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A NOTE ON THE COMMUTING GRAPHS OF A CONJUGACY CLASS IN SYMMETRIC GROUPS

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Abstract. The aim of this paper is to obtain the automorphism group of the commuting graph of a conjugacy class in the symmetric groups. The clique number, coloring number, independence number and diameter of these graphs are also computed.

1. Introduction

Let $L = (V, E)$ be a graph with vertex set V and edge set E. The (*open*) *neighborhood* $N(a)$ of a vertex $a \in V$ is the set of all vertices that are adjacent to a, and the *closed neighborhood* of a is defined as $N[a] = N(a) \cup \{a\}$. The *distance* between vertices x and y in L, denoted by $d(x, y)$, is defined as the length of a shortest path connecting them. Note that $d(x, x) = 0$, and $d(x, y) = \infty$ if there is no path connecting x and y. The *diameter* $diam(L)$ is the maximum of $d(x, y)$ taken over all pairs of vertices of L. A subset $X \subseteq V$ is called an *independence set* if there exists no edge with both endpoints in X. The *independent number* $\alpha(L)$ is the maximum cardinality among all independent sets in L and the *chromatic number* $\omega(L)$ is the maximum number of vertices that are mutually adjacent, that is the order of a maximum complete subgraph of L. The *chromatic number* of L is the minimum number of colors which are used for the coloring of the vertices of L, where any two adjacent vertices has distinct colors. Let us denote the clique number of L by $\chi(L)$. The *Kneser graph* $K_{n:m}$ is the graph whose vertices are the m-subsets of a fixed n-set, and two vertices are adjacent if the

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corresponding m-subsets are disjoint. We refer to [6] for other graph theory notations of this paper.

Let G be a group and $S \subseteq G$. The *commuting graph* $C(G, S)$ is the graph with vertex set S such that two vertices are adjacent if and only if they commute. Akbari and his co-workers [1] studied the commuting graph of a ring and Araújo et al. [2] investigated this graph for semigroups. In [3], Bates et al. determined the diameter of $C(S_n, X)$, where X is the set of $m-$cycles and $n \geq 2m+1 \geq 7$. In some special cases, they obtained upper bounds for diameter of commuting graph. In [4, 5], the authors, among others, obtained a number of results on the diameter of commuting graph of a finite group.

In this paper, we apply the main properties of the Kneser graphs to obtain the automorphism group of the commuting graph of a conjugacy class in symmetric groups and then determine the clique number, the independence number and the diameter of these graphs. For the sake of completeness, we mention here the main properties of Kneser graphs which is crucial throughout this paper.

Theorem 1.1. *Suppose $L = K_{n:m}$, $n > 2m$ and $n = 2m + k$. Then we have*

(1) *$Aut(L) = S_n$, where S_n is the symmetric group of degree n,*
(2) *$\omega(L) = \lfloor \frac{n}{m} \rfloor$,*
(3) *$\chi(L) = n - 2k + 2$,*
(4) *$\alpha(L) = \binom{n-1}{m-1}$,*
(5) *$diam(L) = \lceil \frac{m-1}{k} \rceil + 1$.*

2. PRELIMINARY RESULTS

Let L be a graph. A subgraph H of L is called an *EN-subgraph* if any two vertices of H have equal closed neighborhood. An EN-subgraph of L is called *MEN-subgraph* if it is maximal among the set of all EN-subgraphs of L. It can be seen that any EN-subgraph is complete and the set of vertices of L can be partitioned into the vertex sets of all MEN-subgraphs of L. Suppose $x \in V(L)$. The set of MEN-subgraph B with $x \in V(B)$ is denoted by $\overline{x}$. Define the weighted graph $\overline{L}$ as follows: $V(\overline{L}) = \{\overline{x} | x \in V(L)\}$, $weight(\overline{x}) = |\overline{x}|$ and two vertices $\overline{x}$ and $\overline{y}$ are adjacent if and only if x and y are adjacent in L.

