Journal of Algebraic Systems
Vol. 3, No. 1, (2015), pp 39-47

ANNIHILATING SUBMODULE GRAPHS FOR MODULES OVER COMMUTATIVE RINGS

M. BAZIAR

Abstract. In this article, we give several generalizations of the concept of annihilating an ideal graph over a commutative ring with identity to modules. We observe that, over a commutative ring, R, $\mathbb{AG}_*({}_RM)$ is connected, and $\mathrm{diam}\mathbb{AG}_*({}_RM) \leq 3$. Moreover, if $\mathbb{AG}_*({}_RM)$ contains a cycle, then $\mathrm{gr}\mathbb{AG}_*({}_RM) \leq 4$. Also for an R-module M with $\mathbb{A}_*(M) \neq S(M) \setminus \{0\}$, $\mathbb{A}_*(M) = \emptyset$, if and only if M is a uniform module, and $\mathrm{ann}(M)$ is a prime ideal of R.

1. Introduction

In the literature, there are many papers on assigning a graph to a ring, group, semigroup or module (see for example [1]-[16], [19]and [21]-[25]). The concept of zero-divisor graph of a commutative ring R was first introduced by Beck [11], where he was mainly interested in colorings. In his work, all elements of the ring were vertices of the graph. The investigation of colorings of a commutative ring was then continued by Anderson and Naseer [9]. Let $Z(R)$ be the set of zero-divisors of R. In [8], Anderson and Livingston associated a graph, $\Gamma(R)$, to R with vertices $Z(R) \setminus \{0\}$, the set of non-zero zero-divisors of R, and for distinct x, $y \in Z(R) \setminus \{0\}$, the vertices x, and y are adjacent if and only if $xy = 0$. In [23], Sharma and Bhatwadekar define another graph on R, $G(R)$, with vertices as elements of R, where, two distinct vertices a, and b are adjacent, if and only if $Ra + Rb = R$. (See also [21] and [5], in which, the notion "comaximal graph of commutative

MSC(2010): Primary: 05C25; Secondary: 13C13, 16D60
Keywords: Zero-divisor graph, Annihilating submodule graph, Weakly annihilating submodule.
Received: 27 November 2014, Revised: 9 July 2015.

rings" is investigated.) Recently, Anderson and Badawi, in [6], have introduced and studied the total graph of R, denoted by $T(\Gamma(R))$. It is the (undirected) graph with all elements of R as vertices, and for distinct x, $y \in R$, the vertices x and y are adjacent, if and only if $x+y \in Z(R)$. We denote the set of all proper ideals of R by $\mathbb{I}(R)$. In [13], Behboodi and Rakeei named an ideal, I of R, an *annihilating-ideal* if there exists a non-zero ideal J of R, such that $IJ = (0)$, and used the notation $\mathbb{A}(R)$ for the set of all annihilating-ideals of R. They defined *the annihilating-ideal graph* of R, denoted by $\mathbb{AG}(R)$, as a graph with vertices $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{(0)\}$, where, distinct vertices I and J are adjacent, if and only if $IJ = (0)$. They extensively investigated the interplay between the graph-theoretic properties of $\mathbb{AG}(R)$ and the ring-theoretic properties of R. There are a few papers on annihilating the ideal graph (see [1] ,[13], and [14]). In the next sections, we introduce and study various module generalizations of the annihilating ideal graphs of commutative rings.

Recall that a graph Γ is connected, if there is a path between any two distinct vertices. For the distinct vertices x and y of Γ, let $d(x, y)$ be the length of the shortest path from x to y ($d(x, y) = \infty$, if there is no such path). The diameter of Γ, diam(Γ), is defined as sup $\{d(x, y) \mid x$ and y are distinct vertices of Γ $\}$. The girth of Γ, denoted by $g(\Gamma)$, is defined as the length of the shortest cycle in Γ ($g(\Gamma) = \infty$; if Γ contains no cycles).

2. Annihilating graphs for modules

We begin with the following definition (we note that for any R-module M, $(N : M) := Ann(M/N)$, for $N \leq M$).

