Journal of Algebraic Systems
Vol. 4, No. 1, (2016), pp 65-77
DOI: 10.22044/jas.2016.729

SIGNED ROMAN DOMINATION NUMBER AND JOIN OF GRAPHS

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Abstract. In this work, we study the signed Roman domination number of the join of graphs. Specially, we determine it for the join of cycles, wheels, fans, and friendship graphs.

1. Introduction

Throughout this paper, we consider (non trivial) simple graphs, which are finite and undirected graphs without loops or multiple edges. Let $G = (V(G), E(G))$ be a simple graph of order $n = |V(G)|$ and of size $m = |E(G)|$. When x is a vertex of G, then the open neighborhood of x in G is the set $N_G(x) = \{y : \ xy \in E(G)\}$, and the closed neighborhood of x in G is the set $N_G[x] = N_G(x) \cup \{x\}$. The degree of vertex x is the number of edges adjacent to x, which is denoted by $\deg_G(x)$. The minimum degree and the maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

A set $D \subseteq V(G)$ is called a *dominating set* of G if each vertex outside D has at least one neighbor in D. The minimum cardinality of a dominating set of G is the *domination number* of G, denoted by $\gamma(G)$. For example, the domination numbers of the n-vertex complete graph, path, and cycle are given by $\gamma(K_n) = 1$, $\gamma(P_n) = \lceil \frac{n}{3} \rceil$ and $\gamma(C_n) = \lceil \frac{n}{3} \rceil$, respectively [6]. Domination is a rapidly developing area of research in graph theory, and its various applications to ad hoc

MSC(2010): Primary: 05C69; Secondary: 05C78.
Keywords: Domination, Signed Roman domination, Join, Cycle, Wheel, Fan, Friendship.
Received: 21 February 2016, Revised: 23 June 2016.
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networks, distributed computing, social networks, biological networks, and web graphs partly explain the increased interest. The concept of domination has existed and has been studied for a long time, and early discussions on the topic can be found in the works of Berge [2] and Ore [10]. At present, domination is considered to be one of the fundamental concepts in graph theory with an extensive research activity. Garey and Johnson [4] have shown that determining the domination number of an arbitrary graph is an NP-complete problem. The domination number can be defined equivalently by means of a function, which can be considered as a characteristic function of a dominating set; see [6]. A function $f : V(G) \to \{0,1\}$ is called a *dominating function* on G if, for each vertex $x \in V(G)$, $\sum_{y\in N_G[x]} f(y) \geq 1$. The value $w(f) = \sum_{x\in V(G)} f(x)$ is called the *weight* of f. Now, the domination number of G can be defined as

$$\gamma(G) = \min\{w(f) : \ f \text{ is a dominating function on } G\}.$$

Analogously, a *signed dominating function* of G is a labeling of the vertices of G with $+1$ and -1 such that the closed neighborhood of each vertex contains more $+1$'s than -1's. The *signed domination number* of G is the minimum value of the sum of vertex labels, taken over all signed dominating functions of G. This concept is closely related to the combinatorial discrepancy theory, as shown by Füredi and Mubayi in [3]. In general, many domination parameters are defined by combining domination with other graph theoretical properties; see [5] and [9].

Definition 1.1. [1] Let $G = (V,E)$ be a graph. A signed Roman dominating function (simply, a "SRDF") on the graph G is a function $f : V \to \{-1,1,2\}$, which satisfies the following two following conditions:

(a) For each $x \in V$, $\sum_{y\in N_G[x]} f(y) \geq 1$;
(b) Each vertex x for which $f(x) = -1$ is adjacent to at least one vertex y for which $f(y) = 2$.

The value $f(V) = \sum_{x\in V} f(x)$ is called the weight of the function f, and is denoted by $w(f)$. The signed Roman domination number of G, $\gamma_{sR}(G)$ is the minimum weight of a SRDF on G.

This concept has been introduced by Ahangar, Henning, et al. in [1]. They have described the usefulness of this concept in various applicative areas like graph labeling and "defending the Roman empire" (see [1], [7] and [13] for more details). It is obvious that for every graph G of order n, we have $\gamma_{sR}(G) \leq n$ since assigning $+1$ to each vertex yields a SRDF. In [1], Ahangar et al. have presented various lower and upper bounds on the signed Roman domination number of a graph in terms of its order, size, and vertex degrees. Moreover, they have characterized all graphs that attain these bounds. They have also investigated the relation between γ_{sR} and some other graphical parameters, and the signed Roman domination number of some special bipartite graphs. It has been proved in [1] that $\gamma_{sR}(K_n) = 1$ for each $n \neq 3$, $\gamma_{sR}(K_3) = 2$, $\gamma_{sR}(C_n) = \lceil \frac{2n}{3} \rceil$, $\gamma_{sR}(P_n) = \lfloor \frac{2n}{3} \rfloor$, and that the only n-vertex graph G with $\gamma_{sR}(G) = n$ is the empty graph $\overline{K}_n$.

Henning and Volkmann have studied the signed Roman domination number of trees in [8]. Also the signed Roman domination number of directed graphs has been considered in [11].

