod_{n=1}^{\infty} \mathbb{C}_n}$. Consider the conjugate transpose as an involution for R. By [3, Example 2.35] R is not a generalized quasi-Baer $*$-ring. Thus Proposition 2.9 implies that R is not a generalized π-Baer $*$-ring.

Let R be a ring with an involution $*$. Recall that, $*$ is called a *proper* involution if for every $x \in R$, $xx^* = 0$ implies $x = 0$ [7]. Also, $*$ is called a *semiproper* involution if $xRx^* = 0$ implies $x = 0$ [17]. Recall from [3] that, $*$ is said to be a *quasi-proper* involution if for every $x \in R$, $xRx^* = 0$ implies $x^n = 0$, for some $n \in \mathbb{N}$ depending on x. Observe that if $*$ is a semiproper involution, then it is quasi-proper. Thus the involution of every C*-algebra is quasi-proper since it is a proper involution [7].

Proposition 2.13. *Let R be a generalized π-Baer $*$-ring. Then $*$ is a quasi-proper involution.*

Proof. The proof follows from Proposition 2.9 and [3, Proposition 2.13]. □

Proposition 2.14. *The following are equivalent for a $*$-ring R.*

(1) *R is a generalized π-Baer $*$-ring.*
(2) *For each projection invariant left (right) ideal Y of R, there exist a central projection $p \in R$ and a positive integer n such that $Y^n \subseteq Rp$ and $r_R(Y^n) \cap Rp = 0$ ($\ell_R(Y^n) \cap pR = 0$).*

Proof. (1)$\Rightarrow$(2) Suppose that R is a generalized π-Baer $*$-ring. Let Y be a projection invariant left ideal of R. Then there exist a central projection $p \in R$ and a positive integer n such that $r_R(Y^n) = pR$. So $Y^n \subseteq \ell_R(r_R(Y^n)) = R(1-p)$. Set $q = 1-p$. Then q is a projection and $r_R(Y^n) \cap Rq = (1-q)R \cap Rq = 0$.

(2)$\Rightarrow$(1) Let Y be a projection invariant left ideal of R. Choose a central projection $p \in R$ and an integer $n \geq 1$ such that $Y^n \subseteq Rp$ and $r_R(Y^n) \cap pR = 0$. Then $(1-p)R = r_R(Rp) \subseteq r_R(Y^n)$. Let $a \in r_R(Y^n)$, then $a = ap + a(1-p)$. Since $ap \in r_R(Y^n) \cap Rp$, $ap = 0$. Thus

$$a = a(1-p) = (1-p)a \in (1-p)R.$$

Hence, $r_R(Y^n) \subseteq (1-p)R$. Therefore, R is a generalized π-baer $*$-ring. $\square$

Let M_R be a right R-module. A submodule N_R of M_R is called *essential* in M_R if for any $x \in M \setminus \{0\}$, there exists $r \in R$ such that $0 \neq xr \in N$. Also recall a *right essential overring* T of R is an overring of R such that R_R is essential in T_R. Recall that for a ring R, the *left socle* of R, $\mathrm{Soc}({}_RR)$, is defined as the sum of all minimal left ideals of R. Equivalently, $\mathrm{Soc}({}_RR)$ is the intersection of all essential left ideals of R (see [25, Exercise 6.12]). The *right socle*, $\mathrm{Soc}(R_R)$, is defined similarly. One can easily check that both socles are ideals of R.

Corollary 2.15. *Let R be a generalized π-Baer $*$-ring and Y a projection invariant left ideal of R. Then there exist a central projection $p \in R$ and a positive integer n such that Y^n is left essential in Rp.*

Proof. By Proposition 2.14, there exist a central projection p and a positive integer n such that $Y^n \subseteq Rp$ and $r_R(Y^n) \cap Rp = 0$. Suppose that Y^n is not left essential in Rp. Then there exists a nonzero left ideal X of the ring Rp such that $Y^n \cap X = 0$. So $X \subseteq r_R(Y^n) \cap Rp = 0$, which is a contradiction. $\square$

Proposition 2.16. *Let R be a $*$-ring and $e \in R$ a central projection. If R is a generalized π-Baer $*$-ring, then so is eRe.*

Proof. The proof is straightforward. $\square$

Proposition 2.17. *The center of a generalized π-Baer $*$-ring is a generalized Baer $*$-ring (and hence a generalized π-Baer $*$-ring).*

Proof. Let R be a generalized π-Baer $*$-ring. Then by Proposition 2.7, R is a generalized π-Baer ring in which every left (right) semicentral idempotent is a central projection. Thus every idempotent of $\mathrm{C}(R)$, the center of R, is a projection. Also, by [31, Proposition 2.22] $\mathrm{C}(R)$ is a generalized π-Baer ring. Therefore, $\mathrm{C}(R)$ is a generalized π-Baer $*$-ring. □

Proposition 2.18. *Let $\{R_\gamma \mid \gamma \in \Gamma\}$ be a family of $*$-rings, and $R = \prod_{\gamma\in\Gamma} R_\gamma$. Then we have the following.*

(1) *If R is a generalized π-Baer $*$-ring, then R_γ is a generalized π-Baer $*$-ring for each $\gamma \in \Gamma$.*
(2) *If $|\Gamma| < \infty$ and for each $\gamma \in \Gamma$, R_γ is a generalized π-Baer $*$-ring, then R is a generalized π-Baer $*$-ring.*

Proof. (1) The proof follows immediately from Proposition 2.16.