Definition 2.1. Let M be an R-module. A submodule N of M is called:

- *weakly annihilating submodule*, if either $N = 0$ or $(N : M)(K : M)M = 0$, for some non-zero proper submodule K of M.
- *annihilating sub-module*, if either $N = 0$ or $0 \neq (N : M)$ and $(N : M)(K : M)M = 0$ for some non-zero proper submodule K of M with $0 \neq (K : M)$.
- *strongly annihilating submodule*, if either $N = 0$ or $Ann(M) \subset (N : M)$, and $(N : M)(K : M)M = 0$ for some non-zero proper sub-module K of M with $Ann(M) \subset (K : M)$.

For any module M, we denote $\mathbb{A}_*(M)$, $\mathbb{A}(M)$ and $\mathbb{A}^*(M)$, respectively, for the set of *weakly annihilating submodule, annihilating submodule,* and *strongly annihilating submodule* of M. It is clear that

$$\mathbb{A}^*(M) \subseteq \mathbb{A}(M) \subseteq \mathbb{A}_*(M).$$

The following proposition shows that for any module, we only need to consider strongly annihilating and weakly annihilating submodules.

Proposition 2.2. *Let R be a ring and M be an R-module. Then*
1) If M is a faithful R-module, then $\mathbb{A}^*(M) = \mathbb{A}(M)$;
2) If M is a non-faithful R-module, then $\mathbb{A}(M) = \mathbb{A}_*(M)$.

Proof. By Definition 2.1, the results hold. □

The following proposition shows that, for $M = R$, the three parts of Definitions 2.1 are equivalent and they are the generalizations of annihilating ideal.

Proposition 2.3. *Let R be any ring, and I be an ideal of R. Then the following are equivalent:*
1) I is an annihilating ideal of R;
2) I is a weakly annihilating submodule of ${}_RR$;
3) I is an annihilating submodule of ${}_RR$;
4) I is a strongly annihilating submodule of ${}_RR$.

Proof. The proof is easy. □

Now, for an R-module M, we let $\tilde{\mathbb{A}}_*(M) := \mathbb{A}_*(M) \setminus \{0\}$, $\tilde{\mathbb{A}}(M) := \mathbb{A}(M) \setminus \{0\}$, and $\tilde{\mathbb{A}}^*(M) := \mathbb{A}^*(M) \setminus \{0\}$. Then we associate the three undirected (simple) graphs $\mathbb{AG}_*({}_RM)$, $\mathbb{AG}({}_RM)$, and $\mathbb{AG}^*({}_RM)$ to M with vertices $\tilde{\mathbb{A}}_*(M)$, $\tilde{\mathbb{A}}(M)$, and $\tilde{\mathbb{A}}^*(M)$, respectively, and for which, the vertices N, and K are adjacent, if and only if $(N : M)(K : M)M = 0$. It is clear that we have $\mathbb{AG}^*({}_RM) \subseteq \mathbb{AG}({}_RM) \subseteq \mathbb{AG}_*({}_RM)$, as induced subgraphs. In fact, Proposition 2.2 shows that for any R-module M, either $\mathbb{AG}({}_RM) = \mathbb{AG}^*({}_RM)$ or $\mathbb{AG}({}_RM) = \mathbb{AG}_*({}_RM)$.
Let $\mathbb{AG}(R)$ be the annihilating ideal graph of a ring R. By Proposition 2.3, we have $\mathbb{AG}^*({}_RR) = \mathbb{AG}({}_RR) = \mathbb{AG}_*({}_RR) = \mathbb{AG}(R)$. In the following theorem, we determine when $\mathbb{AG}_*({}_RM) = \mathbb{AG}({}_RM) = \mathbb{AG}^*({}_RM)$.

Theorem 2.4. *Let M be an R-module. Then $\mathbb{AG}_*({}_RM) = \mathbb{AG}({}_RM) = \mathbb{AG}^*({}_RM)$, if and only if $Ann(M) \subset (N : M)$, for every non-zero submodule N of M.*

Proof. ($\Rightarrow$) If for some non-zero proper submodule N of M, $(N : M) = \mathrm{Ann}(M)$, then for every non-zero submodule K of M, we have $(K : M)(N : M)M = 0$, so that N —— K is a path in $\mathbb{AG}_*(M)$, and

hence, is a path in $\mathbb{AG}^*(M)$, which implies $\mathrm{Ann}(M) \subset (N : M)$, which is a contradiction.
($\Leftarrow$) By definition 2.1. $\square$

Recall that an R-module M is called *multiplication,* in case for every non-zero submodule N of M, there exists an ideal I of R, such that $N = IM$. One can show that if M is a multiplication module, then for every submodule N of M, we have $N = (N : M)M$.

