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THE ZERO-DIVISOR GRAPH OF A MODULE

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Abstract. Let R be a commutative ring with identity and M an R-module. In this paper, we associate a graph to M, say $\Gamma({}_RM)$, such that when $M = R$, $\Gamma({}_RM)$ coincide with the zero-divisor graph of R. Many well-known results by D. F. Anderson and P. S. Livingston, have been generalized for $\Gamma({}_RM)$. We will show that $\Gamma({}_RM)$ is connected with $\mathrm{diam}(\Gamma({}_RM)) \leq 3$, and if $\Gamma({}_RM)$ contains a cycle, then $\mathrm{gr}(\Gamma({}_RM)) \leq 4$. We will also show that $\Gamma({}_RM) = \emptyset$ if and only if M is a prime module. Among other results, it is shown that for a reduced module M satisfying DCC on cyclic submodules, $\mathrm{gr}\,(\Gamma({}_RM)) = \infty$ if and only if $\Gamma({}_RM)$ is a star graph. Finally, we study the zero-divisor graph of free R-modules.

1. Introduction

Throughout the paper, R is a commutative ring with identity and ${}_RM$ is a unitary R-module. Let $Z(R)$ be the set of zero-divisors of R. Associating graphs to algebraic structures has become an exciting research topic in the last twenty years. There are many papers on assigning a graph to a ring; see for instance, [2, 3, 6, 9, 10, 18]. Most of the work has focused on the zero-divisor graph. The concept of a zero-divisor graph of a ring R was first introduced by Beck in [7], where he was mainly interested in coloring. This investigation of colorings of a commutative ring was then continued by Anderson and Naseer

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in [5]. In [3], Anderson and Livingston associate a graph, $\Gamma(R)$, with vertices $Z^*(R) := Z(R) \setminus \{0\}$, the set of non-zero zero-divisors of R, and for distinct $x, y \in Z^*(R)$, the vertices x and y are adjacent if and only if $xy = 0$. The zero-divisor graphs of commutative rings have been extensively studied by many authors, and become a major field of research; see for instance, [11, 22] and two survey papers [1, 13]. In [22], Redmond extended the zero-divisor graph of a commutative ring to an ideal-based zero-divisor graph of a commutative ring. This notion of zero-divisor graph was also studied in [19, 21, 25]. The graph of zero-divisors for commutative rings has been generalized to modules over commutative rings; see for instance, [8, 16, 23].

In this paper, we introduce a new (and natural) definition of the zero-divisor graph for modules. As any suitable generalization, many of well known results about zero-divisor graph of rings have been generalized to modules.

The concept of a zero-divisor elements of a ring, has been generalized to a module (see for example [24], or any other book in commutative algebra):

$$\mathrm{Zdv}({}_RM) = \{r \in R | rx = 0 \text{ for some non-zero } x \in M\}.$$

Let N and K be two submodules of an R-module M. Then, $(N : K) := \{r \in R | rK \subseteq N\}$ is an ideal of R. The ideal $(0 : M)$ is called the *annihilator* of M and is denoted by $\mathrm{Ann}(M)$; for $x \in M$, we may write $\mathrm{Ann}(x)$ for the ideal $\mathrm{Ann}(Rx)$. We give a new generalization of the concept of zero-divisor elements in rings to modules:

Definition 1.1. Let M be an R-module. The set of the zero-divisors of M is:

$$Z({}_RM) := \{x \in M | x \in \mathrm{Ann}(y)M \text{ or } y \in \mathrm{Ann}(x)M \text{ for some } 0 \neq y \in M\}.$$

We note that when $M = R$, this concept coincides with the set of zero-divisor elements of R.

Definition 1.2. Let M be an R-module. We define an undirected graph $\Gamma({}_RM)$ with vertices $Z^*({}_RM) := Z({}_RM) \setminus \{0\}$, where x—y is an edge between distinct vertices x and y if and only if $x \in \mathrm{Ann}(y)M$ or $y \in \mathrm{Ann}(x)M$.

We note that, the graph $\Gamma({}_RM)$ is exactly a generalization of the zero-divisor graph of R (i.e., $\Gamma({}_RR) = \Gamma(R)$). As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the integers and integers modulo n, respectively. The zero-divisor graphs of some $\mathbb{Z}$-modules are presented in figure1.

Let G be a graph with the vertex set $V(G)$. For two distinct vertices x and y of $V(G)$ the notation x—y means that x and y are adjacent.

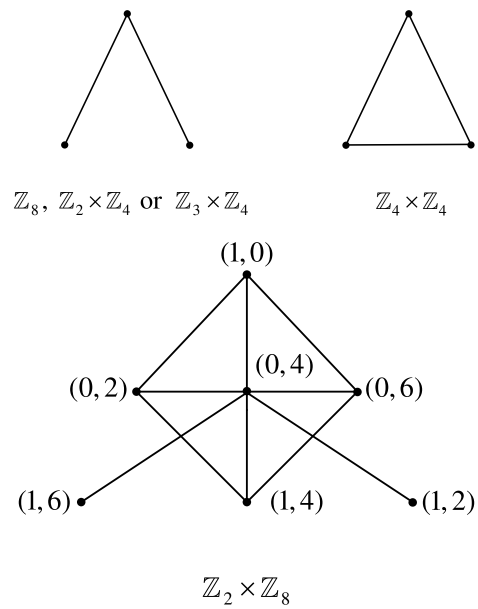

FIGURE 1. The zero-divisor graphs of some $\mathbb{Z}$-modules.

For $x \in V(G)$, we denote by $N_G(x)$ the set of all vertices of G adjacent to x. Also, the size of $N_G(x)$ is denoted by $\deg_G(x)$ and it is called the *degree* of x. A *walk* of length n in a graph G between two vertices x, y is an ordered list of vertices $x = x_0, x_1, ..., x_n = y$ such that x_{i-1} is adjacent to x_i, for $i = 1, ..., n$. We denote this walk by

$$x_0 — x_1 — \cdots — x_n.$$

If the vertices in a walk are all distinct, it defines a *path* in G. A *cycle* is a path $x_0 — \cdots — x_n$ with an extra edge $x_0 — x_n$. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G ($\mathrm{gr}(G) = \infty$, if G has no cycle). A graph G is called *connected* if for any vertices x and y of G there exists a path between x and y. For $x, y \in V(G)$, the *distance* between x and y, denoted by $d(x, y)$, is the length of a shortest path between x and y. The greatest distance between any two vertices in G, is the *diameter* of G, denoted by $\mathrm{diam}(G)$.

A graph G is called *bipartite* if $V(G)$ admits a partition into two classes such that vertices in the same partition class must not be adjacent. A simple bipartite graph in which every two vertices from different partition classes are adjacent, is called a *complete bipartite* graph. Let $K^{m,n}$ denote the complete bipartite graph on two nonempty disjoint sets V_1 and V_2 with $|V_1| = m$ and $|V_2| = n$ (we allow m and n to be infinite cardinals). A $K^{1,n}$ graph will often be called a *star graph*.