(2) It is enough to take $\Gamma = \{1, 2, \ldots, k\}$ for some $k \in \mathbb{N}$. Put $R = \prod_{i=1}^{k} R_i$. Assume that R_i be a generalized π-Baer $*$-ring, for each $i = 1, \ldots, k$. Let Y be a projection invariant left ideal of R. It is easy to see that $Y = \prod_{i=1}^{k} Y_i$, for some projection invariant left ideal Y_i of R_i. Since R_i is a generalized π-Baer $*$-ring, $r_{R_i}(Y_i^{n_i}) = p_i R_i$, for some central projection $p_i \in R_i$ and some positive integers n_i. Put $n = \max\{n_1, n_2, \ldots, n_k\}$. By [31, Lemma 2.18] $r_{R_i}(Y_i^{n}) = p_i R_i$ for each i. Then $r_R(Y^n) = \prod_{i=1}^{k} p_i R_i$. Put $p = (p_1, \ldots, p_k) \in R$, it is clear that p is a projection of R. So $r_R(Y^n) = pR$. Therefore, R is a generalized π-Baer $*$-ring. □

Let R be a $*$-ring. A right (left, two-sided) ideal I of R is said to be a $*$-*essential* right (left, two-sided) ideal in R if $I \neq 0$ and $I \cap J \neq 0$ for any nonzero self-adjoint ideal J of R. An ideal P of R is said to be a $*$-*prime* ideal of R if $IJ \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$, where I and J are self-adjoint ideals of R (see [8]).

Proposition 2.19. *Every $*$-prime (prime) ideal of a generalized π-Baer $*$-ring R is either a $*$-essential (essential) ideal or is generated by a central projection.*

Proof. We prove the case of $*$-prime ideal, the other one can be shown similarly. Let P be a $*$-prime ideal of R. If P is not a $*$-essential ideal of R, then there exists a nonzero self-adjoint ideal I of R such that $P \cap I = 0$. Since R is a generalized π-Baer $*$-ring, there exists a positive integer n such that $r_R(I^n) = qR$ for some central projection

$q \in R$. It is clear that $P \subseteq r_R(I) \subseteq r_R(I^n) = qR$. On the other hand, I^n and qR are self-adjoint ideals and $I^n(qR) = 0$. Then $I^n \subseteq P$ or $qR \subseteq P$, since P is a $*$-prime ideal. If $I^n \subseteq P$ then $I \subseteq P$ and so $I \cap P = I = 0$, which is a contradiction. Hence $q \in P$ and this implies that $P = qR$. $\square$

Proposition 2.20. *Let R be a generalized π-Baer $*$-ring and Y a projection invariant left* (*right*) *ideal of R. If $r_R(Y)$* ($\ell_R(Y)$) *is a $*$-essential ideal of R, then Y is nilpotent.*

Proof. Let Y be a projection invariant left ideal of R such that $r_R(Y)$ is $*$-essential in R. Since R is a generalized π-Baer $*$-ring, there is a central projection $p \in R$ such that $r_R(Y^n) = pR$ for some positive integer n. We show that $Y^n = 0$. Assume to the contrary that $Y^n \neq 0$, so $p \neq 1$. Then $(1-p)R$ is a nonzero self-adjoint ideal and

$$r_R(Y) \cap (1-e)R \subseteq r_R(Y^n) \cap (1-p)R = pR \cap (1-p)R = 0,$$

which is a contradiction as $r_R(Y)$ is a $*$-essential ideal of R. Thus $Y^n = 0$. $\square$

3. Extensions

Let R be a ring and T be an overring of R. Recall that T is called a *right essential overring* of R if R_R is essential in T_R. Also, recall from [31] that, R is said to satisfy the *power intersection of projection invariant right (left) ideals property* (right (left) PII for short) related to T, if for every projection invariant right (left) ideal Y of T and every positive integer n, there exists $m \geq n$ such that $(Y \cap R)^m = Y^m \cap R$.

In the following Theorem, we provide some conditions to ensure the transfer of generalized π-Baer $*$ property between a $*$-ring to its overrings.