Note that each signed Roman dominating function f on G is uniquely determined by the ordered partition (V_{-1}, V_1, V_2) of $V(G)$, where $V_i = \{x \in V(G) : f(x) = i\}$ for each $i \in \{-1, 1, 2\}$. Specially, $w(f) = 2|V_2| + |V_1| - |V_{-1}|$. For convenience, we usually write $f = (V_{-1}, V_1, V_2)$, and when $S \subseteq V$, we denote the summation $\sum_{x \in S} f(x)$ by $f(S)$. If $w(f) = \gamma_{sR}(G)$, then f is called a $\gamma_{sR}(G)$-*function* or an *optimal SRDF* on G. Recall that the join of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1),\ y \in V(G_2)\}$. For example, $K_1 \vee P_n$ is the fan F_n, $K_1 \vee C_n$ is the wheel W_n, and the friendship graph Fr_n, $n = 2m + 1$, is the graph obtained by joining K_1 to m disjoint copies of K_2.

In this paper, we study the signed Roman domination number of the join of graphs. Specially, we determine the signed Roman domination number of $C_m \vee C_n$, W_n, F_n, and friendship graph Fr_n.

2. Wheels, Fans, and Friendship graphs

For investigating γ_{sR} of the join of graphs, the following lemma is useful and will be used frequently.

Lemma 2.1. *If G is a graph with $\Delta(G) = |V(G)| - 1$, then $\gamma_{sR}(G) \geq 1$.*

Proof. Let f be an optimal signed Roman dominating function on G, and let $x \in V(G)$ be a vertex of maximum degree $\Delta(G)$. Since $N_G(x) = V(G) \setminus \{x\}$, using the definition of a $SRDF$, we have

$$\gamma_{sR}(G) = w(f) = \sum_{v \in V(G)} f(v) = f(x) + \sum_{v \in N_G(x)} f(v) = f(N_G[x]) \geq 1.$$

□

Corollary 2.2. *For each graph G, $\gamma_{sR}(G \vee K_1) \geq 1$. Specially, if $\gamma_{sR}(G) = 0$, then $\gamma_{sR}(G \vee K_1) = 1$.*

Proof. The first statement follows directly from Lemma 2.1. Assume that $\gamma_{sR}(G) = 0$, and let f be a $\gamma_{sR}(G)$-function of G. Define $g : V(G \vee K_1) \to \{-1, 1, 2\}$ as $g(x) = f(x)$ when $x \in V(G)$, and $g(y) = 1$ when $y \in V(K_1)$. Since g is a SRDF of weight 1 on $G \vee K_1$, $\gamma_{sR}(G \vee K_1) \leq 1$. Thus $\gamma_{sR}(G \vee K_1) = 1$. □

Proposition 2.3. *Let G and H be two graphs such that $\gamma_{sR}(G) \geq 0$ and $\gamma_{sR}(H) \geq 0$. Then,*

$$\gamma_{sR}(G \vee H) \leq \gamma_{sR}(G) + \gamma_{sR}(H).$$

Proof. Let f_1 be a $\gamma_{sR}(G)$-function on G, and let f_2 be a $\gamma_{sR}(H)$-function on H. Define $f : V(G \vee H) \to \{-1, 1, 2\}$ as $f(x) = f_1(x)$ when $x \in V(G)$, and $f(y) = f_2(y)$ when $y \in V(H)$. For each $x \in V(G)$, $f(N_{G\vee H}[x]) = f(N_G[x]) + w(f_2) \geq 1$. Similarly, for each $y \in V(H)$, $f(N_{G\vee H}[y]) = f(N_H[y]) + w(f_1) \geq 1$. Thus f is a $SRDF$ on $G \vee H$ and $\gamma_{sR}(G \vee H) \leq w(f) = w(f_1) + w(f_2) = \gamma_{sR}(G) + \gamma_{sR}(H)$. □

For $G = K_2$ and $H = K_1$, we have $\gamma_{sR}(G \vee H) = \gamma_{sR}(G) + \gamma_{sR}(H)$. Hence, this bound is attainable.

The following theorem determines the signed Roman domination number of wheels.

Theorem 2.4. *Let $W_n = K_1 \vee C_n$ be a wheel of order $n + 1$. Then, $\gamma_{sR}(W_4) = 2$ and $\gamma_{sR}(W_n) = 1$ for each $n \neq 4$.*

Proof. Let $V(W_n) = \{v_0, v_1, v_2, ..., v_n\}$ and $E(W_n) = \{v_0v_i : 1 \leq i \leq n\} \cup \{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\}$. Since $\Delta(W_n) = |V(W_n)| - 1$, Lemma 2.1 implies that $\gamma_{sR}(W_n) \geq 1$. For the case $n = 4$, it is not hard to check by inspection that there exists no signed Roman dominating function on W_4 of weight 1, while Figure 1 (a) illustrates an $SRDF$ on W_4 of weight 2. Hence, $\gamma_{sR}(W_4) = 2$. To complete the proof, it is sufficient to provide a signed Roman dominating function of weight 1 on W_n for each $n \neq 4$. For this reason, we consider the following different cases.
Case 1. n is odd:
Define the function $f : V(W_n) \to \{-1, 1, 2\}$ as below. Figure 1 (b) illustrates it for the case $n = 5$, where the central vertex is v_0, top one is v_1, and v_2 is the second vertex when the sense of traversal is clockwise.

