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JACOBSON MONOFORM MODULES

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ABSTRACT. In this paper, we introduce and study the concept of Jacobson monoform modules which is a proper generalization of that of monoform modules. We present a characterization of semisimple rings in terms of Jacobson monoform modules by proving that a ring R is semisimple if and only if every R-module is Jacobson monoform. Moreover, we demonstrate that over a ring R, the properties monoform, Jacobson monoform, compressible, uniform and weakly co-Hopfian are all equivalent.

1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unitary R-modules. Let L be an R-module, for submodules A and B of L, $A \leq B$ denotes that A is a submodule of B, $A \leq^{\oplus} L$ denotes that A is a direct summand of L, and $E(L)$, $\mathrm{Rad}(L)$, $\mathrm{Soc}(L)$, $\mathrm{End}_R(L)$ will denote the injective hull, the radical, the socle and the ring of endomorphisms of a module L.

Recall that a submodule N of L is called a small submodule of L if whenever $N + K = L$ for some submodule K of L, we have $L = K$, and in this case we write $N \ll L$. A module L is called small if it is a small submodule of some module. The socle of L is defined as the sum of all its simple submodules and can be shown to coincide with the intersection of all the essential submodules of L. It is a fully invariant submodule of L. Note that L is semisimple precisely when

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$L = \mathrm{Soc}(L)$. The radical of an R-module L defined as a dual of the socle of L, is the intersection of all maximal submodules of L, taking $\mathrm{Rad}(L) = L$ when L has no maximal submodules. A submodule K of L is said to be Jacobson-small in L ($K \ll_J L$), in case $L = K + P$ with $\mathrm{Rad}(L/P) = L/P$, implies $P = L$ (see [5]). It is clear that if A is a small submodule of L, then A is a Jacobson-small submodule of L, but the converse is not true in general. By [5], if $\mathrm{Rad}(L) = L$ and $K \leq L$, then K is small in L if and only if K is Jacobson-small in L. A submodule K of an R-module L is said to be δ-small in L, written $K \ll_\delta L$, if for every submodule N of L such that $K + N = L$ with L/N singular implies $N = L$ (see [17]).

The study of modules by properties of their endomorphisms is a classical research subject. In [16], Zelmanowitz introduced the concept of monoform modules. We call a module monoform if any its nonzero partial endomorphism is monomorphism. Recall that a submodule N of L is called a dense or rational submodule if $\mathrm{Hom}_R(L/N; E(L)) = 0$. An R-module L is monoform if and only if every nonzero submodule of L is rational (see [16]). A monoform module is uniform (i.e., any two nonzero submodules have nonzero intersection). In [4], Inaam Hadi and Hassan Marhun introduced and studied the notion of small monoform modules. An R-module L is called small monoform, if any nonzero partial endomorphism of L has a small kernel. In [7], the concept of δ-weakly Hopfian modules was introduced. A right R-module L is called δ-weakly Hopfian if any δ-small surjective endomorphism of L is an automorphism. In [2], Diop and Diallo introduced and studied the notion of δ-small monoform modules. An R-module L is called δ-small monoform, if any nonzero partial endomorphism of L has a δ-small kernel. In [10], the authors investigated and introduced the concept of Jacobson Hopfian modules. An R-module L is called Jacobson Hopfian if every surjective endomorphism of L has a Jacobson-small kernel.

Motivated by the above-mentioned works, we are interested in introducing a new generalization of monoform modules namely Jacobson monoform modules (JM modules, for short). We call a module L is JM if every nonzero partial endomorphism of L has a Jacobson-small kernel. The concept of JM modules form a proper generalization of monoform modules (Example 2.3). It is obvious that any small monoform module is JM. Example 2.21 demonstrates that the converse is false, in general.

We discuss the following questions:

1) When does a module have the property that every of its nonzero partial endomorphisms has a Jacobson-small kernel?

2) How can Jacobson monoform modules be used to characterize the base ring itself?

Our paper is structured as follows:

In Section 1, we give some known results which we will cite or use throughout this paper.