Theorem 3.1. *Let R be a generalized π-Baer $*$-ring, T a right* (*or left*) *essential overring of R, and the involution of R extends to T. If R has the left* (*right*) *PII property related to T, then T is a generalized π-Baer $*$-ring and R contains all central projections of T.*

Proof. Assume that R is a generalized π-Baer $*$-ring. First, we note that T has a unity element 1_T with $1_T = 1_R$. Indeed, if $t \neq t1_R$ for some $t \in T \setminus \{0\}$, then there exists $r \in R$ such that

$$0 \neq (t - t1_R)r = tr - tr = 0,$$

a contradiction. Let Y be a projection invariant left ideal of T and $X = Y \cap R$. It is easily seen that X is a projection invariant left ideal of R. So there exist a central projection $p \in R$ and a positive integer n

such that $r_R(X^n) = pT$. We claim that $r_T(Y^n) = pT$. Let $a \in r_T(Y^n)$. Assume that $(1-p)a \neq 0$. Since R_R is essential in T_R, there exists $r \in R$ with $0 \neq (1-p)ar \in R$. Then $0 \neq (1-p)ar \in r_R(Y^n) \subseteq r_R(X^n)$, a contradiction. So $(1-p)r_T(Y^n) = 0$. Therefore, $r_T(Y^n) \subseteq pT$. Now assume that there exists $y \in Y^n$ such that $yp \neq 0$. One can show that T is also a left essential overring of R. Then there exists $s \in R$ with $0 \neq syp \in R$. Hence $syp \in Y^n \cap R = (Y \cap R)^n = X^n$. But $sye \in X^n p = 0$, a contradiction. Hence $Y^n p = 0$. Therefore, $r_T(Y^n) = pT$, and so T is a generalized π-Baer $*$-ring.

To prove the last part of the statement, let $p \in T$ be a central projection, and set $Y = T(1-e)$ and $X = Y \cap R$. Then there exist a projection $q \in R$ and an integer n such that $r_R(X^n) = qR$. Then $r_T(Y^n) = qT$. On the other hand, $r_T(Y^n) = r_T(Y) = pT$. So $pT = qT$ and $p = q$. Thus R contains all central projections of T. $\square$

The following Lemma will be useful.

Lemma 3.2 ([9], Theorem 2.3)**.** *For a ring R, let T be $R[X]$ or $R[[X]]$, where X is a nonempty set of not necessarily commuting indeterminates. If $e(x) \in \mathrm{S}_\ell(T)$, then $e_0 \in \mathrm{S}_\ell(R)$, where e_0 is the constant term of $e(x)$. Moreover, $e(x)T = e_0 T$.*

In [5], Armendariz has shown that a reduced ring R is a Baer ring if and only if $R[x]$ is a Baer ring. In the next theorem, we show that being a generalized π-Baer $*$-ring is preserved by polynomial extensions. Note that the involution of a $*$-ring R can be naturally extended to an involution on $R[x]$ and $R[[x]]$.

Theorem 3.3. *Let R be a $*$-ring and let X is an arbitrary nonempty set of commuting indeterminates. Then the following conditions are equivalent.*

(1) *R is a generalized π-Baer $*$-ring;*
(2) *$R[X]$ is a generalized π-Baer $*$-ring;*
(3) *$R[[X]]$ is a generalized π-Baer $*$-ring.*

Proof. We will prove the equivalence (1)$\Leftrightarrow$(2). The other one can be proved similarly. For this, we show that R is generalized π-Baer $*$-ring if and only if $R[x]$ is generalized π-Baer $*$-ring. This result can be generalized to X by analogy. Assume that R is a generalized π-Baer $*$-ring. Let Y be a projection invariant right ideal of $T := R[x]$. By [31, Theorem 4.4], T is a generalized π-Baer ring. Thus, $\ell_T(Y^n) = Te(x)$ for some idempotent $e(x) \in \mathrm{S}_\ell(T)$ and some positive integer n. By Lemma 3.2, $e_0 \in \mathrm{S}_\ell(R)$ where e_0 is the constant term of $e(x)$. By Proposition 2.7, e_0 is a central projection, since R is a generalized

π-Baer $*$-ring. By Lemma 3.2, $e(x)T = e_0T$. Then

$$T(e(x))^* = Te_0^* = e_0T = e(x)T,$$

and hence $e(x) = e(x)(e(x))^* = (e(x))^*$. So $e(x)$ is a projection of T. Thus T is a generalized π-Baer $*$-ring.