Corollary 2.5. *Let M be a multiplication R-module. Then $\mathbb{AG}_*({}_RM) = \mathbb{AG}({}_RM) = \mathbb{AG}^*({}_RM)$.*

Proof. The result holds, since multiplication modules have the property that for every non-zero submodule N of M, $\mathrm{Ann}(M) \subset (N : M)$. $\square$

Proposition 2.6. *Let M be an R-module with $0 \neq I = Ann(M)$. Then the following statements hold.*
(1) $\mathbb{AG}({}_RM) = \mathbb{AG}_*({}_RM) = \mathbb{AG}_*({}_{R/I}M)$;
(2) $\mathbb{AG}^*({}_RM) = \mathbb{AG}^*({}_{R/I}M) = \mathbb{AG}({}_{R/I}M)$.

Proof. Let $N \in \tilde{\mathbb{AG}}_*(M)$. Then there exists $0 \neq K \leq M$ such that $(N : M)(K : M)M = 0$. It is clear that $I = Ann(M) \subseteq (N : M) \cap (K : M)$, $Ann_{R/I}(M/N) = (N : M)/I$, $Ann_{R/I}(M/K) = (K : M)/I$, and $((N : M)/I)(K : M)/I)M = 0$. This follows that $N \in \mathbb{A}_*({}_RM)$, if and only if $N \in \mathbb{A}_*({}_{R/I}M)$, and the vertices N and K are adjacent in $\mathbb{AG}_*({}_RM)$, if and only if N and K are adjacent in $\mathbb{AG}_*({}_{R/I}M)$. Therefore, $\mathbb{AG}_*({}_RM) = \mathbb{AG}_*({}_{R/I}M)$. Similarly, we can show that $\mathbb{AG}^*({}_RM) = \mathbb{AG}^*({}_{R/I}M)$. $\square$

Proposition 2.7. *Let M be a homogeneous sem-isimple R-module. Then $\mathbb{AG}^*({}_RM)$ is the empty graph.*

Proof. Since $Ann(M)$ is a maximal ideal, the result holds. $\square$

Proposition 2.8. *Let M be an R-module. Then $\mathbb{AG}^*({}_RM)$ is the empty graph, if and only if $Ann(M)$ is a prime ideal of R.*

Proof. Since for every non-zero submodules N, K of M, $(N : M)(K : M)M = 0$ if and only if $(N : M)M = 0$ or $(K : M)M = 0$, if and only if $\mathrm{Ann}(M)$ is a prime ideal of R, we are done. $\square$

Corollary 2.9. *Let M be an R-module. Then $\mathbb{AG}_*(M) = \mathbb{AG}(M) = \mathbb{AG}^*(M) = \emptyset$, if and only if $Ann(M)$ is a prime ideal of R, and $Ann(M) \subset (N : M)$, for every non-zero submodule N of M.*

Proof. It follows from Theorem 2.4 and Proposition 2.8. $\square$

3. Weakly annihilating submodule graph

Now, one may ask a question; when two submodules of an R-module M maybe connected to each other in $\mathbb{AG}_*(M)$?

Lemma 3.1. *Let M be an R-module, and N, K be the submodules of M.*
1) If $N \cap K = 0$, then N —— K is a path in $\mathbb{AG}_*(M)$.
2) If N —— K is a path in $\mathbb{AG}_*(M)$, then for each $0 \neq N_1 \leq N$ and $0 \neq K_1 \leq K$, N_1 —— K_1 is also a path in $\mathbb{AG}_*(M)$.

Proof. 1) The result holds, since $(N : M)(K : M)M \subseteq N \cap K$.
2) Let $N, K \in \tilde{\mathbb{AG}}_*(M)$, and $0 \neq N_1 \leq N$. Assume that N —— K is a path in $\mathbb{AG}_*(M)$, and $0 \neq K_1 \leq K$. Then $(N : M)(K : M)M = 0$. It is clear that $(N_1 : M) \subseteq (N : M)$, and $(K_1 : M) \subseteq (K : M)$. Therefore, $(N_1 : M)(K_1 : M)M \subseteq (N : M)(K : M)M = 0$. Thus N_1 —— K_1 is also a path in $\mathbb{AG}_*(M)$. □

Corollary 3.2. *Let M be an R-module. Then $N \in \mathbb{AG}_*(M)$, for every non-zero non-essential submodule N of M.*

In [7, Theorem 2.3], it is shown that, for any commutative ring R, $\Gamma(R)$ is connected, and diam$\Gamma(R) \leq 3$. Furthermore, if $\Gamma(R)$ contains a cycle, then $g(\Gamma(R)) \leq 7$. Moreover, in [22], it is shown that, for any commutative ring R, the girth of the zero-divisor graph of R is less than (or equal to) 4. In the next theorem, we give a generalization of these result for modules.