$$f(v_i) = \begin{cases} 2 & i = 0 \\ 1 & i \geq 3,\ i \equiv 1 \pmod 2. \\ -1 & o.w. \end{cases} \tag{2.1}$$

Note that f is a $SRDF$ on W_n of weight $w(f) = f(N_{W_n}[v_0]) = 1$.
Case 2. n is even and $n \equiv 0 \pmod 3$:

Define the function $f : V(W_n) \to \{-1, 1, 2\}$, as below. Figure 2 (a) depicts it for the case $n = 12$.

$$f(v_i) = \begin{cases} 1 & i = 0 \\ 2 & i \geq 1,\ i \equiv 0 \pmod 3. \\ -1 & o.w. \end{cases} \tag{2.2}$$

It is straightforward to check that f is a $SRDF$ on W_n of weight 1.

Case 3. n is even and $n \equiv 1 \pmod 3$.

Define the function f on $V(W_n)$, as follows. Figure 2 (b) illustrates it for the case $n = 10$.

$$f(v_i) = \begin{cases} 2 & i = 0 \\ 2 & 1 \leq i \leq n-7,\ i \equiv 0 \pmod 3. \\ 1 & i \in \{n-4, n-1, n\} \\ -1 & o.w. \end{cases} \tag{2.3}$$

It is not hard to check that f is a $SRDF$ on W_n and $w(f) = 1$.

Case 4. n is even and $n \equiv 2 \pmod 3$.

Define the function f on $V(W_n)$, as follows. Figure 2 (b) depicts it for the case $n = 8$.

$$f(v_i) = \begin{cases} 2 & i = 0 \\ 2 & 1 \leq i \leq n-5,\ i \equiv 0 \pmod 3. \\ 1 & i \in \{n-2, n\} \\ -1 & o.w. \end{cases} \tag{2.4}$$

It is easy to check that f is a $SRDF$ on W_n and its weight is one. Therefore, in each case, we provide a SRDF on W_n of weight one. This completes the proof. □

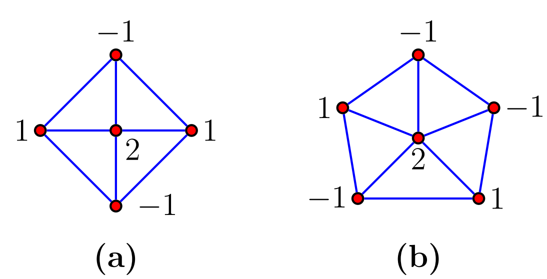

FIGURE 1. Signed Roman domination labeling on W_4 and W_5.

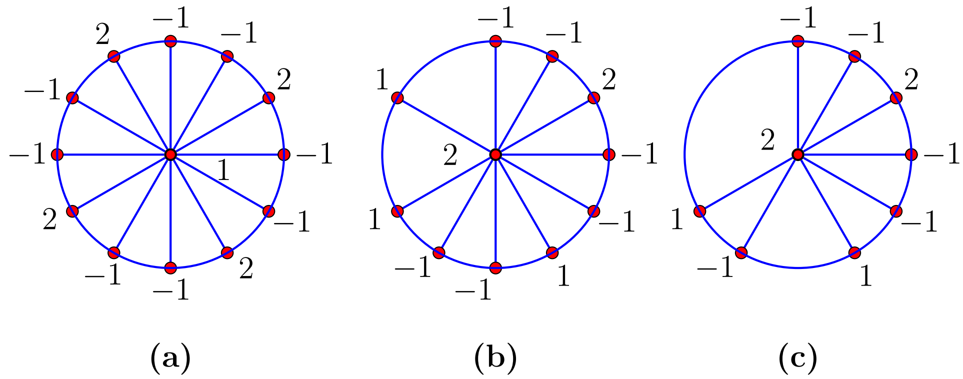

FIGURE 2. Signed Roman domination labeling of W_{12}, W_{10} and W_8.

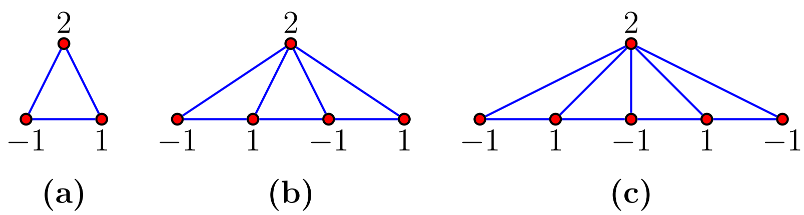

FIGURE 3. Signed Roman domination labeling on F_2, F_4 and F_5, respectively.