Conversely, let Y be a projection invariant left ideal of R. Then by [10, Lemma 4.1(iii)], $Y[x]$ is a projection invariant left ideal of $R[x]$. Since $R[x]$ is a generalized π-Baer $*$-ring, there exists a projection $e(x) \in R[x]$ such that $r_{R[x]}((Y[x])^n) = e(x)R[x]$ for some positive integer n. Assume that e_0 be the constant term of $e(x)$. Since $e(x) \in T$ is a projection, it follows that e_0 is a projection of R. We show that $r_R(Y^n) = e_0R$. Since $e(x)(Y[x])^n = e(x)Y^n[x] = 0$, $e_0Y^n = 0$. Thus, $e_0R \subseteq r_R(Y^n)$. Now, let $a \in r_R(Y^n)$, then

$$a \in r_{R[x]}(Y^n[x]) = r_{R[x]}((Y[x])^n) = e(x)R[x].$$

Hence, $a = e(x)f(x)$. So $e(x)a = a$ and that $a = e_0a$. Hence $a \in e_0R$. Therefore, $r_R(Y^n) = e_0R$, and that R is a generalized π-Baer $*$-ring. □

Proposition 3.4. *Let R be a generalized π-Baer $*$-ring and n be a positive integer. Then $\mathrm{M}_n(R)$, with the $*$-transpose involution, is a generalized π-Baer $*$-ring.*

Proof. Let R be a generalized π-Baer $*$-ring. Then by Proposition 2.9, R is a generalized quasi-Baer $*$-ring. Now [3, Theorem 2.22] implies that $\mathrm{M}_n(R)$ is a generalized quasi-Baer $*$-ring. It is easy to see that $\mathrm{M}_n(R)$ is generated by its idempotents. So [10, Corollary 2.2(iii)] yields that every projection invariant one-sided ideal of $\mathrm{M}_n(R)$ is an ideal of $\mathrm{M}_n(R)$. Therefore, $\mathrm{M}_n(R)$ is a generalized π-Baer $*$-ring. Moreover, this assertion follows from [31, Proposition 3.1] and Proposition 2.7. □

In the remainder of this section, we are concerned with the matrix rings $T_n(R)$, $S_n(R)$, $A_n(R)$, $B_n(R)$, $U_n(R)$, and $V_n(R)$.

We will see that an abelian $*$-ring R is a generalized π-Baer $*$-ring if and only if so are these matrix rings, for $n \geq 2$ (see Theorem 4.4 below).

Definition 3.5 ([3], Definition 3.1)**.** Let $n \geq 2$ be an integer, and let $V_n = \sum_{i=1}^{n-1} E_{i,i+1}$, where the E_{ij}, $1 \leq i, j \leq n$, be the standard matrix units. Following Lee and Zhou [26], we define

$$A_n(R) = R\mathrm{I}_n + \sum_{\ell=2}^{[\frac{n}{2}]} RV_n^{\ell-1} + \sum_{i=1}^{[\frac{n+1}{2}]} \sum_{j=[\frac{n}{2}]+i}^{n} RE_{ij},$$

and

$$B_n(R) = R\mathrm{I}_n + \sum_{\ell=3}^{[\frac{n}{2}]} RV_n^{\ell-2} + \sum_{i=1}^{[\frac{n+1}{2}]+1} \sum_{j=[\frac{n}{2}]+i-1}^{n} RE_{ij}.$$

According to [3] we define

$$U_n(R) = R\mathrm{I}_n + \sum_{i=1}^{[\frac{n-1}{2}]} \sum_{j=[\frac{n}{2}]+1}^{n} RE_{ij} + \sum_{j=[\frac{n-1}{2}]+2}^{n} RE_{[\frac{n-1}{2}]+1,j}.$$

The ring $S_n(R)$ is defined as a subring of $T_n(R)$ by

$$S_n(R) = R\mathrm{I}_n + \sum_{1\le i<j\le n} RE_{ij}.$$

Also, the ring $V_n(R)$ is defined as a subring of $S_n(R)$ by

$$V_n(R) = R\mathrm{I}_n + \sum_{\ell=2}^{n} RV_n^{\ell-1}.$$

There are two involutions $⋇$ and $\star$ on the triangular matrix ring over any unital $*$-ring (see [3, Definition 3.1]).

Let R be a unital ring with an involution $*$, and for each $n \ge 2$ consider the ring $\mathrm{M}_n(R)$. Let $*$ denote the $*$-transpose involution on $\mathrm{M}_n(R)$. Put $E = \sum_{i+j=n+1} E_{ij}$. The involution $⋇$ on $\mathrm{M}_n(R)$ is defined by $A^{⋇} = (EAE)^*$, for $A = (a_{ij}) \in \mathrm{M}_n(R)$. By inspection, we can see that the subrings $T_n(R)$, $S_n(R)$, $A_n(R)$, $B_n(R)$, $U_n(R)$, and $V_n(R)$ of $\mathrm{M}_n(R)$ are self-adjoint with respect to the involution $⋇$. Hence they are $*$-rings with the involution $⋇$.