We start our result, by the following elementary lemma:

Lemma 2.1. *Let L be a graph and $\varphi \in Aut(L)$. Then $\overline{\varphi} \in Aut(\overline{L})$, where $\overline{\varphi}(\overline{x}) = \overline{\varphi(x)}$.*

Proof. Let L be a graph and $\varphi \in Aut(L)$. Since $\varphi(N[x]) = N[\varphi(x)]$ for all x, hence, $B \in \overline{L}$ if and only if $\varphi(B) \in \overline{L}$. On the other hand, if $x \in B$ then $\varphi(B) = \overline{\varphi(x)}$, proving the lemma. $\square$

Theorem 2.2. *For a graph L with $|V(L)| < \infty$, we have $Aut(L) \cong Aut(\overline{L}) \ltimes \prod_{B \in V(\overline{L})} S_{|B|}$.*

Proof. Define $\psi : Aut(L) \to Aut(\overline{L})$, by $\psi(\varphi) = \overline{\varphi}$ such that $\overline{\varphi}(B) = \varphi(B)$. Then ψ is homomorphism and $|Aut(L)| \leq |kerl(\psi)||Aut(\overline{L})|$. But $kerl(\psi) = \{\varphi | \varphi(B) = B$, for all $B \in \overline{L}\}$. On the other hand, all functions where induced a bijection function on each B are in $kerl(\psi)$ and so $kerl(\psi) \cong \prod_{B \in V(\overline{L})} S_{|B|}$, where S_n is the symmetric group of degree n.

We now find a subgroup H of $Aut(G)$ such that $H \cap kerl(\varphi) = 1$. Since L is finite, there exists a total order $\leq$ on $V(L)$. Set $H = \{\varphi | \varphi(B) \in V(\overline{L})$, for all $B \in V(\overline{L})$ and φ preserve the partial order $\}$. It is clear that H is a subgroup of $Aut(L)$. Let $\varphi \in H$, $B = \{x_1, \ldots, x_t\} \in V(\overline{L})$ and $\varphi(B) = B$ where $x_1 < x_2 < \cdots < x_t$. So, $\{\varphi(x_1), \ldots, \varphi(x_t)\} = B$ and consequently $\varphi(x_i) = x_i$, for all i. Thus, if $\varphi \in H$ then $\varphi(B) = B$ if and only if $\varphi(x) = x$, for all $x \in B$. Thus, $H \cap kerl(\psi) = 1$ and then $|Aut(L)| \geq |kerl(\psi)||H|$. Now $f : H \to Aut(\overline{L})$ given by $f(\varphi) = \psi(\varphi)$ is an homomorphism. Assume that $h \in Aut(\overline{L})$. For $B = \{x_1, \ldots, x_t\} \in \overline{L}$, $x_1 < \ldots < x_t$, we define $\varphi(x_i) = y_i$, where $h(B) = \{y_1, \ldots, y_t\}$, $y_1 < \ldots < y_t$. If x, y are adjacent and $x, y \in B$ for some B, then $\varphi(x), \varphi(y) \in \varphi(B)$ and are adjacent. If x, y are adjacent and $x \in B_1, y \in B_2$ for some $B_1 \neq B_2$, then $\varphi(x) \in \varphi(B_1), \varphi(y) \in \varphi(B_2)$ and, since B_1, B_2 are adjacent, $\varphi(B_1), \varphi(B_2)$ are adjacent and consequently $\varphi(x), \varphi(y)$ are adjacent.

Thus $\varphi \in Aut(L)$ and $f(\varphi) = h$. Consequently, f is an automorphism, which completes the proof. $\square$

For a graph L, we define the relation $\sim$ on $V(L)$ as follows: $a \sim b$ if and only if $N(a) = N(b)$. It is easy to see that $\sim$ is an equivalence relation on $V(L)$. Denote the equivalence class of x by $[x]$. Moreover, define the weighted graph $\widetilde{L}$ with vertex set $\{[x] | x \in V(L)\}$, $weight([x]) = |[x]|$ and, $[x]$ is adjacent to $[y]$ if and only if x is adjacent to y. Then we can see that no pair of elements of $[x]$ are adjacent in L. Similar to the Theorem 2.2, we can obtain the following:

Theorem 2.3. *For a finite graph L, we have $Aut(L) \cong Aut(\widetilde{L}) \ltimes \prod_{[x] \in V(\widetilde{L})} S_{|[x]|}$.*

We have the following important theorem:

Theorem 2.4. *For a finite graph L with $|V(L)| \geq 2$, the following hold:*

(1) $\alpha(\overline{L}) = \alpha(L)$ *and if* $L, \widetilde{L}$ *are regular then* $\alpha(L) = r\alpha(\widetilde{L})$, *where* r *is the weight of each vertex of* $\widetilde{L}$.
(2) $\chi(L) = \chi(\widetilde{L})$ *and if* $L, \overline{L}$ *are regular then* $\chi(L) = r\chi(\overline{L})$, *where* r *is the weight of each vertex of* $\overline{L}$.
(3) $\omega(L) = \omega(\widetilde{L})$ *and if* $L, \overline{L}$ *are regular then* $\omega(L) = r\omega(\overline{L})$, *where* r *is the weight of each vertex of* $\overline{L}$.
(4) *If* L *is not complete, then* $diam(L) = diam(\overline{L})$.
(5) *Let* L *be connected but not complete. If* $\widetilde{L}$ *is complete, then* $diam(L) = diam(\widetilde{L}) + 1$ *otherwise* $diam(L) = diam(\widetilde{L})$.

Proof. (1) It is clear that $\alpha(L) \geq \alpha(\overline{L})$. Let $\{x_1, \dots, x_s\} \subseteq V(L)$ be an independent set. Then x_i, x_j are not adjacent and, then $\overline{x_i}, \overline{x_j}$ are not adjacent. Thus $\alpha(L) \leq \alpha(\overline{L})$, as required. Also, if $\{[x_1], \dots, [x_s]\} \subseteq \widetilde{L}$ is an independent set then any element of $[x_i]$ is not adjacent to any element of $[x_j]$, therefore $[x_1] \cup \ldots \cup [x_s]$ is an independent set of L. On the other hand, if A is a maximal independent set of L and $x \in A$, then $[x] \subseteq A$ and $\{[x] | x \in A\}$ is an independent set of $\widetilde{L}$. Thus, $\alpha(\widetilde{L}) \geq s/r$, which completes (1).
(2) The proof of $\chi(L) = \chi(\widetilde{L})$ is trivial. Assume that $\{u_1, \dots, u_t\}$ is a coloring set of $\overline{L}$. We use r distinct colors $u_i^1, \dots, u_i^r$ for vertices in $\overline{x}$, where x has color u_i. Thus, $\chi(L) \leq r\chi(\widetilde{L})$. Let S be a coloring set for L and u_x is the color of x. We consider the color of $b \in \overline{x}$ for $\overline{x}$ and obtain a coloring set for $\overline{L}$. Assume that A is an arbitrary subset of $V(L)$, where $|A \cap \overline{x}| = 1$. Then $\{u_x | x \in A\}$ is a coloring set of $\overline{L}$. Consequently, $|S| \geq r\chi(\overline{L})$, as required.
(3) Is similar to (1).
(4) Is elementary.
(5) We see that $\widetilde{L}$ is complete if and only if L is a complete k-partite graph, where $k = |\widetilde{L}|$. So, we assume that $\widetilde{L}$ is not complete. Let $diam(\widetilde{L}) = s \geq 2$, $d([a], [b]) = s$ and, $[a] = [a_0] - [a_1] - \cdots - [a_s] = [b]$ is a path. Then $a = a_0 - a_1 - \cdots - a_s = b$ is a path and thus $d(a, b) \leq s$. If $d(a, b) = t$ and $a = x_0 - x_1 - \cdots - x_t = b$ is a path, then we can obtain a path with length less than or equal to t. From which $d(a, b) = s$ and $diam(\widetilde{L}) \leq diam(L)$. Since $\widetilde{L}$ is not complete, $diam(\widetilde{L}) \geq 2$. Let $d(x, y) = diam(L) = t$ for $t \geq 2$ and $x = x_0 - x_1 - \cdots - x_t = y$ is a path. Because $d(x, y) \geq 2$, $[x] \neq [y]$ and thus $d(x, y) = d([x], [y])$, which completes the proof. $\square$

Note. For a graph L, if $\overline{L}$ or $\widetilde{L}$ is regular with equal weights, then we can consider this graphs un-weighted.

3. Main results

Let $\sigma = (12\dots m)$ be a cycle of length m in S_n and $S = \sigma^{S_n}$. We obtain the automorphism group, chromatic number, clique number and diameter of $C(S_n, S)$. We start by the following elementary lemma.

Lemma 3.1. $C_{S_n}(\sigma) = \langle\sigma\rangle \times Sym(\{m+1,\dots,n\})$.