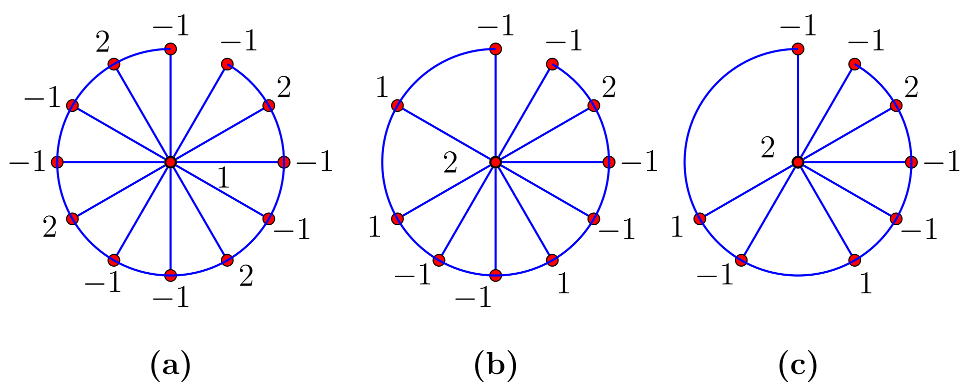

FIGURE 4. Signed Roman domination labeling on F_{12}, F_{10} and F_8, respectively.

Structures of F_n and W_n are similar. This similarity helps us to provide signed Roman dominating functions on F_n using what we construct for W_n.

Theorem 2.5. *Let $F_n = K_1 \vee P_n$ be a fan of order $n+1$. Then*

$$\gamma_{sR}(F_n) = \begin{cases} 2 & n \in \{2,4\} \\ 1 & n \notin \{2,4\}. \end{cases}$$

Proof. Let $V(F_n) = \{v_0, v_1, v_2, ..., v_n\}$ and $E(F_n) = \{v_0v_i : 1 \le i \le n\} \cup \{v_2v_3, v_3v_4, ..., v_{n-1}v_n, v_nv_1\}$, i.e. $F_n = W_n - v_1v_2$. Since $\Delta(F_n) = |V(F_n)| - 1$, Lemma 2.1 implies that $\gamma_{sR}(F_n) \ge 1$. F_2 is a complete graph with three vertices, and hence, $\gamma_{sR}(F_2) = \gamma_{sR}(K_3) = 2$. For the case $n = 4$ it is not hard to check by inspection that there exists no signed Roman dominating function on F_4 of weight 1. Figure 3 (a) and (b) illustrate a $SRDF$ of weight 2 on F_2 and F_4, respectively. Thus for $n \in \{2,4\}$, we have $\gamma_{sR}(F_n) = 2$.

To complete the proof, it is sufficient to provide a signed Roman dominating function of weight 1 on F_n for each $n \notin \{2,4\}$. Regarding to the different possible cases for n, as mentioned in the proof of Theorem 2.4, consider the functions that are defined in the equations 2.1, 2.2, 2.3, and 2.4. For instance, an optimal SRDF on F_5 is depicted in Figure 3 (c), where the top vertex is v_0, and its below lef one is v_1. Also optimal SRDF's on F_{12}, F_{10}, and F_8 are illustrated in Figure 4 (a), (b), and (c), respectively (where the central vertex is v_0, and the top one is v_1). $\square$

Theorem 2.6. *Let $m \ge 2$ be an integer and $n = 2m+1$. Then the signed Roman domination number of the Friendship graph $Fr_n = K_1 \vee (mK_2)$ is given by $\gamma_{sR}(Fr_n) = 2$.*

Proof. Let $V(Fr_n) = \{x\} \cup \{y_i, z_i : 1 \le i \le m\}$ and $E(Fr_n) = \{xy_i, xz_i : 1 \le i \le m\} \cup \{y_iz_i : 1 \le i \le m\}$. Since $\Delta(Fr_n) = |V(Fr_n)| - 1$, Lemma 2.1 implies that $\gamma_{sR}(Fr_n) \ge 1$. Consider the function g defined from $V(Fr_n)$ to the set $\{-1, 1, 2\}$, as follows.

$$g(v) = \begin{cases} 2 & v = x \\ 1 & v \in \{y_1, y_2, ..., y_m\} \\ -1 & v \in \{z_1, z_2, ..., z_m\}. \end{cases}$$

Since g is a $SRDF$ on Fr_n, we get $\gamma_{sR}(Fr_n) \le 2$. Now, let $f = (V_{-1}, V_1, V_2)$ be an optimal signed Roman dominating function on Fr_n. If $V_{-1} = \emptyset$, then $w(f) \ge n \ge 5$, which is a contradiction. Hence, $|V_{-1}| \ge 1$, and this implies that $|V_2| \ge 1$. If $f(y_i) = f(z_i) = -1$ for some i, then $f(N_{Fr_n}[y_i]) \le 0$, which is a contradiction. Thus for each $i \in \{1, 2, ..., m\}$, we have $|V_{-1} \cap \{y_i, z_i\}| \le 1$, and this implies that $|V_{-1}| \le m+1$. If $|V_{-1}| = m+1$, then $x \in V_{-1}$ and $|V_{-1} \cap \{y_i, z_i\}| = 1$ for each $i \in \{1, 2, ..., m\}$. Hence, $f(N_{Fr_n}[y_1]) = f(y_1) + f(z_1) + f(x) \le 0$, which is a contradiction. Therefore, $|V_{-1}| \leq m$, and

$$\gamma_{sR}(Fr_n) = w(f) = 2|V_2| + |V_1| - |V_{-1}| \geq 2\times 1 + m\times 1 + m\times(-1) = 2,$$

which completes the proof. $\square$

3. **Join of cycles**

Since $\Delta(C_m \vee C_n) = \max\{m+2, n+2\}$, the maximum degree of $C_m \vee C_n$ is $m+n-1$ if and only if $3 \in \{m,n\}$. Hence, for $m \geq 4$ and $n \geq 4$, the graph $C_m \vee C_n$ has no vertex of degree $|V(C_m \vee C_n)| - 1$.