The motivation of this paper is the study of interplay between the graph-theoretic properties of $\Gamma({}_RM)$ and the module-theoretic properties of ${}_RM$. The organization of the paper is as follows: In Section 2 of this paper, we give some basic properties of $\Gamma({}_RM)$. In Section 3, we determine when the graph $\Gamma({}_RM)$ is bipartite. In Section 4, we study $\Gamma({}_RF)$, where F is a free R-module. Finally, in Section 5, we study $\Gamma({}_RM)$, where M is a multiplication R-module.

We follow standard notations and terminologies from graph theory [12] and module theory [15].

2. Basic Properties of $\Gamma({}_RM)$

We begin with the following evident proposition.

Proposition 2.1. *Let M be an R-module and I be an ideal of R. Then*
(1) $\Gamma({}_RM) = \Gamma({}_{R/\operatorname{Ann}(M)}M)$,
(2) $\Gamma({}_R(R/I)) = \Gamma(R/I)$.

Let I be an arbitrary index set, and let $\{M_i | i \in I\}$ be a family of R-modules. The direct product $\prod_{i\in I} M_i = \{(x_i)_I | x_i \in M_i\}$ is an R-module. We also note that a direct product of rings is a ring endowed with componentwise operations. We are going to explain the relationship between the zero-divisor graph of $\prod_{i\in I} R/\mathfrak{m}_i$ as R-module and as ring.

Theorem 2.2. *Let $\{\mathfrak{m}_i | i \in I\}$ be a family of maximal ideals of R and ${}_RM = \prod_{i\in I} R/\mathfrak{m}_i$. Then*
(1) *$\Gamma({}_RM)$ is a subgraph of $\Gamma(\prod_{i\in I} R/\mathfrak{m}_i)$,*
(2) *$\Gamma({}_RM) = \Gamma(\prod_{i\in I} R/\mathfrak{m}_i)$ if and only if the $\mathfrak{m}_i$'s are distinct ideals.*

Proof. (1): Let $(x_i)_I$ and $(y_i)_I$ be two adjacent vertices of $\Gamma({}_RM)$. Without loss of generality, we may assume that $(x_i)_I \in \operatorname{Ann}(y_i)_I M$. Then, $x_i \in \cap_{j\in I}\operatorname{Ann}(y_j)M \subseteq \operatorname{Ann}(y_i)M$, for all $i \in I$. It then follows that $x_i y_i = 0$ and hence $(x_i)_I(y_i)_I = 0$. So, $(x_i)_I$ and $(y_i)_I$ are two adjacent vertices of $\Gamma(\prod_{i\in I} R/\mathfrak{m}_i)$.

(2)$\Rightarrow$: Suppose $\mathfrak{m}_r = \mathfrak{m}_s$, for some distinct elements $r, s \in I$. Let $x_r := 1$ and $x_i := 0$ for all $i \neq r$ and let $y_s := 1$ and $y_i := 0$ for all $i \neq s$. Then, $(x_i)_I(y_i)_I = 0$, and hence $(x_i)_I$—$(y_i)_I$ is an edge in $\Gamma(\prod_{i\in I} R/\mathfrak{m}_i)$. But $(x_i)_I$ and $(y_i)_I$ are not adjacent in $\Gamma({}_RM)$, since $(x_i)_I \notin \operatorname{Ann}(y_i)_I M$ and $(y_i)_I \notin \operatorname{Ann}(x_i)_I M$.
$\Leftarrow$: Let $(x_i)_I$ and $(y_i)_I$ be two adjacent vertices of $\Gamma(\prod_{i\in I} R/\mathfrak{m}_i)$. We have

$$\operatorname{Ann}(x_i)_I M = \bigcap_{\imath\in I}\operatorname{Ann}(x_i)M = \prod_{i\in I} N_i,$$

where $N_i = 0$ if $x_i \neq 0$ and $N_i = M_i$ if $x_i = 0$. Since $(x_i)_I (y_i)_I = 0$, the assumption $y_i \neq 0$ implies that $x_i = 0$ and hence $(y_i)_I \in \prod_{i\in I} N_i$. It then follows that $(x_i)_I$ and $(y_i)_I$ are two adjacent vertices of $\Gamma({}_RM)$. Hence, the assertion follows from Part (1). □

It is well-known that a ring R is a domain if and only if the zero-divisor graph $\Gamma(R)$ is empty. The following proposition is a natural generalization of this fact. We recall that an R-module $M \neq 0$ is called a *prime* module if its zero submodule is prime, i.e., $rx = 0$ for $x \in M$, $r \in R$ implies that $x = 0$ or $rM = (0)$ (see [14] and [20]).

Proposition 2.3. *Let M be an R-module. Then the following are equivalent:*

1. $\Gamma({}_RM) = \emptyset$ *i.e.,* $Z(M) = \{0\}$,
2. $\mathrm{Zdv}(M) = \mathrm{Ann}(M)$,
3. M *is a prime R-module.*

Proof. (1)⇒(3) Suppose that $\Gamma({}_RM) = \emptyset$. If M is not a prime module, then there exist $r \in R \setminus \mathrm{Ann}(M)$ and non-zero element $x \in M$ such that $rx = 0$. Since $r \notin \mathrm{Ann}(M)$, there exists a non-zero element $y \in M$ such that $ry \neq 0$. It follows that ry—x is an edge of $\Gamma({}_RM)$ and hence $\Gamma({}_RM) \neq \emptyset$, which is a contradiction.

(3)⇒(1) Suppose that M is a prime R-module. If $\Gamma({}_RM) \neq \emptyset$, then there exist $x, y \in Z^*({}_RM)$ such that $x \in \mathrm{Ann}(y)M$. Therefore, there exist $r_1, \ldots, r_n \in \mathrm{Ann}(y)$ and $z_1, \ldots, z_n \in M$ such that $x = r_1z_1 + \cdots + r_nz_n$. Since, $r_iy = 0$ for all $1 \leq i \leq n$ and M is prime, we have $r_iM = 0$ for all $1 \leq i \leq n$. This implies that $x = 0$, which is a contradiction.

(2)⇔(3) Follows easily from the definition of prime modules. □

Corollary 2.4. *Let R be a ring. Then R is a field if and only if $\Gamma({}_RM) = \emptyset$ for every R-module M.*

Proof. If R is field, then proposition 2.3 implies that $\Gamma({}_RM) = \emptyset$. Now, suppose that $\Gamma({}_RM) = \emptyset$, for every R-module M. Let $\mathfrak{m}$ be a non-zero maximal ideal of R and $0 \neq x \in \mathfrak{m}$. Set $M := R/\mathfrak{m} \times R$. Then, $(0, x) \in \mathrm{Ann}(1 + \mathfrak{m}, 0)M$. Therefore, $(0, x)$ is adjacent to $(1 + \mathfrak{m}, 0)$. Thus, $\Gamma({}_RM) \neq \emptyset$, which is a contradiction. Therefore, $\mathfrak{m} = 0$ and hence R is a field. □

A semisimple module M is said to be *homogeneous* if M is a direct sum of pairwise isomorphic simple submodules.