Theorem 3.3. *Let M be any R-module.*
1) The graph $\mathbb{AG}_*(_RM)$ is a connected graph, and diam$\mathbb{AG}_*(_RM) \leq 3$.
2) If $\mathbb{AG}_*(_RM)$ contains a cycle, then $g(\mathbb{AG}_*(_RM)) \leq 4$.

Proof. (1) Let $N, K \in \tilde{\mathbb{AG}}_*(M)$ be distinct. If $(N : M)(K : M)M = 0$, then $d(N, K) = 1$. So suppose that $(N : M)(K : M)M \neq 0$. Hence, there are $A, B \in \tilde{\mathbb{AG}}_*(M) \setminus \{N, K\}$ with $(A : M)(N : M)M = (B : M)(K : M)M = 0$. If $(A : M)(B : M)M = 0$, then N —A —B —K is a path of length 3. Thus we may assume that $(A : M)(B : M)M \neq 0$; then $T = A \cap B \neq 0$. Hence by Lemma 2.1, N —T —K is a path of length 2, and hence, $d(N, K) \leq 3$. Thus $diam(\mathbb{AG}(_RM)) \leq 3$.
(2) Let N_1 —N_2 —... —N_{k-1} —N_k be a cycle with length $k \geq 3$. Put $N_{k+1} := N_1$, and $N_0 := N_k$. If N_i has a proper non-zero submodule T_i (for some $1 \leq i \leq k$), then, by Lemma 2.1, N_{i-1} —T_i —N_{i+1} is a path, and N_{i-1} —T_i —N_{i+1} —N_i —N_{i-1} is a cycle of length at most 4. If every N_i has no proper non-zero submodule, then every N_i is a simple module. If $N_1 \cap N_4 = 0$ then N_1 —N_2 —N_3 —N_4 —N_1

is a cycle of length 4. If $N_1 \cap N_4 \neq 0$, then $N_1 = N_4$, and $N_1 — N_2 - -N_3 — N_4$ is a cycle of length 3.Thus $g(\mathbb{AG}_*(_RM)) \leq 4$. □

Corollary 3.4. *Let M be any non-faithful R-module. Then $\mathbb{AG}(_RM)$ is connected, and diam$\mathbb{AG}(_RM) \leq 3$. Moreover, if $\mathbb{AG}(_RM)$ contains a cycle, then $g(\mathbb{AG}(_RM)) \leq 4$.*

Proof. If M is a non-faithful R-module, then, by Proposition 2.2, $\mathbb{AG}(_RM) = \mathbb{AG}_*(_RM)$. Now, apply Theorem 3.3. □

The following result assures us when $\mathbb{AG}_*(_RM)$ contains a cycle. As we can see it happens when $\mathbb{AG}_*(_RM)$ contains a path of length 4. In fact, when $\mathbb{AG}_*(_RM)$ has a path of length 4, then gr$(\mathbb{AG}_*(_RM)) \leq 4$.

Proposition 3.5. *Let M be an R-module. If $\mathbb{AG}_*(_RM)$ contains a path of length 4, then $\mathbb{AG}_*(_RM)$ contains a cycle.*

Proof. Let $N_1 - N_2 - N_3 - N_4 - N_5$ be a path of length 4. If $N_2 \cap N_4 = 0$, then N_2 and $N_4 = 0$ are adjacent, and hence, $N_2 - N_3 - N_4 - N_2$ is a cycle. Now, assume that $0 \neq K \leqslant N_2 \cap N_4$. One of the following cases holds:
(Case 1). If $K = N_1$, then, by Lemma 3.1, $N1 - N_2 - N3 - N_1$ is a cycle.
(Case 2). If $K = N_2$, then, by Lemma 3.1, $N_2 - N_3 - N_4 - N_5 - N_2$ is a cycle.
(Case 3). If $K = N_3$, then, by Lemma 3.1, $N_1 - N_2 - N_3 - N_1$ is a cycle.
(Case 4). If $K = N_4$, then by Lemma 3.1, $N_3 - N_4 - N_1 - N_2 - N_3$ is a cycle.
(Case 5). If $K = N_5$, then by Lemma 3.1, $N_3 - N_4 - N_5 - N_3$ is a cycle.
(Case 6). If $K \notin \{N_1, N_2, N_3, N_4, N_5\}$, then by Lemma 3.1, $N_1 - K - N_3 - N_2 - N_1$ is a cycle. □