In Section 2, we present some equivalent properties and characterizations of JM modules. A nonzero R-module L is called compressible provided for each nonzero submodule N of L there exists a monomorphism $f : L \to N$. We have the obvious implications: compressible $\Rightarrow$ monoform $\Rightarrow$ JM. We will see later that under certain conditions the properties monoform, uniform, compressible and JM are coincide (Theorem 2.10). The dual of an R-module L is $Hom_R(L, R)$, this will be denoted by L^*. If the natural map $L \to (L^*)^*$ is bijective, L will be called reflexive. We prove that for a quasi-Frobenius principal ring R, if L is a JM R-module such that for each $K \leq L$, $\mathrm{Rad}(K) = K$, then L is reflexive and $E(L^*)$ is finitely generated (Proposition 2.15). In proposition 2.18, we obtain that if L is a fully retractable R-module such that for every nonzero submodule N of L, the kernel of any nonzero endomorphism of N is Jacobson-small, then L is JM.

Moreover, we present a characterization of semisimple rings in terms of Jacobson monoform modules by proving that a ring R is semisimple if and only if every R-module is Jacobson monoform (Theorem 2.26)

Let R be a ring and let L be an R-module. We now state a few well known preliminary results:

Remark 1.1.
1. Let R be a ring and L be a right R-module. Then L is nonsingular monoform if and only if L is uniform [12].
2. Let R be a commutative ring and L be an R-module. Then L is monoform if and only if L is uniform prime [12].
3. It is clear that every monoform R-module is small monoform. However the converse in general is not true. $\mathbb{Z}_4$ is a small monoform $\mathbb{Z}$-module but it is not monoform [4].
4. The epimorphic image of small monoform module is not necessarily small monoform [4].
5. Every nonzero submodule of small monoform module is small monoform module [4].
6. Let L be a semisimple R-module. Then the following are equivalent [4].
   (a) L is small monoform.
   (b) L is monoform.
   (c) L is simple.

We list some properties of Jacobson-small submodules that will be used in the paper.

Lemma 1.2. [5]. *Let L be an R-module.*

(1) *Let $A \leq B \leq M$. Then $B \ll_J M$ if and only if $A \ll_J M$ and $B/A \ll_J M/A$.*
(2) *Let A_1, A_2, ...,A_n are submodules of M. Then $A_i \ll_J M$, $\forall$ $i = 1, ..., n$ if and only if $\sum_{i=1}^{n} A_i \ll_J M$.*
(3) *Let A, B be submodules of M with $A \leq B$, if $A \ll_J B$, then $A \ll_J M$.*
(4) *Let $f : M \to N$ be a homomorphism such that $A \ll_J M$, then $f(A) \ll_J N$.*
(5) *Let $M = M_1 \oplus M_2$ be an R-module and let $A_1 \leq M_1$ and $A_2 \leq M_2$. Then $A_1 \oplus A_2 \ll_J M_1 \oplus M_2$ if and only if $A_1 \ll_J M_1$ and $A_2 \ll_J M_2$.*
(6) *Let L be a module and let $X \leq Y \leq L$. If $Y \leq^{\oplus} L$ and $X \ll_J L$, then $X \ll_J Y$.*

Lemma 1.3. [10] *Let K be a submodule of a module L. Then the following statements are equivalent.*

(1) *$K \ll_J L$.*
(2) *If $X + K = L$, then $X \leq^{\oplus} L$ and L/X is semisimple.*

2. Modules in which every partial endomorphism has a Jacobson-small kernel

Definition 2.1. An R-module L is called Jacobson monoform modules (JM modules, for short) if every its nonzero partial endomorphism has a Jacobson-small kernel.

Remark 2.2. By the definitions, every hollow module is JM, but the converse is not hold in general. Note that $L = \mathbb{Z}_6$ is a semisimple $\mathbb{Z}$-module. Since for any semisimple module L, $\mathrm{Rad}(L) = 0$, so every proper submodule is Jacobson-small in L, thus L is JM while it has no nonzero small submodule then it is not hollow.

Example 2.3. Let $H = \mathbb{Z}_{q^\infty}$. Since H is a hollow group, H is a JM group. But H is not monoform because the multiplication by q induces an endomorphism of H which is not a monomorphism.

Theorem 2.4. *The following are equivalent for an R-module L:*

(1) *L is JM.*
(2) *For every nonzero partial endomorphism $f \in Hom(N, L)$ where $0 \neq N \leq L$, if there exists $P \leq N$ such that $f(P) = f(N)$, then*

there exists a semisimple direct summand H of N such that $N = H \oplus P$.