Corollary 2.5. *Let M be a homogeneous semisimple R-module. Then, $\Gamma({}_RM) = \emptyset$.*

Proof. Since $\mathrm{Ann}(M)$ is a maximal ideal of R, M is vector space over $R/\mathrm{Ann}(M)$. Hence, the assertion follows easily from Proposition 2.3. □

We are now in a good position to bring a generalization of [3, Theorem 2.2].

Theorem 2.6. *Let M be an R-module. Then, $\Gamma({}_RM)$ is finite if and only if either M is finite or a prime module. In particular, if $1 \leq |\Gamma({}_RM)| < \infty$, then M is finite and is not a prime module.*

Proof. ($\Rightarrow$): Suppose that $\Gamma({}_RM)$ is finite and nonempty. Then, there are non-zero elements $x, y \in M$ such that $x \in \mathrm{Ann}(y)M$. Therefore, there exist $r_1, \ldots, r_n \in \mathrm{Ann}(y)$ and $z_1, \ldots, z_n \in M$ such that $x = r_1z_1 + \cdots + r_nz_n$. Since $x \neq 0$, we have $r_iz_i \neq 0$ for some $1 \leq i \leq n$. Let $L = r_iM$. Then $L \subseteq Z({}_RM)$ is finite. If M is infinite, then there exists $x_0 \in L$ such that $A := \{m \in M | r_im = x_0\}$ is infinite. If m_0 is a fixed element of A, then $N := \{m_0 - m | m \in A, m \neq m_0\}$ is an infinite subset of A. For any element $m_0 - m \in N$, we have $r_i(m_0 - m) = 0$. Thus x_0—$(m_0 - m)$ is an edge in $\Gamma({}_RM)$ and hence $\Gamma({}_RM)$ is infinite, a contradiction. Thus M must be finite.
($\Leftarrow$): If M is finite, there is nothing to prove, also if M is prime, then the assertion follows from Proposition 2.3. □

Corollary 2.7. *Let M be an R-module such that $\Gamma({}_RM) \neq \emptyset$. If every vertex of $\Gamma({}_RM)$ has finite degree, then M is a finite module.*

Proof. The assertion follows from the proof of the theorem 2.6. □

The following lemma has a key role in the proof of our main results in the sequel.

Lemma 2.8. *Let M be an R-module, $x, y \in M$ and $r \in R$. If x—y is an edge in $\Gamma({}_RM)$, then either $ry \in \{0, x\}$ or x—ry is an edge in $\Gamma({}_RM)$.*

Proof. Let x and y be two adjacent vertices of $\Gamma({}_RM)$ and let $ry \notin \{0, x\}$. If $x \in \mathrm{Ann}(y)M$, then $x \in \mathrm{Ann}(ry)M$, and hence, x and ry are adjacent. If $y \in \mathrm{Ann}(x)M$, then $ry \in \mathrm{Ann}(x)M$, and hence, x and ry are adjacent. This completes the proof. □

The next result is a generalization of [3, Theorem 2.3].

Theorem 2.9. *Let M be an R-module. Then $\Gamma({}_RM)$ is connected with* $\mathrm{diam}(\Gamma({}_RM)) \leq 3$.

Proof. Let x and y be distinct vertices of $\Gamma({}_RM)$. If either $x \in \mathrm{Ann}(y)M$ or $y \in \mathrm{Ann}(x)M$, then $d(x, y) = 1$. So, suppose that $d(x, y) \neq 1$. There

exists a vertex x' of $\Gamma({}_RM)$ such that $x \in \mathrm{Ann}(x')M$ or $x' \in \mathrm{Ann}(x)M$. We consider the following two cases:
Case 1: There exists a vertex y' of $\Gamma({}_RM)$ such that $y \in \mathrm{Ann}(y')M$. Then, there exist $r_1, \ldots, r_n \in \mathrm{Ann}(y')$ and $z_1, \ldots, z_n \in M$ such that $y = r_1z_1 + \cdots + r_nz_n$. If $r_ix' = 0$ for all i, then $x—x'—y$ is a path of length 2. If $r_ix' \neq 0$ for some $1 \leq i \leq n$, then by Lemma 2.8, $x—r_ix'—y'—y$ is a walk, and hence $d(x, y) \leq 3$.
Case 2: There exists a vertex y' of $\Gamma({}_RM)$ such that $y' \in \mathrm{Ann}(y)M$. Then, there exist $r_1, \ldots, r_n \in \mathrm{Ann}(y)$ and $z_1, \ldots, z_n \in M$ such that $y' = r_1z_1 + \cdots + r_nz_n$. If $r_ix = 0$ for all i, then $x—y'—y$ is a path of length 2. If $r_ix \neq 0$ for some $1 \leq i \leq n$, then $x—x'—r_ix—y$ is a walk, and hence $d(x, y) \leq 3$. □

Theorem 2.10. *Let M be an R-module. If $\Gamma({}_RM)$ contains a cycle, then*

$$\mathrm{gr}(\Gamma({}_RM)) \leq 4.$$

Proof. Let $x_0—x_1—x_2— \cdots —x_n—x_0$ be a cycle in $\Gamma({}_RM)$. If $n \leq 4$, we are done. So, suppose that $n \geq 5$. We consider the following two cases:
Case 1: $x_{n-1} \in \mathrm{Ann}(x_n)M$. Then, there exist $r_1, \ldots, r_m \in \mathrm{Ann}(x_n)$ and $z_1, \ldots, z_m \in M$ such that $x_{n-1} = r_1z_1 + \cdots + r_mz_m$. If $r_ix_1 = 0$ for all $1 \leq i \leq n$, then $x_1—x_{n-1}$ is an edge, and hence $x_1—x_{n-1}—x_n—x_0$ $—x_1$ is a cycle of length 4. Suppose that $r_ix_1 \neq 0$ for some $1 \leq i \leq m$. If $r_ix_1 = x_0$, then $x_0—x_2$ is an edge and hence $x_0—x_1—x_2—x_0$ is a cycle of length 3. If $r_ix_1 = x_n$, then $x_2—x_n$ is an edge and hence $x_2—x_1—x_0—x_n—x_2$ is a cycle of length 4. So, suppose that $r_ix_1 \notin \{x_0, x_n\}$. Then $x_0—r_ix_1—x_n—x_0$ is a cycle of length 3.
Case 2: $x_n \in \mathrm{Ann}(x_{n-1})M$. Then, there exist $r_1, \ldots, r_m \in \mathrm{Ann}(x_n)$ and $z_1, \ldots, z_m \in M$ such that $x_n = r_1z_1 + \cdots + r_mz_m$. If $r_ix_1 = 0$ for all $1 \leq i \leq m$, then $x_1—x_n$ is an edge and hence $x_n—x_0—x_1—x_n$ is a cycle of length 3. Suppose that $r_ix_1 \neq 0$ for some $1 \leq i \leq m$. If $r_ix_1 = x_0$, then $x_0—x_2$ is an edge and hence $x_0—x_1—x_2—x_0$ is a cycle of length 3. If $r_ix_1 = x_{n-1}$, then $x_0—x_{n-1}$ is an edge and hence $x_0—x_n—x_{n-1}—x_0$ is a cycle of length 3. So, suppose that $r_ix_1 \notin \{x_0, x_{n-1}\}$. Then, $x_0—r_ix_1—x_{n-1}—x_n—x_0$ is a cycle of length 4. □

In the following theorem, we answer to the question that "when does $\Gamma({}_RM)$ contain a cycle?".