Corollary 3.6. *Let R be a ring. If $\mathbb{AG}(R)$ contains a path of length 4, then $\mathbb{AG}(R)$ contains a cycle.*

Proof. By Proposition 3.5, the verification is immediate. □

Let Γ be a graph with vertices V, and let $\emptyset \neq A,\ B \subseteq V$. Then $A \leftrightsquigarrow B$ means that, for each $a \in A$, $b \in B$, $a —— b$ is a path in Γ. Also, for each non-zero R-module M, we denote the set of all non-zero proper submodules of M by $\tilde{S}(M)$ (i.e., $\tilde{S}(M) = S(M) \setminus \{0\}$). Let $M = M_1 \oplus M_2$, where $M_i \neq 0,\ i = 1, 2$. Then $\tilde{S}(M_1), \tilde{S}(M_2) \subseteq \tilde{\mathbb{AG}_*}(M)$, and $\tilde{S}(M_1) \leftrightsquigarrow \tilde{S}(M_2)$ in $\mathbb{AG}_*(M)$.

Theorem 3.7. *Let M be an R-module with $\mathbb{A}_*(M) \neq S(M) \setminus \{0\}$. Then $\mathbb{A}_*(M) = \emptyset$, if and only if M is a uniform module, and $Ann(M)$ is a prime ideal of R.*

Proof. Let $\mathbb{A}_*(M) = \emptyset$. Then, by Lemma 3.1 for non-zero elements $K, N \in S(M)$, $N \cap K$ must be non-zero. This implies that M is a uniform R-module. Now, suppose that I and J are ideals of R, such that $IJ \subseteq$Ann(M), but neither $I \subseteq$Ann(M) nor $J \subseteq$Ann(M). Therefore,

$$(JM : M)(IM : M)M \subseteq (JM : M)IM \subseteq IJM = 0.$$

Hence, IM and JM belong to $\mathbb{A}_*(M)$. This is a contradiction. Conversely, assume that M is a uniform module with, prime annihilator such that $0 \neq N \in \mathbb{A}_*(M)$. There exists $0 \neq K \in \mathbb{A}_*(M)$, such that $(N : M)(K : M)M = 0$. Therefore $(N : M)(K : N) \subseteq$Ann(M), and hence, either $(N : M) \subseteq$Ann(M) or $(K : M) \subseteq$Ann(M) because Ann(M) is a prime ideal. Hence, for each non-zero submodule T of M, either $(T : M)(N : M)M = 0$ or $(T : M)(K : M)M = 0$. Thus $\mathbb{A}_*(M) = S(M) \setminus \{0\}$. This is a contradiction. ☐

Corollary 3.8. *Let R be a ring. R is a domain, if and only if there exists a faithful R-module M with $\Gamma_*(M) = \emptyset$.*

Proof. By Theorem 3.7, the verification is immediate. ☐

Proposition 3.9. *Let M be a non-simple semisimple R-module. Then $\mathbb{AG}_*({}_RM)$ is a connected graph with vertex set $\tilde{S}(M)$.*

Proof. Since every proper submodule of a semisimple module M is a direct summand of M, by Lemma 3.1 is evident. ☐

Lemma 3.10. *Let $M = M_1 \oplus M_2$, and $0 \neq N \in \tilde{\mathbb{A}}_*(M_1)$. Then $N \oplus 0 \in \tilde{\mathbb{A}}_*(M)$. Moreover, if the vertices N and K are adjacent in $\mathbb{AG}_*(M_1)$, then $N \oplus 0$, $K \oplus 0$ are adjacent in $\mathbb{AG}_*({}_RM)$.*