Theorem 3.1. *If $n \geq 3$ is an integer, then $\gamma_{sR}(C_3 \vee C_n) = 1$.*

Proof. Let $V(C_3) = \{x_1, x_2, x_3\}$ and $V(C_n) = \{y_1, y_2, ..., y_n\}$, each one arranged consecutively on a circle, and consider the following cases.
Case 1. $n \equiv 0$ $(mod\ 3)$:
Define $f : V(C_3 \vee C_n) \to \{-1, 1, 2\}$ as $f(x_1) = f(x_2) = 1$, $f(x_3) = -1$, $f(y_j) = 2$ when $i \equiv 1$ $(mod\ 3)$, and $f(y_j) = -1$ otherwise. Note that $f(V(C_3)) = 1$ and $f(V(C_n)) = 0$.
Case 2. $n \equiv 1$ $(mod\ 3)$:
Define f as $f(x_1) = f(x_2) = 2$, $f(x_3) = 1$, $f(y_1) = f(y_2) = \cdots = f(y_{\frac{n-4}{3}}) = 2$, and $f(y_j) = -1$ for each $j > \frac{n-4}{3}$. Note that $f(V(C_3)) = 5$, $f(V(C_n)) = -4$ and $f(N_{C_3\vee C_n}[y_j]) \geq -3+5 \geq 1$ for each j.
Case 3. $n \equiv 2$ $(mod\ 3)$:
Define f as $f(x_1) = f(x_2) = f(x_3) = 2$, $f(y_1) = f(y_2) = \cdots = f(y_{\frac{n-5}{3}}) = 2$, and $f(y_j) = -1$ for each $j > \frac{n-5}{3}$. Note that $f(V(C_3)) = 6$, $f(V(C_n)) = -5$, and $f(N_{C_3\vee C_n}[y_j]) \geq -3+6 \geq 1$ for each j.

In each case, it is easy to check that f is a SRDF (of weight 1) on $C_3 \vee C_n$. Now Lemma 2.1 completes the proof. $\square$

The following theorem considers the general case.

Proposition 3.2. *For each pair of positive integers $m \geq 3$ and $n \geq 3$, we have $1 \leq \gamma_{sR}(C_m \vee C_n) \leq 4$.*

Proof. Assume that $V(C_m) = \{x_1, x_2, ..., x_m\}$ and $V(C_n) = \{y_1, y_2, ..., y_n\}$, which are arranged consecutively on a circle, respectively. Without loss of generality, assume that m is odd and n is even (other cases are similar). Define the two functions $f_o : V(C_m) \to \{-1, 1, 2\}$ and $f_e : V(C_n) \to \{-1, 1, 2\}$ as

$$f_o(x_i) = \begin{cases} 2 & i = 1 \\ -1 & i \in \{2, 4, ..., m-1\} \\ 1 & i \in \{3, 5, ..., m\}, \end{cases} \qquad f_e(y_j) = \begin{cases} 2 & j \in \{1, 3\} \\ -1 & j \in \{2, 4, ..., n\} \\ 1 & j \in \{5, 7, ..., n-1\}. \end{cases}$$

Now, define $f : V(C_m \vee C_n) \to \{-1, 1, 2\}$ as $f(v) = f_o(x_i)$ when $v = x_i$, and $f(v) = f_e(y_j)$ when $v = y_j$. Note that $f(x_1) = f(y_1) = 2$, and each vertex in $C_m \vee C_n$ is adjacent to x_1 or y_1. Also, $f(V(C_m)) = f(V(C_n)) = 2$, and for each i, j, we have $f_o(N_{C_m}[x_i]) \geq -1$ and $f_e(N_{C_n}[y_j]) \geq -1$. Hence, for $i = 1, 2, ..., m$, we have

$$f(N_{C_m \vee C_n}[x_i]) = f_o(N_{C_m}[x_i]) + f_e(V(C_n)) \geq -1 + 2 = 1,$$

and for $j = 1, 2, ...n$, we have

$$f(N_{C_m \vee C_n}[y_j]) = f_e(N_{C_n}[y_j]) + f_o(V(C_m)) \geq -1 + 2 = 1.$$

Thus, f is a SRDF on $C_m \vee C_n$ and $w(f) = f_o(C_m) + f_e(C_n) = 4$, the upper bound follows.