Theorem 2.11. *Let M be an R-module. If $\Gamma({}_RM)$ has a path of length four, then $\Gamma({}_RM)$ has a cycle.*

Proof. Let $x_1—x_2—x_3—x_4—x_5$ be a path of length four. We consider the following two cases:

Case 1: $x_1 \in \mathrm{Ann}(x_2)M$. Then, there exist $r_1, \ldots, r_n \in \mathrm{Ann}(x_2)$ and $y_1, \ldots, y_n \in M$ such that $x_1 = r_1y_1 + \cdots + r_ny_n$. If $r_ix_4 = 0$ for all $1 \leq i \leq n$, then x_1 and x_4 are adjacent and hence $x_1—x_2—x_3—x_4—x_1$ is a cycle. Now, let $z := r_ix_4 \neq 0$ for some $1 \leq i \leq n$. Then, we have the following subcases:
Subcase 1.1: $z = x_1$. Then, $x_1—x_2—x_3—x_4—x_5—x_1$ is a cycle.
Subcase 1.2: $z = x_2$. Then, $x_2—x_3—x_4—x_5—x_2$ is a cycle.
Subcase 1.3: $z = x_3$. Then, $x_3—x_4—x_5—x_3$ is a cycle.
Subcase 1.4: $z = x_4$. Then, $x_2—x_3—x_4—x_2$ is a cycle.
Subcase 1.5: $z = x_5$. Then, $x_2—x_3—x_4—x_2$ is a cycle.
Subcase 1.6: $z \notin \{x_1, x_2, x_3, x_4, x_5\}$. Then, $x_2—x_3—x_4—x_5—z—x_2$ is a cycle.
Case 2: $x_2 \in \mathrm{Ann}(x_1)M$. So there exist $r_1, \ldots, r_n \in \mathrm{Ann}(x_1)$ and $y_1, \ldots, y_n \in M$ and such that $x_2 = r_1y_1 + \cdots + r_ny_n$. If $r_ix_4 = 0$ for all $1 \leq i \leq n$, then x_2 and x_4 are adjacent and hence $x_2—x_3—x_4—x_2$ is a cycle. Now, let $z := r_ix_4 \neq 0$ for some $1 \leq i \leq n$. Then, we have the following subcases:
Subcase 2.1: $z = x_1$. Then, $x_1—x_2—x_3—x_1$ is a cycle.
Subcase 2.2: $z = x_2$. Then, $x_2—x_3—x_4—x_5—x_2$ is a cycle.
Subcase 2.3: $z = x_3$. Then, $x_3—x_4—x_5—x_3$ is a cycle.
Subcase 2.4: $z = x_4$. Then, $x_1—x_2—x_3—x_4—x_1$ is a cycle.
Subcase 2.5: $z = x_5$. Then, $x_1—x_2—x_3—x_4—x_5—x_1$ is a cycle.
Subcase 2.6: $z \notin \{x_1, x_2, x_3, x_4, x_5\}$. Then, $x_3—x_4—x_5—z—x_3$ is a cycle.

So, the proof is complete. □

3. Bipartite Graphs

In [17], the authors showed that a zero-divisor semigroup graph is bipartite if and only if it contains no triangles. The following theorem is an analogous of this result.

Theorem 3.1. *Let M be an R-module. Then $\Gamma(_RM)$ is bipartite if and only if it contains no triangles.*

Proof. ⇒: Follows immediately from the fact that any bipartite graph contains no cycles of odd length.

⇐: We will show that for every cycle of odd length $2n+1 \geq 5$, there exists a cycle with length $2m+1$ such that $m < n$. Suppose that $n \geq 2$ and $x_1—x_2—\cdots—x_{2n+1}—x_1$ is a cycle with odd length $2n+1$. Since x_1 is adjacent to x_2, we have the following two cases:
Case 1: $x_1 \in \mathrm{Ann}(x_2)M$. So, there exist $r_1, \ldots, r_t \in \mathrm{Ann}(x_2)$ and $y_1, \ldots, y_t \in M$ such that $x_1 = r_1y_1 + \cdots + r_ty_t$. If $r_ix_4 = 0$ for all $1 \leq i \leq t$, then x_1 is adjacent to x_4 and hence $x_1—x_4—x_5—\cdots—x_{2n+1}—x_1$ is

a cycle with odd length $2n-1$. Now, suppose that $r_jx_4 \neq 0$ for some $1 \leq j \leq t$. Let $z := r_jx_4$. We consider the following three subcases:
Subcase 1.1: $z = x_2$. Then $x_1—z—x_5—\cdots—x_{2n+1}—x_1$ is a cycle with odd length $2n-1$.
Subcase 1.2: $z = x_3$. Then $x_3—x_4—x_5—x_3$ is a triangle.
Subcase 1.3: $z \notin \{x_2, x_3\}$. Then $x_3—z—x_2—x_3$ is a triangle.
Case 2: $x_2 \in \mathrm{Ann}(x_1)M$. So, there exist $r_1, \ldots, r_t \in \mathrm{Ann}(x_1)$ and $y_1, \ldots, y_t \in M$ such that $x_2 = r_1y_1 + \cdots + r_ty_t$. If $r_ix_4 = 0$ for all $1 \leq i \leq t$, then x_2 is adjacent to x_4 and hence we have a triangle. Now suppose that $r_jx_4 \neq 0$ for some $1 \leq j \leq t$. Let $z := r_jx_4$. Then, $x_1—z—x_5—\cdots—x_{2n+1}—x_1$ is a cycle with odd length $2n-1$.

So, by induction on n, $\Gamma({}_RM)$ contains a triangle. □

We recall that an R-module M is called *reduced* if whenever $r^2x = 0$ (where $r \in R$ and $x \in M$), then $rx = 0$. A submodule N of an R-module M is called *essential* (or *large*) in M if, for every non-zero submodule K of M, we have $N \cap K \neq 0$.