Proof. It is clear that for every $N \leq M_1$;

$$\frac{M_1 \oplus M_2}{N \oplus 0} \cong \frac{M_1}{N} \oplus M_2.$$

Therefore, if $N \in \tilde{\mathbb{A}}_*(M_1)$, then there exists $0 \neq K \leq M_1$, such that $(N : M_1)(K : M_1)M_1 = 0$. Now, $(N \oplus 0 : M_1 \oplus M_2) = Ann(\frac{M_1}{N} \oplus M_2)$, and $(K \oplus 0 : M_1 \oplus M_2) = Ann(\frac{M_1}{K} \oplus M_2)$. Thus $(N \oplus 0 : M_1 \oplus M_2)(K \oplus 0 : M_1 \oplus M_2)M = 0$, and it follows that $N \oplus 0 \in \tilde{\mathbb{A}}_*(M)$. Now, the "moreover" statement is clear. ☐

Theorem 3.11. *Let $M = M_1 \oplus M_2$, such that $\mathbb{AG}_*(M_1) \neq \emptyset$. Then $\mathbb{AG}_*(M_1) \cong G$, where G is an induced subgraph of $\mathbb{AG}_*({}_R M)$ with vertex set $\{N \oplus 0 \in \tilde{\mathbb{A}}_*(M) | \ N \in \tilde{\mathbb{A}}_*(M_1)\}$.*

Proof. The result is a consequence of Lemma 3.10. □

Lemma 3.12. *Let M be an R-module, and $f \in \mathrm{End}_R(M)$ be a non-monic and non-zero endomorphism. Then $\ker(f)$ is adjacent to $\mathrm{Im}(f)$ in $\mathbb{AG}_*(M)$.*

Proof. Let $K = \ker(f)$, and $I = \mathrm{Im}(f)$. Then:

$$(K : M)(I : M)M \subseteq (K : M)f(M) \subseteq f((K : M)M) \subseteq f(K) = 0.$$

Thus $\ker(f)$ is adjacent to $\mathrm{Im}(f)$. □

Corollary 3.13. *Let M be an R-module, and f be a non-monic epimorphism of M. Then $\mathbb{A}_*(M) = S(M) \setminus \{0\}$.*

Proof. Since f is non-monic, $\ker(f) \neq 0$. By Lemma 3.12, $\mathrm{Im}(f) = M$ is adjacent to $\ker(f)$. Now, by Lemma 3.1, any sub-module of M is adjacent to $\ker(f)$. Therefore, $\mathbb{A}_*(M) = S(M) \setminus \{0\}$.

□

Corollary 3.14. *Let M be an R-module. If $\mathbb{A}_*(M) \neq S(M) \setminus \{0\}$, then M is a Hopfian module.*

Proof. Let $f : M \to M$ be a non-zero epimorphism. Then f must be monic. Otherwise, by Corollary 3.13, $\mathbb{A}_*(M) = S(M) \setminus \{0\}$, which is a contradiction. □

Acknowledgment

The author wishes to thank the referees for their comments that improved the paper.

References

1. G. Aalipour, S. Akbari, M. Behboodi, R. Nikandish, M.J. Nikmehr and F. Shaveisi, The Classification of the Annihilating-Ideal Graphs of Commutative Rings, *Algebra Colloquium* **21**(2) (2014), 249-256.
2. S. Akbari and H. R. Maimani, Zero-divisor graphs of non-commutative rings, *J. Algebra* **296**(2) (2006), 462-479.
3. S. Akbari, H. R. Maimani and S. Yassemi, When a zero-divisor graph is planar or a complete r-partite graph, *J. Algebra* **270** (2003), 169-180.
4. S. Akbari and A. Mohammadian, On the zero-divisor graph of a commutative ring, *J. Algebra* **274** (2004), 847-855.
5. A. Amini, B. Amini, E. Momtahan and M. H. Shirdareh Haghigh, On a graph of ideals, *Acta Math. Hungar.* **134**(3) (2012), 369-384.