Now, let $n = 2m$ with $m \in \mathbb{N}$. Consider

$$D = \sum_{i=1}^{m} E_{ii} - \sum_{i=m+1}^{n} E_{ii} \in \mathrm{M}_n(R).$$

The involution $\star$ on $\mathrm{M}_n(R)$ is defined by $A^\star = DA^{⋇}D^{-1}$, for each $A \in \mathrm{M}_n(R)$. It is easy to see that the subrings $T_n(R)$, $S_n(R)$, and $U_n(R)$ of $\mathrm{M}_n(R)$ are self-adjoint with respect to the involution $\star$. Hence these algebras are also $*$-rings with the involution $⋇$.

Theorem 3.6. *Let R be an abelian $*$-ring and let $n \ge 2$ be an integer. Then the following conditions are equivalent.*

(1) *R is a generalized π-Baer $*$-ring;*
(2) *$S_n(R)$ with the involution $⋇$ is a generalized π-Baer $*$-ring;*
(3) *$A_n(R)$ with the involution $⋇$ is a generalized π-Baer $*$-ring;*
(4) *$B_n(R)$ with the involution $⋇$ is a generalized π-Baer $*$-ring;*
(5) *$U_n(R)$ with the involution $⋇$ is a generalized π-Baer $*$-ring;*

(6) *$V_n(R)$ with the involution $\circledast$ is a generalized π-Baer $*$-ring.*

Proof. This follows by using [31, Lemma 3.7] and a similar argument as in the proof of [3, Theorem 3.4]. $\square$

A similar proof as that of Theorem 3.6, proves the following.

Theorem 3.7. *Let R be an abelian $*$-ring and let $n \geq 2$ be an even integer. Then the following conditions are equivalent.*

(1) *R is a generalized π-Baer $*$-ring;*
(2) *$S_n(R)$ with the involution $\star$ is a generalized π-Baer $*$-ring;*
(3) *$U_n(R)$ with the involution $\star$ is a generalized π-Baer $*$-ring.*

4. Banach $*$-algebras

In this section we give examples of Banach $*$-algebras which are generalized π-Baer $*$-rings but are not π-Baer $*$-rings. To motivate the significance of these examples, observe that every Baer $*$-ring is a π-Baer $*$-ring, and every π-Baer $*$-ring is a generalized π-Baer $*$-ring. There are algebraic examples distinguishing these classes. Since these algebraic notions have their roots (and some applications) in Functional Analysis, it is important to have analytic examples distinguishing these classes. We show that if a pre-C*-algebra is a generalized π-Baer $*$-ring, then it is a π-Baer $*$-ring (Theorem 4.6). Thus, a C*-algebra is a generalized π-Baer $*$-ring if and only if it is a π-Baer $*$-ring. We construct examples of Banach $*$-algebras which are generalized π-Baer $*$-rings but are not π-Baer $*$-rings. At the end of this section, we characterize locally compact abelian groups G where the group algebra $L^1(G)$ and the group C*-algebra $C^*(G)$ are generalized π-Baer $*$-rings.

Let us first recall some definitions from Functional Analysis for the convenience of the reader.

Definition 4.1 ([28])**.** Let $\mathscr{A}$ be an algebra over $\mathbb{C}$. Then $\mathscr{A}$ is called a *normed algebra* if there is a norm $\|\cdot\|$ on $\mathscr{A}$ such that $\|xy\| \leq \|x\|\|y\|$, for all $x, y \in \mathscr{A}$. A normed algebra $\mathscr{A}$ is called a *Banach algebra* if it is complete in the norm. If, in addition, $\mathscr{A}$ has a unity element 1 such that $\|1\| = 1$, it is called a *unital Banach algebra*. If a normed algebra $\mathscr{A}$ admits an involution $*$ such that $(\alpha x)^* = \overline{\alpha} x^*$, and $\|x^*\| = \|x\|$, for every $x \in \mathscr{A}$ and every $\alpha \in \mathbb{C}$, then $\mathscr{A}$ is called a *normed $*$-algebra*. A *Banach $*$-algebra* (or *involutive normed algebra*) is a complete normed $*$-algebra. A *C*-norm* on a complex $*$-algebra $\mathscr{A}$ is a norm $\|\cdot\|$ such that $\|xy\| \leq \|x\|\|y\|$, and $\|x^*x\| = \|x\|^2$, for all $x, y \in \mathscr{A}$. A complex $*$-algebra equipped with a C*-norm is called a *pre-C*-algebra*. If $\mathscr{A}$ is a pre-C*-algebra which is complete in the norm then it is called a

C^*-algebra. Also, if $\mathscr{A}$ is a pre-C*-algebra then there is a unique C*-algebra $\mathscr{B}$, called the *enveloping* C*-algebra of $\mathscr{A}$, containing $\mathscr{A}$ as a dense $*$-subalgebra.