Theorem 3.2. *Let M be a reduced R-module satisfying* DCC *on cyclic submodules and let $\Gamma({}_RM)$ be a bipartite graph with parts V_1 and V_2. Let $\overline{V}_1 = V_1 \cup \{0\}$ and $\overline{V}_2 = V_2 \cup \{0\}$. Then*

(1) *$\overline{V}_1$ and $\overline{V}_2$ are submodules of M,*
(2) *$\overline{V}_1 \oplus \overline{V}_2$ is an essential submodule of M.*

Proof. (1): We will show that $\overline{V}_1$ is a submodules of M. Let $x, y \in \overline{V}_1$. First we show that $x - y \in \overline{V}_1$. If $x = y$, we are done. Now, let $x \neq y$. If x or y is equal to zero, then $x - y \in \overline{V}_1$. So, we may assume that neither x nor y is zero. There exist $x', y' \in V_2$ such that x, y are adjacent to x', y', respectively. We consider the following two cases:
Case 1: $x' \in \mathrm{Ann}(x)M$ and $y' \in \mathrm{Ann}(y)M$. Without loss of generality, we may assume that $x' = rx_1$ and $y' = sy_1$, where $r \in \mathrm{Ann}(x)$, $s \in \mathrm{Ann}(y)$ and $x_1, y_1 \in M$. Let $z := srx_1$. We claim that $z \neq 0$. If $z = 0$, then x' and y' are adjacent and hence $x' = y'$, since $x', y' \in V_2$. It then follows that $s^2y_1 = srx_1 = 0$ and hence $y' = sy_1 = 0$, which is a contradiction. So, $z \neq 0$. Since $z \in \mathrm{Ann}(x)M \cap \mathrm{Ann}(y)M$, we must have $z \in V_2$. If $z = x - y$, then $r^2s^2x_1 = rsx - rsy = 0$. Since M is reduced, we have $z = 0$, a contradiction. So $z \neq x - y$. On the other hand, $z \in \mathrm{Ann}(x - y)M$, and hence $x - y \in \overline{V}_1$.
Case 2: $x \in \mathrm{Ann}(x')M$ and $y' \in \mathrm{Ann}(y)M$. Then there exist are $r_1, \ldots, r_n \in \mathrm{Ann}(x')$ and $x_1, \ldots, x_n \in M$ such that $x = r_1x_1 + \cdots + r_nx_n$ and again without loss of generality, we may assume that $y' = sy_1$, for some $s \in \mathrm{Ann}(y)$ and $y_1 \in M$. Let $z_0 := sx$. If $z_0 = 0$, then $0 \neq y' \in \mathrm{Ann}(x - y)M$ and hence $x - y \in \overline{V}_1$. Now, let $z_0 \neq 0$.

Consider the following ascending chain of cyclic submodules:

$$Rz_0 \supseteq Rr_1z_0 \supseteq Rr_1^2z_0 \supseteq \cdots .$$

Suppose that $Rz_0 = Rr_1z_0$. Then, there exists $a \in R$ such that $z_0 = ar_1z_0$. Since M is reduced, $z_0 \neq y$ and hence z_0—y is an edge in V_1, which is a contradiction. Let $n_1 \geq 1$ be the smallest integer number such that $Rr_1^{n_1}z_0 = Rr_1^{n_1+1}z_0$. There exists $a_1 \in R$ such that $r_1^{n_1}z_0 = a_1r_1^{n_1+1}z_0$. Set $z_1 = (r_1^{n_1-1} - a_1r_1^{n_1})z_0$. Then, $z_1 \neq 0$ and we have the following ascending chain of cyclic submodules:

$$Rz_1 \supset Rr_1z_1 \supseteq Rr_1^2z_1 \supseteq \cdots .$$

Let $n_2 \geq 1$ be the smallest integer number such that $Rr_2^{n_2}z_1 = Rr_2^{n_2+1}z_1$. There exists $a_2 \in R$ such that $r_2^{n_2}z_1 = a_2r_2^{n_2+1}z_1$. Set $z_2 = (r_2^{n_2-1} - a_2r_2^{n_2})z_1$. By continuing this process, we have $z_n = (r_n^{n_n-1} - a_nr_n^{n_n})z_{n-1}$. We have $z_n \neq 0$ and

$$\begin{aligned} z_n &\in (\mathrm{Ann}(r_1x_1) \cap \cdots \cap \mathrm{Ann}(r_nx_n) \cap \mathrm{Ann}(y))M \\ &\subseteq (\mathrm{Ann}(r_1x_1 + \ldots r_nx_n) \cap (\mathrm{Ann}(y))M \\ &\subseteq (\mathrm{Ann}(x) \cap \mathrm{Ann}(y))M \\ &\subseteq \mathrm{Ann}(x-y)M. \end{aligned}$$

It follows that $z_n \in V_2$ and hence $x - y \in \overline{V}_1$.

Case 3: $x' \in \mathrm{Ann}(x)M$ and $y \in \mathrm{Ann}(y')M$. The proof of this case is similar to that of Case 2.

Case 4: $x \in \mathrm{Ann}(x')M$ and $y \in \mathrm{Ann}(y')M$. Then there exist $r_1, \ldots, r_n \in \mathrm{Ann}(x')$ and $x_1, \ldots, x_n \in M$ such that $x = r_1x_1 + \cdots + r_nx_n$. Let $z_0 := y'$. Consider the following ascending chain of cyclic submodules:

$$Rz_0 \supseteq Rr_1z_0 \supseteq Rr_1^2z_0 \supseteq \cdots .$$

Suppose that $Rz_0 = Rr_1z_0$. Then, there exists $a \in R$ such that $z_0 = ar_1z_0$. Since M is reduced, $z_0 \neq y'$ and hence z_0—y' is an edge in V_2, which is a contradiction. Let $n_1 \geq 1$ be the smallest integer number such that $Rr_1^{n_1}z_0 = Rr_1^{n_1+1}z_0$. There exists $a_1 \in R$ such that $r_1^{n_1}z_0 = a_1r_1^{n_1+1}z_0$. Set $z_1 = (r_1^{n_1-1} - a_1r_1^{n_1})z_0$. We have $z_1 \neq 0$ and the following ascending chain of cyclic submodules:

$$Rz_1 \supset Rr_1z_1 \supseteq Rr_1^2z_1 \supseteq \cdots .$$

Let $n_2 \geq 1$ be the smallest integer number such that $Rr_2^{n_2}z_1 = Rr_2^{n_2+1}z_1$. There exists $a_2 \in R$ such that $r_2^{n_2}z_1 = a_2r_2^{n_2+1}z_1$. Set $z_2 = (r_2^{n_2-1} - a_2r_2^{n_2})z_1$. By continuing this process we have $z_n = (r_n^{n_n-1} - a_nr_n^{n_n})z_{n-1}$. We have $x \in \mathrm{Ann}(z_n)M$, $r_i \in \mathrm{Ann}(z_n)$, for all $1 \leq i \leq n$. We also have $y \in \mathrm{Ann}(y')M \subseteq \mathrm{Ann}(z_n)M$. Therefore, $z_n \in V_2$. If $z_n \neq x - y$, then

$x-y \in V_1$, since $x-y \in \mathrm{Ann}(z_n)M$. If $z_n = x-y$, then $x \in \mathrm{Ann}(x-y)$ and $y \in \mathrm{Ann}(x-y)$. It follows that $x-y \in V_1$.