In order to obtain the lower bound, let g be an optimal SRDF on $C_m \vee C_n$. If $g(V(C_m)) \geq 1$ and $g(V(C_n)) \geq 1$, then the result follows. Assume that $g(V(C_n)) = \alpha \leq 0$. Since g is a SRDF, for each $x \in V(C_m)$, we have $g(N_{C_m \vee C_n}[x]) \geq 1$. Using the fact $g(N_{C_m \vee C_n}[x]) = g(N_{C_m}[x]) + g(V(C_n))$, we see that $g(N_{C_m}[x]) \geq 1 - \alpha$. Hence,

$$g(V(C_m)) = \sum_{x \in V(C_m)} g(x) = \frac{1}{3} \sum_{x \in V(C_m)} g(N_{C_m}[x]) \geq \frac{1}{3} \sum_{x \in V(C_m)} (1 - \alpha) \geq \frac{m}{3}(1 - \alpha).$$

Thus

$$\begin{aligned} \gamma_{sR}(C_m \vee C_n) = w(g) &= g(V(C_m)) + g(V(C_n)) \\ &\geq \frac{m}{3}(1 - \alpha) + \alpha = \frac{m}{3} + (\frac{m}{3} - 1)(-\alpha) \geq 1. \end{aligned}$$

A similar argument holds for the situation $g(V(C_m)) \leq 0$. This completes the proof. □

After some required lemmas and in Corollary 3.6, we will see that the exact and sharp value for the upper bound of $\gamma_{sR}(C_m \vee C_n)$ is 3.

Lemma 3.3. *Let $m \geq 13$ and $n \geq 13$ be two integers. If f is an optimal SRDF on $C_m \vee C_n$, then $f(V(C_m)) > 0$ and $f(V(C_n)) > 0$. Specially, $\gamma_{sR}(C_m \vee C_n) \geq 2$.*

Proof. Suppose, to the contrary, that f is an optimal SRDF on $C_m \vee C_n$ and $f(V(C_n)) = \alpha \leq 0$. Since f is a SRDF, for each $x \in V(C_m)$, we have $f(N_{C_m \vee C_n}[x]) \geq 1$, which implies that $f(N_{C_m}[x]) \geq |\alpha| + 1$. Hence,

$$f(V(C_m)) = \frac{1}{3} \sum_{x \in V(C_m)} f(N_{C_m}[x]) \geq \frac{1}{3} \sum_{x \in V(C_m)} (|\alpha| + 1) \geq \frac{1}{3} m(|\alpha| + 1).$$

Therefore,

$$\begin{aligned}\gamma_{_{sR}}(C_m \vee C_n) &= f(V(C_m)) + f(V(C_n)) \\ &\geq \frac{m}{3}(|\alpha|+1)+\alpha \geq \frac{13}{3}(-\alpha+1)+\alpha > 4.\end{aligned}$$

This contradicts Proposition 3.2. Thus, $f(V(C_n)) \geq 1$. Similarly, we can prove that $f(V(C_m)) \geq 1$. □

Lemma 3.4. *Let $n \geq 13$ be an integer such that $n \not\equiv 2 \pmod 3$. If $f : V(C_n) \to \{-1, 1, 2\}$ is a function for which $f(V(C_n)) = 1$, then there exists $y \in V(C_n)$ such that $f(N_{C_n}[y]) < 0$.*

Proof. Since $1 = f(V(C_n)) = \frac{1}{3}\sum_{x\in V(C_n)} f(N_{C_n}[x])$, the summation $\sum_{x\in V(C_n)} f(N_{C_n}[x])$ is equal to 3. Assume, to the contrary, that $f(N_{C_n}[y]) \geq 0$ for each $y \in V(C_n)$. Thus, one of the following cases should be happen.

- i) There exists $y \in V(C_n)$ such that $f(N_{C_n}[y]) = 3$ and $f(N_{C_n}[y']) = 0$ for each $y' \neq y$.
- ii) There exist $y, y' \in V(C_n)$ such that $f(N_{C_n}[y]) = 2$, $f(N_{C_n}[y']) = 1$ and $f(N_{C_n}[y'']) = 0$ for each $y'' \notin \{y, y'\}$.
- iii) There exist $y, y', y'' \in V(C_n)$ such that $f(N_{C_n}[y]) = f(N_{C_n}[y']) = f(N_{C_n}[y'']) = 1$ and $f(N_{C_n}[\bar{y}]) = 0$ for each $\bar{y} \notin \{y, y', y''\}$.

Claim. There exists no vertex with label 1.
In order to prove this claim, suppose (to the contrary) that $f(y_j) = 1$ for some $y_j \in V(C_n) = \{y_1, y_2, ..., y_n\}$. We consider the following possibilities for the labels of the neighbours of y_j.

1) $f(y_{j-1}) = 1$ and $f(y_{j+1}) = 1$:
   This implies that $f(N_{C_n}[y_j]) = 3$ and $f(N_{C_n}[y_{j-1}]) \geq 1$, which contradicts the above three possible cases (i), (ii), and (iii).
2) $f(y_{j-1}) = 2$ and $f(y_{j+1}) = 2$:
   This implies that $f(N_{C_n}[y_j]) = 5$, which is a contradiction.
3) $f(y_{j-1}) = 2$ and $f(y_{j+1}) = 1$:
   Hence, $f(N_{C_n}[y_j]) = 4$, which is a contradiction.
4) $f(y_{j-1}) = 2$ and $f(y_{j+1}) = -1$:
   This implies that $f(N_{C_n}[y_j]) = 2$ and $f(N_{C_n}[y_{j-1}]) \geq 2$, which is a contradiction.
5) $f(y_{j-1}) = -1$ and $f(y_{j+1}) = -1$:
   Thus $f(N_{C_n}[y_j]) = -1$, which is a contradiction.
6) $f(y_{j-1}) = 1$ and $f(y_{j+1}) = -1$:
   Since $f(N_{C_n}[y_{j+1}]) \geq 0$, $f(y_{j+2}) \in \{1, 2\}$. Since $f(N_{C_n}[y_j]) = 1$, $f(N_{C_n}[y_{j-1}]) \geq 1$ and $f(N_{C_n}[y_{j+1}]) \geq 1$, we should have $f(N_{C_n}[y_{j+1}]) = 1$ and $f(y_{j+2}) = 1$. Therefore, $f(N_{C_n}[y_{j'}]) = 0$

for each $j' \notin \{j-1, j, j+1\}$, and specially, $f(N_{C_n}[y_{j+2}]) = 0$, which is impossible.