Definition 4.2 ([21])**.** Let $\mathscr{A}$ be a Banach algebra. An *approximate identity* for $\mathscr{A}$ is a net $(u_\gamma)_{\gamma\in\Gamma}$ of elements of $\mathscr{A}$, possessing the following properties:

(1) $\|u_\gamma\| \leq 1$ for every $\gamma \in \Gamma$;
(2) $\|u_\gamma x - x\| \to 0$ and $\|x u_\gamma - x\| \to 0$ for every $x \in \mathscr{A}$.

We say that $(u_\gamma)_{\gamma\in\Gamma}$ is a *left approximate identity* for $\mathscr{A}$ if (1) holds and instead of (2) we have $\|u_\gamma x - x\| \to 0$ for every $x \in \mathscr{A}$. Similarly, one defines the notion of a right approximate identity.

Note that an approximate identity $(u_\gamma)_{\gamma\in\Gamma}$ for a C*-algebra $\mathscr{A}$ is defined as in Definition 4.2 except that we also require that each u_λ is positive and that the net $(u_\gamma)_{\gamma\in\Gamma}$ is increasing [28].

Lemma 4.3 ([3], Corollary 5.10 and Theorem 5.14)**.** *Let A be a unital C^*-algebra. Then we have the following.*

(1) *For each $n \geq 2$, the algebras $S_n(\mathscr{A})$, $A_n(\mathscr{A})$, $B_n(\mathscr{A})$, $U_n(\mathscr{A})$ and $V_n(\mathscr{A})$ with involution $⋇$ are Banach $*$-subalgebra of $T_n(\mathscr{A})$.*
(2) *For each even integer $n \geq 2$, the algebras $S_n(\mathscr{A})$and $U_n(\mathscr{A})$ with involution $\star$ are Banach $*$-subalgebra of $T_n(\mathscr{A})$.*

In the next theorem we obtain classes of finite dimensional Banach $*$-algebras which are generalized π-Baer $*$-rings, but they are not π-Baer $*$-rings.

Theorem 4.4. *Let $\mathscr{A}$ be a C^*-algebra. If $\mathscr{A}$ is a generalized π-Baer $*$-ring, then*

(1) *for each $n \geq 2$, the Banach $*$-algebras $S_n(\mathscr{A})$, $A_n(\mathscr{A})$, $B_n(\mathscr{A})$, $U_n(\mathscr{A})$ and $V_n(\mathscr{A})$ with involution $⋇$ are generalized π-Baer $*$-rings, but they are not π-Baer $*$-rings. In particular, the Banach $*$-algebras $S_n(\mathbb{C})$, $A_n(\mathbb{C})$, $B_n(\mathbb{C})$ and $U_n(\mathbb{C})$ are generalized π-Baer $*$-rings, but they are not π-Baer $*$-rings.*
(2) *for each even integer $n \geq 2$, the Banach $*$-algebras $S_n(\mathscr{A})$ and $U_n(\mathscr{A})$ with involution $\star$ are generalized π-Baer $*$-rings, but they are not π-Baer $*$-rings. In particular, the Banach $*$-algebras $S_n(\mathbb{C})$ and $U_n(\mathbb{C})$ are generalized π-Baer $*$-rings, but they are not π-Baer $*$-rings.*

Proof. We prove part (1), the other part can be shown similarly. By Lemma 4.3, the normed $*$-algebras $S_n(\mathscr{A})$, $A_n(\mathscr{A})$, $B_n(\mathscr{A})$ and $U_n(\mathscr{A})$ are Banach $*$-algebras. Now let $n \geq 2$ and R be any of the $*$-rings

$S_n(\mathscr{A})$, $A_n(\mathscr{A})$, $B_n(\mathscr{A})$, $U_n(\mathscr{A})$ and $V_n(\mathscr{A})$. Then by Theorem 3.6, R is a generalized π-Baer $*$-ring. The involution $\circledast$ on R is not semiproper. To see this, we have $E_{1n}RE_{1n}^{\circledast} = E_{1n}RE_{1n} = 0$ but $E_{1n} \neq 0$. Therefore, by [30, Proposition 2.9], R is not a π-Baer $*$-ring. $\square$

Lemma 4.5. *Let $\mathscr{A}$ be a pre-C*-algebra, Y a left (right) ideal Y of $\mathscr{A}$, and $n \in \mathbb{N}$. Then*

$$r_{\mathscr{A}}(Y) = r_{\mathscr{A}}(\overline{Y}) = r_{\mathscr{A}}(Y^n)$$

$(\ell_{\mathscr{A}}(Y) = \ell_{\mathscr{A}}(\overline{Y}) = \ell_{\mathscr{A}}(Y^n))$, *where $\overline{Y}$ denotes the closure of Y in $\mathscr{A}$.*

Proof. First, we can easily see that for any nonempty subset S of $\mathscr{A}$, $r_{\mathscr{A}}(S) = r_{\mathscr{A}}(\overline{S})$. Indeed, if $r \in r_{\mathscr{A}}(S)$ and $s \in \overline{S}$, then there exists a sequence $(s_n)_{n\in\mathbb{N}}$ in S such that $s_n \to s$. Thus $sr = \lim_{n\to\infty} s_n r = 0$, and so $r \in r_{\mathscr{A}}(\overline{S})$. Hence $r_{\mathscr{A}}(S) = r_{\mathscr{A}}(\overline{S})$.