Proof. (1) $\Rightarrow$ (2) Assume that $f \in Hom(N, L)$ where $0 \neq N \leq L$ be a nonzero partial endomorphism, if there exists $P \leq N$ such that $f(P) = f(N)$, then $Kerf + P = N$. Since L is JM, $Kerf \ll_J N$. By Lemma 1.3 $N = H \oplus P$ for some semisimple $H \leq N$.

(2) $\Rightarrow$ (1) Let $f \in Hom(N, L)$ where $0 \neq N \leq L$ be a nonzero partial endomorphism and $Ker(f) + P = N$ for some $P \leq N$, where $Rad(N/P) = N/P$. Then $f(P) = f(N)$. By (2), there exists a semisimple direct summand H of N such that $N = H \oplus P$, then $Rad(N/P) = 0$. Thus $N/P = 0$. Therefore $N = P$ and $Ker(f) \ll_J N$. □

Proposition 2.5. *Let L be a JM R-module such that for each $K \leq L$,* $\mathrm{Rad}(K) = K$. *Then L is uniform.*

Proof. Let L be a JM R-module and N be any nonzero proper submodule of L. If N is not essential. So, there exists a relative complement K of N in L such that $N \oplus K$ is essential in L. Let

$$f : N \oplus K \to L$$

define by $f(n + k) = n$ for all $n + k \in N \oplus K$. It is clear that f is well defined and $f \neq 0$. Since L is JM, $Kerf = \{0\} \oplus K \ll_J N \oplus K$. So according to Lemma 1.3, K is semisimple and $\mathrm{Rad}(K) = 0$. Then by hypothesis K must be zero. Contradiction with $N \oplus K$ is essential in L.

□

Definition 2.6. [3]. A right R-module L is called weakly co-Hopfian if any injective endomorphism of L is essential.

Example 2.7. The following facts are well known:[9]

(1) Any Artinian R-module M (i.e., M has DCC on submodules), is weakly co-Hopfian.
(2) The additive group $\mathbb{Q}$ of rational numbers is a non-Artinian $\mathbb{Z}$-module, which is weakly co-Hopfian.

Definition 2.8. [12]. A nonzero right R-module L is called prime if, whenever N is a nonzero submodule of L and A is an ideal of R such that $NA = 0$, then $LA = 0$.

Remark 2.9. [12].

(1) For any ring R, every compressible right R-module is prime.

(2) Let R be a commutative ring. Then a finitely generated nonzero R-module L is compressible if and only if L is a uniform prime module.
(3) Let R be a commutative ring. Then an R-module L is monoform if and only if L is a uniform prime module.
(4) Let R be a commutative ring. An R-module M is compressible if and only if M is isomorphic to a nonzero submodule of a finitely generated monoform R-module
(5) Let R be a commutative ring. Then every compressible R-module is monoform.

Recall that an Artinian principal ideal ring is a left and right Artinian, left and right principal ideal ring.

Theorem 2.10. *Let R be an Artinian principal ring and L be a prime R-module such that for each $K \leq L$, $\text{Rad}(K) = K$. The following statements are equivalent:*

(1) *L is JM.*
(2) *L is monoform.*
(3) *L is compressible.*
(4) *L is uniform.*
(5) *L is weakly co-Hopfian.*

Proof. (1) $\Rightarrow$ (2) Let L be a JM module and $0 \neq N \leq L$ and $f : N \to L$ be a homomorphism. By Proposition 2.5 L is uniform, then L is weakly co-Hopfian. Since R is an Artinian principal ring, L is finitely generated by [1, Theorem 3.8]. So, there exists an epimorphism $g : R \to L$ such that $R/ann_R(L) \cong L$. Since L is prime, $ann_R(L)$ is a prime ideal of R. Hence, $ann_R(L)$ is a maximal ideal of R because R is Artinian. This implies that L is simple. Hence, L is monoform.

(2) $\Rightarrow$ (1) It is clear.

(1) $\Rightarrow$ (3) By (1) $\Rightarrow$ (2) we obtain that L is a uniform prime finitely generated module, hence by [12, Lemma 26.2.9] L is compressible.

(3) $\Rightarrow$ (1) By Remark 2.9, every compressible R-module is monoform, then it is JM.