Now, let $r \in R$ and $x \in V_1$ such that $rx \neq 0$. We show that $rx \in V_1$. There exists $y \in V_2$ such x is adjacent to y. We have the following two cases:

Case 1: $y \in \mathrm{Ann}(x)M$. Without loss of generality, we may assume that $y = r_1z_1$, for some $r_1 \in \mathrm{Ann}(x)$ and $z_1 \in M$. If $rx = r_1z_1$, then $r_1^2z_1 = rr_1x = 0$ and hence $rx = 0$, which is a contradiction. So, $rx \neq r_1z_1$. Since rx is adjacent to r_1z_1 and $r_1z_1 \in V_2$, we have $rx \in V_1$.

Case 2: $x \in \mathrm{Ann}(x)M$. Then there exist $r_1, \ldots, r_n \in \mathrm{Ann}(x)$ and $z_1, \ldots, z_n \in M$ such that $x = r_1z_1 + \cdots + r_nz_n$. We may assume $r_iz_i \neq 0$ for all $1 \leq i \leq n$. Let $1 \leq i \leq n$. We claim that $r_iz_i \in V_1$. If $r_iz_i = y$, then $r_i^2z_i = 0$, and hence $r_iz_i = 0$, a contradiction. Since r_iz_i is adjacent to y, we must have $r_iz_i \in V_1$. So, $x = r_1z_1 + \cdots + r_nz_n \in \overline{V}_1$. It then follows that $\overline{V}_1$ is a submodule of M and a similar argument shows that $\overline{V}_2$ is a submodule of M.

(2): Let $x \in M \setminus (\overline{V}_1 \oplus \overline{V}_2)$. Since $\Gamma({}_RM)$ is bipartite, there exist $x_0, y_0 \in \overline{V}_1 \cup \overline{V}_2$ such that $x_0 \in \mathrm{Ann}(y_0)M$. So, there exist $r_1, \ldots, r_n \in \mathrm{Ann}(y_0)$ and $x_1, \ldots, x_n \in M$ such that $x_0 = r_1x_1 + \cdots + r_nx_n$. There exists $1 \leq i \leq n$ such that $r_ix_i \neq 0$. Since M is reduced, the assumption $r_ix = 0$ implies that $x \in V_1 \cup V_2$, which is a contradiction. So, $r_ix \neq 0$. Consider the following ascending chain of cyclic submodules:

$$Rx \supseteq Rr_ix \supseteq Rr_i^2x \supseteq \cdots .$$

Suppose that $Rx = Rr_ix$. Then, $x \in V_1 \cup V_2$, which is a contradiction. Let $n \geq 1$ be the smallest integer number such that $Rr_i^nx = Rr_i^{n+1}x$. There exists $a \in R$ such that $r_i^nx = ar_i^{n+1}x$. Set $z = (r_i^{n-1} - ar_i^n)x$. We have $0 \neq z \in (V_1 \cup V_2)$ and so $\overline{V}_1 \oplus \overline{V}_2$ is an essential submodule of M. □

Theorem 3.3. *Let M be a reduced R-module satisfying* DCC *on cyclic submodules. If $\Gamma({}_RM)$ is a bipartite graph, then it is a complete bipartite graph.*

Proof. Let $\Gamma({}_RM)$ be a bipartite graph with parts V_1 and V_2. Let $x \in V_1$ and $y \in V_2$. We will show that x and y are adjacent. We consider the following three cases:

Case 1: $\mathrm{Ann}(x) \nsubseteq \mathrm{Ann}(y)$. Let $r \in \mathrm{Ann}(x)$ such that $r \notin \mathrm{Ann}(y)$. If $Ry = Rry$, then $y = ray$ for some $a \in R$ and hence x is adjacent to y. Now, suppose that $Ry \neq Rry$. Consider the following ascending chain of cyclic submodules:

$$Ry \supseteq Rry \supseteq Rr^2y \supseteq \cdots .$$

Let $n \geq 1$ be the smallest integer number such that $Rr^n y = Rr^{n+1}y$. There exists $b \in R$ such that $r^n y = br^{n+1}y$. Set $z = (r^{n-1} - br^n)y$. By the definition of n, we have $0 \neq z \in V_2$. Now, we consider the following two subcases:
Subcase 1.1: $z = ry$. Then, $r^2 y = 0$ and hence $z = 0$, which is a contradiction.
Subcase 1.2: $z \neq ry$. Then, z and ry are adjacent vertices of V_2, which is again a contradiction.
Case 2: Ann$y \not\subseteq$ Ann(x). The proof of this case is similar to that of Case 1.
Case 3: $\text{Ann}(x) = \text{Ann}(y)$. There exists $\alpha \in V_2$ such that α is adjacent to x. Since $\alpha, y \in V_2$, the assumption $\alpha \in \text{Ann}(x)M = \text{Ann}(y)M$, implies that $\alpha = y$. Hence, x and y are adjacent. Now, suppose that $x \in \text{Ann}(\alpha)M$. Then, there exist $r_1, \ldots, r_n \in \text{Ann}(\alpha)$ and $x_1, \ldots, x_n \in M$ such that $x = r_1x_1 + \cdots + r_nx_n$. If $r_iy = 0$ for all $1 \leq i \leq n$, then x and y are adjacent, and we are done. Now, suppose that there exists $1 \leq i \leq n$ such that $r_iy \neq 0$. Since M is reduced, r_iy and α are adjacent vertices in V_2, which is a contradiction. This completes the proof. □

If $M = R = \mathbb{Z}_3 \times \mathbb{Z}_4$, then $\Gamma({}_RM)$ is bipartite which is not complete bipartite. So, the reduced condition in Theorem 3.3 is essential. We have not found any example of a module M to show that the DCC condition in Theorem 3.3 is essential, which motivates to ask the following question.

Question 3.4. Let M be a reduced R-module such that $\Gamma({}_RM)$ is a bipartite graph. Is $\Gamma({}_RM)$ a complete bipartite graph?

In [4, Theorem 2.2], it has been proved that for a reduced commutative ring R, $\text{gr}(R) = 4$ if and only if $\Gamma(R) = K^{m,n}$ with $m, n \geq 2$. In the following corollary, we prove an analogous result for $\Gamma({}_RM)$.

Corollary 3.5. *Let M be a reduced R-module satisfying* DCC *on cyclic submodules. Then,* $\text{gr}(\Gamma({}_RM)) = 4$ *if and only if* $\Gamma({}_RM) = K^{m,n}$ *with* $m, n \geq 2$.