Proof. It is enough to see that $n(n-1)\dots(n-m+1)/m = |\sigma^{S_n}| = [S_n : C_{S_n}(\sigma)]$ and $\langle\sigma\rangle \times Sym(\{m+1,\dots,n\}) \subseteq C_{S_n}(\sigma)$. □

Corollary 3.2. *Consider the graph $C(S_n, S)$. If $(m,n) \neq (2,4)$, then the vertices of any MEN-subgraph are the generators of $\langle\alpha\rangle$, for some $\alpha \in S$.*

Proof. Suppose $\alpha \in S$. Without loss of generality, we can assume that $\alpha = (12\dots m)$. By Lemma 3.1, $N[\alpha] = S \cap (\langle\alpha\ \rangle \cup Sym(\{m+1,\dots,n\})) = (S\cap\langle\alpha\rangle)\cup(S\cap Sym(\{m+1,\dots,n\}))$. If $\beta \in N(\alpha) - \langle a\rangle$, then $\beta \in Sym(\{m+1,\dots,n\})$. We see that $\overline{\alpha} = \overline{\beta}$ if and only if $(m,n) = (2,4)$, and hence the result follows. □

We are now ready to present our main result.

Theorem 3.3. *Suppose $n > 2m$ and $n = 2m+k$. Then,*

1. $\alpha(C(S_n,S)) = \frac{\binom{n-1}{m-1}(m-1)!}{\phi(m)}$,
2. $\chi(C(S_n,S) = \phi(m)(n-2m+2)$,
3. $\omega(C(S_n,S) = \phi(m)\lfloor\frac{n}{m}\rfloor$,
4. $diam(C(S_n,S)) = \lceil\frac{m-1}{k}\rceil + 1$,
5. $Aut(C(S_n,S)) = (S_n \ltimes S_c^d) \ltimes S_a^b$, *where* $a = \phi(m)$, $b = \frac{n(n-1)\cdots(n-m+1)}{m\phi(m)}$, $c = \frac{(m-1)!}{\phi(m)}$, $d = \frac{b}{c}$ *and ϕ is the Euler function.*

Proof. Since $\overline{C(S_n,S)}$ is a graph with equal weights $\phi(m)$, we can assume that it is un-weighted. Let L be such a graph and $x = (a_1\dots a_m) \in S$. Since $(m,n) \neq (2,2)$, hence $\overline{x} = gen(x)$, where $gen(x)$ is the set of all generators of $\langle x\rangle$. Since $N(\overline{(12\dots m)}) = \{\overline{\beta} \mid \beta \in Sym(m+1, \dots, n)\}$, $N(\overline{(a_1\dots a_m)}) = N(\overline{(b_1\dots b_m)})$ if and only if $\{a_1,\dots,a_m\} = \{b_1,\dots,b_m\}$. Therefore, $[\overline{x}] = \{\overline{(b_1\dots b_m)} | \{b_1,\dots,b_m\} = \{a_1,\dots,a_m\}\}$. We now consider the graph $\widetilde{L}$. Then two vertices $a = [\overline{(a_1\dots a_m)}]$ and $b = [\overline{(b_1\dots b_m)}]$ are adjacent if and only if $\{a_1,\dots,a_m\}$ and $\{b_1,\dots,b_m\}$ are disjoint. But $|S \cap S_m| = (m-1)!$ and $|\overline{(12\dots m)}| = \varphi(m)$, thus all weights of vertices of $\widetilde{L}$ are equal to $\frac{(m-1)!}{\phi(m)}$ and so we can consider this graph to be un-weighted, say L'. Hence, $V(L')$ is all subsets of $\{1,\ldots,n\}$ with m elements such that two vertices are adjacent if and only if they are disjoint. This concludes that L' is the Kneser graph $K_{n:m}$. By above assumption and Theorems 1.1, 2.2, 2.3 and 2.4, the proof will be proved. $\square$

It is merit to mention here that the part (d) of Theorem 3.3 is a main result of [3].

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گراف جابجایی یک رده تزویج در گروه‌های متقارن

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گراف جابجایی یک گروه، گرافی است که مجموعه‌ی رأسی آن زیر مجموعه‌ای از گروه بوده و دو رأس در گراف مجاورند اگر و تنها اگر دو عنصر در گروه با هم جابجا شوند. هدف اصلی ما در این مقاله محاسبه گروه خودریختی‌های گراف جابجایی برای یک رده تزویج در گروهای متقارن است. همچنین در این مقاله عدد خوشه‌ای، عدد رنگی، عدد استقلال و قطر این گراف‌ها محاسبه شده است.

کلمات کلیدی: گروه متقارن، گروه خودریختی‌ها، گراف جابجایی.