Now we prove that $r_{\mathscr{A}}(Y) = r_{\mathscr{A}}(Y^n)$, where Y is a left ideal of $\mathscr{A}$. Let $\mathscr{B}$ denote the enveloping C*-algebra of $\mathscr{A}$. Let X be the closure of Y in $\mathscr{B}$. Then $r_{\mathscr{A}}(Y) = r_{\mathscr{B}}(Y) \cap \mathscr{A} = r_{\mathscr{B}}(X) \cap \mathscr{A}$ and

$$r_{\mathscr{A}}(Y^n) = r_{\mathscr{B}}(Y^n) \cap \mathscr{A} = r_{\mathscr{B}}(X^n) \cap \mathscr{A}.$$

Thus we need to show that $r_{\mathscr{B}}(X) = r_{\mathscr{B}}(X^n)$. Since X is a closed left ideal of $\mathscr{B}$, it has a right approximate identity $(x_\gamma)_{i\in\Gamma}$ (see [18, II.5.3.3, p. 96]). Let $a \in r_{\mathscr{A}}(X^n)$. We show that $a \in r_{\mathscr{A}}(X)$. Let $x \in X$. Then $xx_{\gamma_1}x_{\gamma_2}\cdots x_{\gamma_{n-1}}a = 0$ for each $\gamma_1, \gamma_2, \ldots, \gamma_{n-1} \in \Gamma$. Since $\lim_{\gamma_{n-1}\to\infty} xx_{\gamma_{n-1}} = x$, we get

$$xx_{\gamma_1}x_{\gamma_2}\cdots x_{\gamma_{n-2}}a = 0.$$

Continuing this procedure, we obtain $xa = 0$. Thus $a \in r_{\mathscr{A}}(X)$, and so $r_{\mathscr{A}}(X^n) \subseteq r_{\mathscr{A}}(X)$. The reverse inclusion is obvious.

Similarly, if Y is a right ideal we get $\ell_{\mathscr{A}}(Y) = \ell_{\mathscr{A}}(\overline{Y}) = \ell_{\mathscr{A}}(Y^n)$. $\square$

As a consequence of Lemma 4.5, we obtain the following result.

Proposition 4.6. *Let $\mathscr{A}$ be a pre-C*-algebra. Then $\mathscr{A}$ is a generalized π-Baer $*$-ring if and only if it is a π-Baer $*$-ring.*

The following proposition shows that the previous result can be extended to certain normed $*$-algebras.

Proposition 4.7. *Let $\mathscr{A}$ be a normed $*$-algebra such that every closed projection invariant one-sided ideal of $\mathscr{A}$ has a left (or right) approximate identity. Then $\mathscr{A}$ is a generalized π-Baer $*$-ring if and only if it is a π-Baer $*$-ring.*

Proof. It is enough to prove that $r_{\mathscr{A}}(Y^n) = r_{\mathscr{A}}(Y)$ for every projection invariant left ideal Y of $\mathscr{A}$ and every positive integer n. For this, let Y be a projection invariant left ideal of $\mathscr{A}$ and let $n \in \mathbb{N}$. Since $r_{\mathscr{A}}(Y) = r_{\mathscr{A}}(\overline{Y})$ and $r_{\mathscr{A}}(Y^n) = r_{\mathscr{A}}((\overline{Y})^n)$, we may assume that Y is a projection invariant closed left ideal of $\mathscr{A}$. Now a similar argument as in the proof of Lemma 4.5 shows that $r_{\mathscr{A}}(Y^n) = r_{\mathscr{A}}(Y)$. $\square$

In the next result, we characterize locally compact abelian groups G where the group algebra $L^1(G)$ and the group C*-algebra $C^*(G)$ are generalized π-Baer $*$-rings. Recall the definition of the group algebra $L^1(G)$ and the group C*-algebra $C^*(G)$ for a locally compact Hausdorff group G [22]. Specially, recall that every such group has a unique (up to a positive constant) nonzero left invariant Borel measure μ, called the Haar measure, and $L^1(G)$ is the set

$$\left\{ f\colon G \to \mathbb{C} \mid f \text{ is mesurable and } \int_G |f| d\mu < \infty \right\}$$

modulo the equivalence relation of being equal almost everywhere.