Proof. Let $\text{gr}(\Gamma({}_RM)) = 4$. By Theorem 3.1, $\Gamma({}_RM)$ has no cycle of odd length, and hence it is a bipartite graph. Now, by Theorem 3.3, we observe that $\Gamma({}_RM)$ is a complete bipartite graph. Since $\Gamma({}_RM)$ has a cycle of length four, we have $\Gamma({}_RM) = K^{m,n}$ with $m, n \geq 2$. The converse is trivial. □

In [4, Theorem 2.4], it has been proved that for a reduced commutative ring R, $\Gamma(R)$ is nonempty with $\text{gr}(R) = \infty$ if and only if

$\Gamma(R) = K^{1,n}$ for some $n \geq 1$. In the following corollary, we prove an analogous result for $\Gamma({}_RM)$.

Corollary 3.6. *Let M be a reduced R-module satisfying* DCC *on cyclic submodules. Then,* $\mathrm{gr}(\Gamma({}_RM)) = \infty$ *if and only if* $\Gamma({}_RM)$ *is a star graph.*

Proof. Let $\mathrm{gr}(\Gamma({}_RM)) = \infty$. Then, $\Gamma({}_RM)$ has no cycle and hence it is a bipartite graph. By Theorem 3.3, $\Gamma({}_RM)$ is a complete bipartite graph. Let $\Gamma({}_RM) = K^{m,n}$, where $m, n \geq 1$. Since $\Gamma({}_RM)$ has no cycle, then either $m = 1$ or $n = 1$, which implies that $\Gamma({}_RM)$ is a star graph. The converse is trivial. □

4. Zero-divisor graphs of free modules

We recall that an R-module F is called *free* if it is isomorphic to a direct sum of copies of R. We write $R^{(I)}$ for the direct sum $\bigoplus_{i\in I} R_i$, where each R_i is a copy of R, and I is an arbitrary indexing set. If I is a finite set with n elements, then the direct sum and the direct product coincide; in this case, we write R^n for $R^{(I)} = R \times \cdots \times R$ (n times).

We begin this section with the following useful and evident proposition.

Proposition 4.1. *Let ${}_RF = R^{(I)}$ be a free R-module and $(x_i)_I, (y_i)_I \in Z^*({}_RF)$. Then*

1. $Z({}_RF) = \{(x_i)_I \in F \mid \exists\, 0 \neq y \in R$ *such that* $yx_i = 0$ *for all* $i \in I\}$,
2. $(x_i)_I$—$(y_i)_I$ *is an edge in* $\Gamma({}_RF)$ *if and only if* $x_iy_j = 0$ *for all* $i, j \in I$.

Theorem 4.2. *Let $F = R^{(I)}$ be a free R-module. Then, $\Gamma({}_RF)$ is complete if and only if $F = R = \mathbb{Z}_2 \times \mathbb{Z}_2$ or $(Z(R))^2 = 0$.*

Proof. If $F = R = \mathbb{Z}_2 \times \mathbb{Z}_2$ or $(Z(R))^2 = 0$, then it is easy to see that $\Gamma({}_RF)$ is complete.

Conversely, suppose that $\Gamma({}_RF)$ is complete. Let $i_0 \in I$ and x, y be two distinct elements of $Z^*(R)$. Let $x_i = y_i = 0$ for all $i \in I\setminus\{i_0\}$, $x_{i_0} = x$ and $y_{i_0} = y$. Then, $(x_i)_I, (y_i)_I \in Z^*({}_RF)$ and hence $xy = 0$. Thus, $\Gamma(R)$ is complete. Then, [3, Theorem 2.8] implies that $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ or $(Z(R))^2 = 0$. We show that $|I| = 1$, if $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Suppose on the contrary that $|I| \geq 2$. Let i_1, i_2 be two distinct elements of I. Put

$$x_i := \begin{cases} (1,0) & \text{if } i = i_1, \\ (1,0) & \text{if } i = i_2, \\ (0,0) & \text{otherwise,} \end{cases}$$

and

$$y_i := \begin{cases} (1,0) & \text{if } i = i_1, \\ (0,0) & \text{otherwise.} \end{cases}$$

Then, $x := (x_i)_I, y := (y_i)_I \in Z^*({}_RF)$ and x and y are not adjacent in $\Gamma({}_RF)$, a contradiction. This completes the proof. □

Let $F = R^{(I)}$. In the following three theorems, we study the relationship between the properties of $\Gamma({}_RF)$ and $\Gamma(R)$.

Theorem 4.3. *Let $F = R^n$ be a finitely generated free R-module. Let $a \in Z^*(R)$, $t = \deg_{\Gamma(R)} a$, $A = \{(x_1, \ldots, x_n) \in Z^*({}_RF) | x_i = 0$ or $x_i = a\}$ and $x \in A$. Then,*

$$\deg_{\Gamma({}_RF)}(x) = \begin{cases} (t+1)^n - 1 & \textit{if } a^2 \neq 0, \\ (t+2)^n - 2 & \textit{otherwise.} \end{cases}$$

Proof. Let $t = \deg_{\Gamma(R)}(a)$ and $N_{\Gamma(R)}(a) = \{a_1, \ldots, a_t\}$. If $a^2 \neq 0$, then

$$N_{\Gamma({}_RF)}(x) = \{(x_1, \ldots, x_n) | x_i \in \{0, a_1, \ldots, a_t\}\} \setminus \{0\}.$$

Therefore, $\deg_{\Gamma({}_RF)}(x) = |N_{\Gamma({}_RF)}(x)| = (t+1)^n - 1$. Now, suppose that $a^2 = 0$. Then,

$$N_{\Gamma({}_RF)}(x) = \{(x_1, \ldots, x_n) | x_i \in \{0, a, a_1, \ldots, a_t\}\} \setminus \{0, x\}.$$

Hence, $\deg_{\Gamma({}_RF)}(x) = |N_{\Gamma({}_RF)}(x)| = (t+2)^n - 2$. □

Theorem 4.4. *Let $F = R^{(I)}$ such that $|I| \geq 2$. Then*

$$\mathrm{gr}(\Gamma({}_RF)) = \begin{cases} \mathrm{gr}(\Gamma(R)) & \textit{if } R \textit{ is reduced,} \\ 3 & \textit{otherwise.} \end{cases}$$