(2) $\Rightarrow$ (4) It is clear.

(4) $\Rightarrow$ (2) Suppose L is a uniform module. According to the proof of (1) $\Rightarrow$ (2), L is simple. Therefore, L is monoform.

(4) $\Rightarrow$ (5) It is clear.

(5) $\Rightarrow$ (4) Assume that L is a weakly co-Hopfian module. Then L is simple. Therefore, L is monoform.

□

Corollary 2.11. *Let R be an Artinian principal ring and L be an R-module such that for each $K \leq L$, $\mathrm{Rad}(K) = K$. The following statements are equivalent:*

(1) *L is JM prime.*
(2) *L is simple.*

Proof. (1) $\Rightarrow$ (2) Suppose L is a JM prime module. Then $ann_R(L)$ is a prime ideal of R. Then by Theorem 2.10, L is simple.
(2) $\Rightarrow$ (1) It is clear.

□

Corollary 2.12. *Let R be an Artinian principal ring and L be a JM R-module such that for each $K \leq L$, $\mathrm{Rad}(K) = K$. Then* $\mathrm{End}(L)$ *is a local ring.*

Proof. Since L is a finitely generated module over an Artinian ring, L is of finite length. Thus, L is an indecomposable module of finite length because L is uniform. Therefore, $\mathrm{End}(L)$ is a local ring.

□

Example 2.13. It is clear that a simple module is JM. But in general the converse is not true. For example, $\mathbb{Z}$ is a JM $\mathbb{Z}$-module. However, $\mathbb{Z}$ is not simple.

Example 2.14. Every compressible R-module is JM. In general the converse is not true. For example, $\mathbb{Q}$ is a JM $\mathbb{Z}$-module. But it is not compressible because $Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = \{0\}$.

Recall that a ring R is called quasi-Frobenius if it is right or left Artinian and right or left self-injective.

Proposition 2.15. *Let R be a principal quasi-Frobenius ring and L be a JM R-module such that for each $K \leq L$, $\mathrm{Rad}(K) = K$. Then the following statements are verified:*

(1) *L is reflexive.*
(2) *L^* and $E(L^*)$ are finitely generated.*

Proof. 1) According to Theorem 2.10, L is a finitely generated R-module. Thus, by [6, Theorem 15.11], L is reflexive.
2) Since R is Artinian and L^* is finitely generated, $E(L^*)$ is finitely generated.

□

Proposition 2.16. *Let L be a JM R-module and f be a surjective endomorphism of L. If $N \leq L$, then $f(N) \ll_J L$ if and only if $N \ll_J L$.*

Proof. $\Rightarrow$) Let $N+Y=L$ with $Rad(L/Y)=L/Y$ for some $Y\leq L$. Then $f(N)+f(Y)=L$. Then $f(Y)=L$ because $f(N)\ll_J L$ and $Rad(L/f(Y))=L/f(Y)$. This implies that $Kerf+Y=L$. Since L is JM, $Kerf\ll_J L$. Hence $Y=L$. Therefore $N\ll_J L$.

$\Leftarrow$) By Lemma 1.2. □

Definition 2.17. [11]. A module L is said to be fully retractable if for any nonzero submodule N of L and every nonzero element $g\in Hom_R(N,L)$ we have $Hom_R(L,N)g\neq 0$.

Proposition 2.18. *Let L be a fully retractable R-module such that for every nonzero submodule N of L, the kernel of any nonzero endomorphism of N is Jacobson-small. Then L is JM.*

Proof. Let $0\neq N\leq L$ and $f:N\to L$ such that $f\neq 0$. Since L is fully retractable, there exists $g:L\to N$, $g\neq 0$. Consider

$$N\xrightarrow{\quad f\quad}L\xrightarrow{\quad g\quad}N$$

We have $gf\neq 0$ because L is fully retractable. By hypothesis, $Ker(gf)\ll_J N$. Since $Kerf\subseteq Ker(gf)$, thus according to Lemma 1.2, $Kerf\ll_J N$. Therefore L is JM.

□

Proposition 2.19. *Let L be a semisimple quasi-injective R-module. Then the following statements are equivalent:*

(1) *L is JM;*
(2) *L is Jacobson Hopfian.*

Proof. (1) $\Rightarrow$ (2) Is clear.