6. D. F. Anderson and A. Badawi, The total graph of a commutative ring, *J. Algebra* **320**(7) (2008), 2706-2719.
7. D. F. Anderson, A. Frazier, A. Lauve and P. S. Livingston, The zero-divisor graph of a commutative ring, II, in: *Lecture Notes in Pure and Appl. Math.*, vol. 220, Dekker, New York, 2001, pp. 61-72.
8. D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* **217** (1999), 434-447.
9. D. D. Anderson and M. Naseer, Becks coloring of a commutative ring, *J. Algebra* **159** (1993), 500-514.
10. M. Axtell, J. Coykendall and J. Stickles, Zero-divisor graphs of polynomials and power series over commutative rings, *Comm. Algebra* **33**(6) (2005), 2043-2050.
11. I. Beck, Coloring of commutative rings, *J. Algebra* **116** (1988), 208-226.
12. M. Behboodi, Zero divisor graphs for modules over commutative rings, *J. Commut. Algebra* **4**(2) (2012), 175–197.
13. M. Behboodi and Z. Rakeei, The annihilating-ideal graph of commutative rings I, *J. Algebra Appl.* **10**(4) (2011), 727-739.
14. M. Behboodi and Z. Rakeei, The annihilating-ideal graph of commutative rings II, *J. Algebra Appl.* **10**(4) (2011), 741-753.
15. F. DeMeyer, T. McKenzie and K. Schneider, The zero-divisor graph of a commutative semigroup, *Semigroup Forum* **65** (2002), 206-214.
16. F. DeMeyer and K. Schneider, Automorphisms and zero-divisor graphs of commutative rings, *Commutative rings*, 25-37, Nova Sci. Publ., Hauppauge, NY, 2002.
17. Z. A. EL-Bast and P. F. Smith, Multiplication modules, *Comm. Algebra* **16** (1988), 755-779.
18. K. R. Goodearl and R. B. Warfild, *An introduction to non-commutative Noetherian rings*, London Math. Soc. Student Texts 16, Camberidge University Press, Cambrige, 1989.
19. R. Levy and J. Shapiro, The zero-divisor graph of von Neumann regular rings, *Comm. Algebra* **30**(2) (2002), 745-750.
20. J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, Wily, Chichester, 1987.
21. H. R. Maimani, M. Salimi, A. Sattari and S. Yassemi, Comaximal graph of commutative rings, *J. Algebra* **319** (2008), 1801-1808.
22. S. B. Mulay, Cycles and symmetries of zero-divisors, *Comm. Algebra* **30**(7) (2002), 3533-3558.
23. P. K. Sharma and S. M. Bhatwadekar, A note on graphical representation of rings, *J. Algebra* 176 (1995), 124-127.
24. S.P. Redmond, The zero-divisor graph of a non-commutative ring, *Internat. J. Commutative Rings* **1**(4) (2002), 203-211.
25. R. Wisbauer, *Modules and Algebras: Bimodule Structure and Group Action on Algebras*, Pitman Monograhs 81, Addison-Wesley-Longman (1996).
26. R. Wisbauer, *Foundations of Modules and Rings Theory*, Gordon and Breach Reading 1991.
27. T. Wu, On directed zero-divisor graphs of finite rings, *Discrete Math.* **296** (2005), 73-86.

Mohammad Baziar
Department of Mathematics, University of Yasouj, P.O.Box 75914, Yasouj, Iran.
Email: mbaziar@yu.ac.ir

Journal of Algebraic Systems

ANNIHILATING SUBMODULE GRAPHS FOR MODULES OVER COMMUTATIVE RINGS

M. BAZIAR

گراف های زیر مدول پوچ ساز برای مدول ها روی حلقه های جابجایی

محمد بازیار
ایران، یاسوج، دانشگاه یاسوج، دانشکده علوم پایه، گروه ریاضی

در این مقاله چند تعمیم از مفهوم گراف ایده ال های پوچ ساز روی حلقه های جابجایی و یکدار به مدول ها ارائه خواهیم داد. ما دریافتیم که روی یک حلقه R گراف $AG_*({}_RM)$ همبند و $\operatorname{diam}AG_*({}_RM) \leq$ ۳ است. به عبارت بیشتر، اگر $AG_*({}_RM)$ شامل یک دور باشد آن گاه $\operatorname{gr}AG_*({}_RM) \leq$ ۴ است. همچنین برای هر R –مدول مانند M با این خاصیت که $A_*(M) \neq S(M) \setminus \{۰\}$ داریم $A_*(M) = \emptyset$ اگر و تنها اگر M یک مدول یکنواخت و $\operatorname{Ann}(M)$ یک ایده ال اول از حلقه R باشد.

کلمات کلیدی: گراف مقسوم علیه صفر، گراف زیر مدول های پوچ ساز، زیر مدول بطور ضعیف پوچ ساز.