Also, recall the definition of the Pontrjagin dual $\widehat{G}$ of a locally compact Hausdorff abelian group G [22, Section 4.1]. In particular, $C_c(G) \subseteq L^1(G) \subseteq C^*(G)$ and $C_c(G)$ and $L^1(G)$ are dense $*$-subalgebras of $C^*(G)$ in the universal norm.

Theorem 4.8. *Let G be a locally compact Hausdorff abelian group. The following statements are equivalent:*

(1) *$L^1(G)$ is a generalized π-Baer $*$-ring;*
(2) *$L^1(G)$ is a π-Baer $*$-ring;*
(3) *$L^1(G)$ is a quasi-Baer $*$-ring;*
(4) *$L^1(G)$ is a generalized quasi-Baer $*$-ring;*
(5) *$L^1(G)$ is a Baer $*$-ring;*
(6) *$C^*(G)$ is a generalized π-Baer $*$-ring;*
(7) *$C^*(G)$ is a π-Baer $*$-ring;*
(8) *$C^*(G)$ is a quasi-Baer $*$-ring;*
(9) *$C^*(G)$ is a generalized quasi-Baer $*$-ring;*
(10) *$C^*(G)$ is a Baer $*$-ring;*
(11) *$\widehat{G}$ is a Stonean space;*
(12) *G is finite.*

Proof. Since G is abelian, $C^*(G)$ and $L^1(G)$ are commutative. Also, since $L^1(G)$ is a $*$-subalgebra of $C^*(G)$, it is a pre-C*-algebra with the induced norm of $C^*(G)$.

(1)$\Leftrightarrow$(2) and (6)$\Leftrightarrow$(7) follow from Proposition 4.6.

(2)$\Leftrightarrow$(3) and (7)$\Leftrightarrow$(8) follow from the fact that a commutative $*$-ring is a π-Baer $*$-ring if and only if it is a quasi-Baer $*$-ring.

(3)$\Leftrightarrow$(4)$\Leftrightarrow$(5), (8)$\Leftrightarrow$(9)$\Leftrightarrow$(10)$\Leftrightarrow$(11)$\Leftrightarrow$(12), and (3)$\Leftrightarrow$(8) follow from [3, Theorem 5.18]. $\square$

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GENERALIZED π–BAER $*$–RINGS

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$*$–حلقه‌های π–بئر تعمیم‌یافته

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$*$–حلقه‌ی R را $*$–حلقه‌ی π–بئر تعمیم‌یافته گوییم هرگاه برای هر ایده‌آل چپ پایای تصویر Y از R، پوچساز راست Y^n به عنوان یک ایده‌آل راست توسط یک تصویر تولید شود، برای عدد صحیح مثبت n، وابسته به Y. در این مقاله، برخی از ویژگی‌های $*$–حلقه‌های π–بئر تعمیم‌یافته را بررسی می‌کنیم. نشان می‌دهیم این ویژگی نسبت به توسیع‌های چندجمله‌ای، حلقه‌های ماتریسی، و چندین کلاس از حلقه‌های ماتریس‌های مثلثی خوش‌رفتار است. روابط بین $*$–حلقه‌های π–بئر تعمیم‌یافته و کلاس‌های مرتبط با آن از جمله حلقه‌های π–بئر تعمیم‌یافته، $*$–حلقه‌های بئر تعمیم‌یافته، $*$–حلقه‌های شبه–بئر تعمیم‌یافته و $*$–حلقه‌های π–بئر را بررسی می‌کنیم. مثال‌های جبری از $*$–حلقه‌های π–بئر تعمیم‌یافته ارائه می‌دهیم که $*$–حلقه‌ی π–بئر نیستند. نشان می‌دهیم برای پیش–C^*–جبرها این دو مفهوم معادل هستند. کلاس‌هایی از $*$–جبرهای باناخ به دست می‌آوریم که $*$–حلقه‌ی π–بئر تعمیم‌یافته هستند اما $*$–حلقه‌ی π–بئر نیستند. مقاله را با اثبات نتیجه‌ی زیر به پایان می‌رسانیم که برای گروه آبلی موضعاً فشرده‌ی G، جبر گروهی $L^1(G)$ یک $*$–حلقه‌ی π–بئر تعمیم‌یافته است، اگر و تنها اگر C^*–جبر گروهی $C^*(G)$ یک $*$–حلقه‌ی π–بئر تعمیم‌یافته باشد، اگر و تنها اگر G متناهی باشد.

کلمات کلیدی: حلقه‌ی π–بئر تعمیم‌یافته، $*$–حلقه‌ی π–بئر تعمیم‌یافته، $*$–حلقه‌ی شبه–بئر تعمیم‌یافته، حلقه‌ی بئر تعمیم‌یافته، $*$–حلقه‌ی بئر تعمیم‌یافته.