(2) $\Rightarrow$ (1) Let $0\neq N\leq L$ and $f:N\to L$ such that $f\neq 0$. Since L is quasi-injective, there exists $g\in End_R(L)$ such that $gi=f$ where i is the inclusion map. Hence, $g(x)=f(x)$ for each $x\in N$ and so $Kerf\leq Kerg$. Since L is Jacobson Hopfian, $Kerg\ll_J L$. So $Kerf\ll_J L$. On the other hand, $Kerf\leq N$ and L is semisimple, then N is a direct summand of L. Hence by Lemma 1.2, $Kerf\ll_J N$. This shows that L is JM.

□

Corollary 2.20. *If R is a semisimple ring, then every R-module is JM.*

Proof. By [10, Theorem 5] and Proposition 2.19 . □

It is obvious that every small monoform module is JM. The following example shows that the converse is false, in general.

Example 2.21. Let R be a semisimple ring, hence according to Corollary 2.20, $R^{(\mathbb{N})}$ is JM. But the kernel of every endomorphism of $R^{(\mathbb{N})}$ is not small by [8, Example 2.11]. Thus $R^{(\mathbb{N})}$ is not small monoform.

For a right R-module L, Talebi and Vanaja [15], defined the submodule

$$\begin{aligned}\overline{Z}(L) &= \cap\{Kerf : f \in Hom(L,N), N \in S\}\\ &= \cap\{K \subset L, L/K \in S\}\end{aligned}$$

as a dual of singular submodule, where S denotes the class of all small right R-modules. A module L is called cosingular (resp. noncosingular) if $\overline{Z}(L) = 0$ (resp. $\overline{Z}(L) = L$). Recall that a ring R is called CP in case every cosingular right R-module is projective. (see [14]).

Proposition 2.22. *Let R be a CP ring such that has no nonzero semisimple projective R-module and L be a cosingular R-module. Then the following statements are equivalent:*

(1) *L is JM.*
(2) *L is small monoform.*

Proof. (1) ⇒ (2) Let L be a JM R-module, N be a nonzero submodule of L and $f \in Hom(N, L)$ be a nonzero partial endomorphism. Assume $Kerf + K = N$ for some $K \leq N$. Since L is JM, $Kerf \ll_J N$. Then by Theorem 2.4, $N = K \oplus H$ for some semisimple submodule H of N. Since L is cosingular, H is cosingular. And since R is CP, H is projective. By hypothesis, $H = 0$. This implies that $N = K$ and so $Kerf \ll N$. Hence L is small monoform.
(2) ⇒ (1) Is clear.

□

Recall that a ring R is right CD if and only if every cosingular right R-module is discrete (see [13]).

Proposition 2.23. [13, Proposition 2.26] *Let R be a commutative domain. Then the following are equivalent:*

(1) *R is CD;*
(2) *Every cosingular R-module is projective.*

Corollary 2.24. *Let R be a commutative domain and L be a cosingular R-module. If R is right CD such that has no nonzero semisimple projective R-module. Then the following statements are equivalent:*

(1) *L is JM.*
(2) *L is small monoform.*

Lemma 2.25. *For an R-module L, consider the following assertions.*

(1) *L is JM.*
(2) *For every right R-module Y, if there exists an epimorphism $L \to L \oplus Y$, then Y is semisimple.*

Then (1) $\Rightarrow$ (2).

Proof. (1) $\Rightarrow$ (2) Let $g : L \to L \oplus Y$ be a surjective homomorphism, and let $\pi : L \oplus Y \to L$ the natural projection. It is obvious that $Ker(\pi g) = g^{-1}(0 \oplus Y)$. Since L is JM, $Ker(\pi g) \ll_J L$. According to Lemma 1.2,

$$0 \oplus Y = g[g^{-1}(0 \oplus Y)] = g(Ker(\pi g)) \ll_J L \oplus Y.$$

Thus $Y \ll_J Y$ by Lemma 1.2. Therefore, Y is semisimple by Lemma 1.3. □

In the following, we characterize the class of rings R for which every (free) R- module is JM.