Proof. First suppose that R is not reduced. Then, there exists $0 \neq a \in R$ such that $a^2 = 0$. Let i_1, i_2 be two distinct elements of I. Put

$$x_i := \begin{cases} a & \text{if } i = i_1, \\ 0 & \text{otherwise,} \end{cases} \qquad y_i := \begin{cases} a & \text{if } i = i_2, \\ 0 & \text{otherwise.} \end{cases}$$

and $z_i := a$ for all $i \in I$. Then $(x_i)_I$—$(y_i)_I$—$(z_i)_I$—$(x_i)_I$ is a cycle of length three and hence $\mathrm{gr}(\Gamma({}_RF)) = 3$. Now, suppose that R is reduced. Let a_1—a_2—$\cdots$—a_t—a_1 be a cycle in $\Gamma(R)$. Let $j \in \{1, 2, \ldots, t\}$ and $i_0 \in I$. Put

$$x_i^j := \begin{cases} a_j & \text{if } i = i_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $(x_i^1)_I$—$(x_i^2)_I$—$\cdots$—$(x_i^t)_I$—$(x_i^1)_I$ is a cycle in $\Gamma({}_RF)$ and hence, $\mathrm{gr}(\Gamma(R)) \leq \mathrm{gr}(\Gamma({}_RF))$. Now, let

$$(x_i^1)_I—(x_i^2)_I—\cdots—(x_i^t)_I—(x_i^1)_I,$$

be a cycle in $\Gamma(F)$. For all $j \in \{1, 2, \dots, t\}$, there exists $i_j \in I$ such that $x_{i_j}^j \neq 0$. Then, $x_{i_1}^1 — x_{i_2}^2 — \cdots — x_{i_t}^t — x_{i_1}^1$ is a cycle in $\Gamma(R)$ and hence, $\mathrm{gr}(\Gamma({}_RF)) \leq \mathrm{gr}(\Gamma(R))$. This completes the proof. □

A *clique* in a graph G is a subset of pairwise adjacent vertices. The supremum of the size of cliques in G, denoted by $\omega(G)$, is called the *clique number* of G.

Theorem 4.5. *Let $F = R^n$ be a finitely generated free R-module. Then $\omega(\Gamma({}_RF)) = \omega(\Gamma(R))$.*

Proof. Let $\{(x_i^1)_I, (x_i^2)_I, \dots, (x_i^t)_I\}$ be a clique in $\Gamma({}_RF)$. For each $1 \leq j \leq t$, there exists $i_j \in I$ such that $x_{i_j}^j \neq 0$. Then, $\{x_{i_1}^1, x_{i_2}^2, \dots, x_{i_t}^t\}$ is a clique in $\Gamma(R)$ and hence $\omega(\Gamma({}_RF)) \leq \omega(\Gamma(R))$. Now, let $\{x_1, x_2, \dots, x_t\}$ be a clique in $\Gamma(R)$. Let $1 \leq j \leq t$ and $i_0 \in I$. Put

$$x_i^j := \begin{cases} x_j & \text{if } i = i_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\{(x_i^1)_I, (x_i^2)_I, \dots, (x_i^t)_I\}$ is a clique in $\Gamma({}_RF)$ and hence $\omega(\Gamma(R)) \leq \omega(\Gamma({}_RF))$. This completes the proof. □

The next theorem shows that the structure of a finitely generated free R-module F can be determined by $\Gamma(F)$. We denote the maximum degree of vertices of a graph G by $\Delta(G)$.

Theorem 4.6. *Let M and N be two finitely generated free R-module. If $\Gamma({}_RM) \cong \Gamma({}_RN)$, then $M \cong N$ as R-modules.*

Proof. Let $M = R^m$ and $N = R^n$, for some natural numbers m, n. Suppose that $m > n$. Let $x = (x_1, x_2, \dots, x_n)$ be a vertex of $\Gamma({}_RN)$ such that $\deg_{\Gamma({}_RN)}(x) = \Delta(\Gamma({}_RN))$. Since $x \in Z^*(\Gamma(N))$, there exists $0 \neq a \in R$ such that $ax_1 = ax_2 = \cdots = ax_n = 0$. Let $y = (x_1, x_2, \dots, x_n, 0, \dots, 0) \in M$. Then, the set

$$\left\{(y_1, \dots, y_n, z_1, \dots, z_{m-n}) \in {}_RM | (y_1, \dots, y_n) \in N_{\Gamma({}_RN)}(x), z_i \in \{0, a\}\right\},$$

is a subset of $N_{\Gamma({}_RN)}(y)$. It then follows that $\Delta(\Gamma({}_RM)) \geq \deg_{\Gamma({}_RM)}(y) > \deg_{\Gamma({}_RN)}(x) = \Delta(\Gamma({}_RN))$, a contradiction. So, $m \leq n$. A similar argument shows that $n \leq m$. This completes the proof. □

5. Further Notes

In this short section, we study $\Gamma({}_RM)$, where M is a multiplication R-module. We recall that an R-module M is called a *multiplication* module if for each submodule N of M, there exists an ideal I of R such that $N = IM$. Let $N = IM$ and $K = JM$, for some ideals I and J of R. The product of N and K, is denoted by $N * K$, and defined by IJM. It is easy to see that the product of N and K, is independent of presentations of N and K. In [16], Lee and Varmazyar have given a generalization of the concept of zero-divisor graph of rings to multiplication modules. For a multiplication R-module M, they defined an undirected graph $\Gamma_*({}_RM)$, with vertices $\{0 \neq x \in M | Rx * Ry = 0 \text{ for some non-zero } y \in M\}$, where distinct vertices x and y are adjacent if and only if $Rx * Ry = 0$.

The following theorem shows that, in multiplication modules, this generalization and the one given in this paper are the same.

Theorem 5.1. *Let M be a multiplication R-module. Then,* $\Gamma({}_RM) = \Gamma_*({}_RM)$.

Proof. Let x and y be two non-zero element of M and suppose that $Rx = IM$ and $Ry = JM$, for some ideals I and J of R. Let x—y be an edge in $\Gamma_*({}_RM)$. Since $Rx * Ry = 0$, we have $IJM = 0$ and hence $I \subseteq \text{Ann}(JM)$. It then follows that $IM \subseteq \text{Ann}(JM)M$. Therefore, $Rx \subseteq \text{Ann}(Ry)M$ and hence, x—y is an edge in $\Gamma({}_RM)$.

Now, suppose that x—y is an edge in $\Gamma({}_RM)$. It then follows that $Rx \subseteq \text{Ann}(Ry)M$. So $IM \subseteq \text{Ann}(JM)M$. In view of [26, Theorem 9], we have the following two cases:

Case 1: $I \subseteq \text{Ann}(JM) + \text{Ann}(M)$. In this case, $I \subseteq \text{Ann}(JM)$, since $\text{Ann}(M) \subseteq \text{Ann}(JM)$. It then follows that $IJM = 0$ and hence, x—y is an edge in $\Gamma_*({}_RM)$.

Case 2: $M = ((\text{Ann}(JM) + \text{Ann}(M)) : I)M$. In this case, we have $M = (\text{Ann}(JM) : I)M$ and hence, $IJM = [(\text{Ann}(JM) : I)I](JM) \subseteq \text{Ann}(JM)JM = 0$. Therefore, x—y is an edge in $\Gamma_*({}_RM)$. This completes the proof. □

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