This completes the proof of the claim. Therefore, the label of each vertex in C_n is -1 or 2. Let t be the number of vertices whose label is 2. If $n = 3k$, then $1 = f(V(C_n)) = 2t + (3k - t)(-1) = 3(t - k)$, which is a contradiction (3 is not a divisor of 1). If $n = 3k + 1$, then $1 = 2t + (3k + 1 - t)(-1)$. Hence, $2 = 3(t - k)$, which is a contradiction. □

Theorem 3.5. *Let $m \geq 13$ and $n \geq 13$ be two integers. Then we have*

$$\gamma_{sR}(C_m \vee C_n) = \begin{cases} 2 & m \equiv 2\ (mod\ 3),\ n \equiv 2\ (mod\ 3) \\ 3 & o.w. \end{cases}$$

Proof. At first, assume that $m \equiv 2$ $(mod\ 3)$ and $n \equiv 2$ $(mod\ 3)$. Define the function f from $V(C_m) \cup V(C_n) = \{x_1, ..., x_m\} \cup \{y_1, ..., y_n\}$ to $\{-1, 1, 2\}$, as follows.

$$f(x_i) = \begin{cases} 2 & i \equiv 1\ (mod\ 3) \\ -1 & o.w. \end{cases} \quad , \quad f(y_j) = \begin{cases} 2 & j \equiv 1\ (mod\ 3) \\ -1 & o.w. \end{cases}$$

Hence, $f(V(C_m)) = f(V(C_n)) = 1$, $f(N_{C_m}[x_m]) = f(N_{C_n}[y_n]) = 3$, and for each $1 \leq i < m$ and each $1 \leq j < n$, we have $f(N_{C_m}[x_i]) = f(N_{C_n}[y_j]) = 0$. Thus f is a SRDF of weight 2. Therefore, Lemma 3.3 completes the proof (in this case).

Now, assume that $m \equiv 2 \pmod 3$ and $n \not\equiv 2 \pmod 3$.
Define the function g on $V(C_m) = \{x_1, ..., x_m\}$ as $g(x_i) = 2$ when $i \equiv 1 \pmod 3$, and $g(x_i) = -1$, otherwise. Thus $g(N_{C_m}[x_m]) = 3$, $g(N_{C_m}[x_i]) = 0$ for each $i \neq m$, and $g(V(C_m)) = 1$. When $n \equiv 0 \pmod 3$ (or $n \equiv 1 \pmod 3$), define the function h_1 (or h_2) on $V(C_n) = \{y_1, ..., y_n\}$ as follows

$$h_1(y_j) = \begin{cases} 1 & j = n \\ 2 & j \equiv 1 \pmod 3 \\ -1 & o.w. \end{cases} \quad , \quad h_2(y_j) = \begin{cases} 2 & j \equiv 1 \pmod 3 \\ -1 & o.w. \end{cases}$$

Note that $h_1(V(C_n)) = 2$ and $h_1(N_{C_n}[y_j]) \geq 0$ for each j (similarly, $h_2(V(C_n)) = 2$ and $h_2(N_{C_n}[y_j]) \geq 0$ for each j). Now, g using h_1 (or h_2) induces a labelling on $V(C_m \vee C_n)$, which is a SRDF of weight 1+2=3. Hence, $\gamma_{sR}(C_m \vee C_n) \leq 3$. Let f be an optimal SRDF on $C_m \vee C_n$. By Lemma 3.3, $f(V(C_m)) \geq 1$ and $f(V(C_n)) \geq 1$. If $f(V(C_n)) \geq 2$, then we are done. Else $f(V(C_n)) = 1$ and Lemma 3.4 imply that there exists $y \in V(C_n)$ such that $f(N_{C_n}[y]) \leq -1$. Since $f(N_{C_m \vee C_n}[y]) \geq 1$, we should have $f(V(C_m)) \geq 2$. Thus $w(f) = f(V(C_m)) + f(V(C_n)) \geq 3$, which completes the proof (for this case).