Theorem 2.26. *Let R be a ring. The following assertions are equivalent:*

(1) *R is semisimple.*
(2) *Any R-module is JM.*
(3) *Any projective R-module is JM.*
(4) *Any free R-module is JM.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (1) Let $L = R^{(\mathbb{N})}$, by (4) L is a JM R-module. Since $L \cong L \oplus L$, L is semisimple by Lemma 2.25. Therefore R is semisimple. □

Proposition 2.27. *Every nonzero submodule of JM module is JM.*

Proof. Let N be a nonzero submodule of a JM module L. For any $0 \neq K \leq N$, let $f : K \to N$ be a nonzero partial endomorphism of N, then $if \neq 0$ where $i : N \to L$ is the inclusion mapping. Since L is JM, $Ker(if) \ll_J K$, hence $Ker f \ll_J K$, and so N is JM. □

Remark 2.28. Let $\pi : \mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$, where π is the natural projection. However $\mathbb{Z}/12\mathbb{Z}$ is not JM $\mathbb{Z}$-module because

$$\overline{0} \neq f = 4\overline{x} \in \text{End}(\mathbb{Z}/12\mathbb{Z})$$

and $Ker f =< \overline{3} >$ is not Jacobson-small in $\mathbb{Z}/12\mathbb{Z}$.

(1) Let $L = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Each of $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$ is a JM module (because every of each is small monoform). Since $L \cong \mathbb{Z}/12\mathbb{Z}$. Then the direct sum of JM modules is not necessarily JM.

(2) Since $\mathbb{Z}$ is a JM $\mathbb{Z}$-module and $\mathbb{Z}/12\mathbb{Z}$ is not JM $\mathbb{Z}$-module. Then the homomorphic image of JM module is not necessarily JM.

Proposition 2.29. *Let L be a Noetherian R-module. Then L is JM iff any non-zero 3-generated submodule of L is JM.*

Proof. $\Rightarrow$) Clear from Proposition 2.27.

$\Leftarrow$) suppose that any non-zero 3-generated submodule of L is JM. Let N be a non-zero submodule of L and $f : N \to L$ such that $f \neq 0$. If $Kerf = 0$ then $Kerf \ll_J N$. If $Kerf \neq 0$, let $x \in Kerf$. Let $y \in N$ and $z = f(y)$. Put $P = Rx + Ry + Rz$ is 3-generated submodule of L. Let $H = Rx + Ry$ and $h = f\mid_H : H \to P$. By hypothesis P is JM, hence $Kerh \ll_J H \leq N$. But $x \in Kerh$, so

$$< x > \subseteq Kerh \ll_J N.$$

Since L is Noetherian R-module, $Kerf$ is finitely generated, hence

$$Kerf = \sum_{i=1}^{n} Rx_i,$$

for some $x_i \in L$, $1 \leq i \leq n$. We have $< x_i > \ll_J N$ for every $1 \leq i \leq n$. Thus according to Lemma 1.2, $Kerf = \sum_{i=1}^{n} Rx_i \ll_J N$. Therefore L is JM.

□

Corollary 2.30. *Let R be an Artinian principal ideal ring and L be a weakly co-Hopfian R-module. Then the following are equivalent:*

(1) *L is JM.*
(2) *Any non-zero 3-generated submodule of L is JM.*

Proof. By [1, Theorem 3.8], L must be finitely generated module. Then L is a Noetherian since R is Artinian principal ideal ring. Thus by Propostion 2.29 the result is obtained. □

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JACOBSON MONOFORM MODULE

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مدول های جیکبسون تک-فرم

عبدالرحیم الموسوی

در این مقاله مدول های جیکبسون تک-فرم که تعمیمی سره از مدول های تک-فرم هستند را معرفی و مورد مطالعه قرار می‌دهیم. یک مشخصه‌سازی از حلقه‌های نیمه ساده با استفاده از مدول‌های جیکبسون تک-فرم ارائه می‌دهیم. در واقع نشان می‌دهیم حلقه R نیمه ساده است اگر و تنها اگر هر R-مدول جیکبسون تک-فرم باشد. بعلاوه نشان می‌دهیم مفاهیم تک-فرم، جیکبسون تک-فرم، تراکم‌پذیری، یکنواختی و هم-هاپفیان ضعیف در حلقه‌ها معادل هستند.

کلمات کلیدی: مدول‌های تک-فرم، مدول‌های تک-فرم کوچک، مدول‌های جیکبسون تک-فرم.