Finally, assume that $m \not\equiv 2 \pmod 3$ and $n \not\equiv 2 \pmod 3$.
Let f be an optimal SRDF on $C_m \vee C_n$. By Lemma 3.3, $f(V(C_m)) \geq 1$ and $f(V(C_n)) \geq 1$. Lemma 3.4 implies that the case $f(V(C_m)) = f(V(C_n)) = 1$ is impossible. Thus $\gamma_{sR}(C_m \vee C_n) \geq 3$. Using h_1 or h_2, as defined in the previous paragraph, we obtain a labeling on $V(C_n)$ with total weight 2. For the case $m \equiv 0 \pmod 3$ (or $m \equiv 1 \pmod 3$), define the function g_1 (or g_2) on $V(C_m)$, as follows.

$$
\begin{aligned}
g_1(x_i) &= \begin{cases} 1 & i \in \{m-2, m-1\} \\ 2 & i \neq m-2,\ i \equiv 1\ (mod\ 3), \\ -1 & o.w. \end{cases} \\
g_2(x_i) &= \begin{cases} 1 & i = m \\ 2 & i \neq m,\ i \equiv 1\ (mod\ 3). \\ -1 & o.w. \end{cases}
\end{aligned}
$$

Note that $g_k(V(C_m)) = 1$, and for each $1 \leq i \leq m$, we have $g_k(N_{C_m}(x_i)) \geq -1$, $k \in \{1, 2\}$. Now, regarding the possible cases for m and n, and using one of the two functions g_1, g_2 and one of the two functions h_1, h_2, we obtain a labelling on $V(C_m) \cup V(C_n)$, which induces a SRDF of weight 3 on $C_m \vee C_n$. $\square$

By considering the proof of Theorem 3.5, we see that the condition $m, n \geq 13$ is used just for providing a suitable lower bound for $\gamma_{sR}(C_m \vee C_n)$ in different cases of m and n (in module 3). Throughout the proof and in each case, a SRDF of weight 2 or 3 is constructed for $C_m \vee C_n$ (without considering the condition $m, n \geq 13$), which implies that the value of $\gamma_{sR}(C_m \vee C_n)$ is at most three in that case.

Corollary 3.6. *For each pair of integers $m \geq 3$ and $n \geq 3$, we have $\gamma_{sR}(C_m \vee C_n) \leq 3$.*

Also by studying the small cases, we see that the condition $m, n \geq 13$ is redundant in Theorem 3.5 for the lower bounds, and we suggest the following conjecture:

Conjecture 1. For each pair of integers $m \geq 4$ and $n \geq 4$ we have

$$
\gamma_{sR}(C_m \vee C_n) = \begin{cases} 2 & m \equiv 2\ (mod\ 3),\ n \equiv 2\ (mod\ 3) \\ 3 & o.w. \end{cases}
$$

Acknowledgments

We would like to express our deepest gratitudes to the referees for their invaluable comments and suggestions, which improve the quality of this paper.

References

1. H. A. Ahangar, M. A. Henning, Y. Zhao, C. Löwenstein, V. Samodivkin, Signed Roman domination in graphs, *J. Comb. Optim.* **27** (2014) 241–255.
2. C. Berge, Graphs and hypergraphs, North Holland, Amsterdam, 1973.
3. Z. Füredi and D. Mubayi, Signed domination in regular graphs and setsystems, *J. Combin. Theory Ser. B* **76** (1999) 223–239.
4. M. R. Garey and D. S. Johnson, Computers and Intractability: A Guide to the theory of NP-completeness, W.H. Freeman, San Francisco 1979.
5. W. Goddard, M. A. Henning, Restricted domination parameters in graphs, *J. Comb. Optim.* **13** (2007) 353–363.
6. T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Domination in Graphs, Advanced Topics, Marcel Dekker, New York, 1998.
7. M. A. Henning, S. T. Hedetniemi, Defending the Roman empire – a new strategy, *Discrete Math.* **266** (2003), 239–251.
8. M. A. Henning, L. Volkmann, Signed Roman $k-$domination in trees, *Discrete Applied Mathematics* **186** (2015), 98–105.
9. M. A. Henning, Graphs with large paired-domination number, *J. Comb. Optim.* **13** (2007) 61–78.
10. O. Ore, Theory of graphs, Vol. 38. Providence: Amer. Math. Soc. Colloq. Publ. 1962.
11. S. M. Sheikholeslami, L. Volkmann, Signed Roman domination in digraphs, *J. Comb. Optim.* **30** (2015) 456–467.
12. S. M. Sheikholeslami, L. Volkmann, The signed Roman domatic number of a graph, *Ann. Math. Inform.* **40** (2012) 105–112.
13. I. Stewart, Defend the Roman Empire, *Sci. Amer.* **281** (1999) 136–139.

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Journal of Algebraic Systems

SIGNED ROMAN DOMINATION NUMBER AND JOIN OF GRAPHS

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هر مجموعه احاطه‌گر در یک گراف را می‌توان با یک تابع به نام تابع احاطه‌گر به طور یکتا مشخص کرد. تابع احاطه‌گر رومی علامت‌دار در واقع تعمیمی از تابع احاطه‌گر معمولی است که بر اساس آن عدد احاطه‌گر رومی علامت‌دار تعریف می‌گردد. در این مقاله عدد احاطه‌گر رومی علامت‌دار الحاق گراف‌ها را مورد بررسی قرار می‌دهیم. به ویژه مقدار دقیق این پارامتر را برای گراف چرخ، گراف بادبزن، گراف دوستی و الحاق دورها مشخص می‌نمائیم.

کلمات کلیدی: احاطه‌گری، عدد احاطه‌گر رومی علامت‌دار، الحاق، چرخ، گراف دوستی.