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TABLE OF MARKS OF FINITE GROUPS

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ABSTRACT. Let G be a finite group and $\mathcal{C}(G)$ be a family of representative conjugacy classes of subgroups of G. The matrix whose H,K-entry is the number of fixed points of the set G/K under the action of H is called the table of marks of G, where H, K run through all elements in $\mathcal{C}(G)$. In this paper, we compute the table of marks and the markaracter table of groups of order pqr, where p, q, r are prime numbers.

1. Introduction

All groups considered in this paper are finite. The concept of table of marks was introduced by William Burnside [3], as a tool to classify G-sets up to equivalence. Similar to the character table of G which classifies the matrix representations of G up to isomorphism, the table of marks of G classifies permutation representations of G up to equivalence. This table encodes a wealth of information about the subgroup structure of G in a compact way. In other words, the table of marks of a group is a useful invariant that provides a considerable amount of data about the group.

Let G be a finite group acting transitively on a finite set X. Then, it is a well-known fact that X is $G-$isomorphic to a set of right cosets $G/H = \{H(e = g_1), \ldots, Hg_m\}$, for some subgroup H of G. Moreover, two transitive $G-$sets G/H and G/K are $G-$isomorphic if and only if H and K are conjugate (see [4] for more details). For every element $g \in G$, the fixed point of g in X is defined as $Fix_X(g) = \{x \in X; x^g = x\}$.

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Similarly, for a subgroup H of G the fixed points of H is $Fix_X(H) = \{x \in X; \forall h \in H, x^h = x\}$. In this context, the mark of a subgroup H of G on X is the number of fixed points of H under the action of G on X, denoted by $\beta_X(H)$. If $H_1, \ldots, H_r$ is a list of representatives of the subgroups of G up to conjugacy, the table of marks of G is then the $r \times r$-matrix

$$M(G) = (\beta_{G/H_j}(H_i))_{i,j=1,2,\ldots,r}.$$

In other words, assume that the set of orbits of this action is $\{G_i^G\}_{i=1}^r$, where $G_1(= e), G_2, \ldots, G_r(= G)$ are representatives of the conjugacy classes of subgroups of G and $|G_1| \leq |G_2| \leq \cdots \leq |G_r|$. The *table of marks* of G is the square matrix $M(G) = (m_{ij})_{i,j=1}^r$, where $m_{ij} = \beta_{G/G_j}(G_i)$. This table has substantial applications in chemistry, specially in isomer counting [1]. For the main properties of this matrix, we refer the reader to the interesting paper of Pfeifier [14].

Let G and H be finite groups and α be a function from $\mathcal{C}(G)$ to $\mathcal{C}(H)$. We say that α is an isomorphism between the tables of marks of G and H if α is a bijection and also $\beta_{H/H_i}(H_j) = \beta_{G/G_i}(G_j)$ for all subgroups H_i of H and G_i of G. An isomorphism between tables of marks of two groups preserves many algebraic properties of related groups, such as the order of subgroups, the order of their normalizers, the number of elements of a given order, the number of subgroups of a given order, the number of normal subgroups of a given order, etc. It sends cyclic groups to cyclic groups and elementary abelian groups to elementary abelian groups. It also sends the derived subgroup of G to the derived subgroup of H, maximal subgroups of G to maximal subgroups of H, Sylow p-subgroups to Sylow p-subgroups and the Frattini subgroup of G to the Frattini subgroup of H.

Suppose G is a finite group, H is a subgroup of G and $\{e = g_1, \ldots, g_m\}$ is a transversal of G with respect to the subgroup H. Define the permutation $\rho_g : G/H \longrightarrow G/H$ $(g \in G)$ given by $\rho_g(Hg_i) = Hg_ig$. Set $R(G/H) = \{\rho_g \mid g \in G\}$. Then, the permutation representation $R(G/H)$ of degree $m = |G|/|H|$ is called a coset representation of G by H. Clearly, this representation is transitive.

Pfeiffer [14] described a procedure for the construction of the table of marks of a finite group from the table of marks of its maximal subgroups. This semi- automatic procedure has proven for simple groups up to a certain order, and has been used extensively in building the GAP library of tables of marks, see [11]. Here, we compute the table of marks of groups of order pqr and then we determine all isomorphisms between them.

2. Main Results

At the beginning of this section, we study some elementary properties of the table of marks.

Theorem 2.1. [2] *Suppose G is a finite group, $M(G) = (m_{ij})$ and $\mathcal{C}(G) = \{G_1, G_2, \ldots, G_r\}$ are all non-conjugate subgroups of G, where $|G_1| \leq |G_2| \leq \cdots \leq |G_r|$. Then;*

a) *The matrix $M(G)$ is a lower triangular matrix,*
b) *m_{ij} divides m_{i1}, for all $1 \leq i, j \leq r$,*
c) *$m_{i1} = [G : G_i]$, for all $1 \leq i \leq r$,*
d) *$m_{ii} = [N_G(G_i) : G_i]$,*
e) *if G_i is a normal subgroup of G, then $m_{ij} = [G : G_i]$ whenever $G_j \subseteq G_i$ and zero otherwise.*

As an immediate consequence of Theorem 2.1, the table of marks of the cyclic group $\mathbb{Z}_p$ is as reported in Table 1.

Table 1. The Table of Marks of Cyclic Group $\mathbb{Z}_p$.

$M(G)$	G_1	G_2
G/G_1	p	0
G/G_2	1	1

Let p be a prime number and q be a positive integer such that $q|p-1$. Define the group $F_{p,q}$ to be presented by $F_{p,q} = \langle a, b : a^p = b^q = 1, b^{-1}ab = a^u \rangle$, where u is an element of order q in multiplicative group $\mathbb{Z}_p^*$ [13, Page 290]. It is easy to see that $F_{p,q}$ is a Frobenius group of order pq.

Theorem 2.2. *Let p and q be two prime numbers such that $p > q$. The table of marks of group $F_{p,q}$ is as reported in Table 2.*

Proof. It is not difficult to see that the group $F_{p,q}$ has four non-conjugate subgroups $G_1 = \langle e \rangle$, $G_2 = \langle a \rangle$, $G_3 = \langle b \rangle$, and $G_4 = G$. By using Theorem 2.1 (c), we have $m_{11} = pq$, $m_{21} = p$, $m_{31} = q$ and $m_{41} = 1$. By Theorem 2.1 (a), we have $m_{12} = m_{13} = m_{23} = 0$ and by Theorem 2.1 (b), one can deduce that $m_{42} = m_{43} = m_{44} = 1$. On the other hand, by using Sylow Theorem, it is clear that the Sylow p-subgroup of $F_{p,q}$ is normal and by using Theorem 2.1 (e), we get $m_{32} = 0$ and $m_{33} = pq/p = q$. □

Table 2. The Table of Marks of Group $F_{p,q}$.

$M(G)$	G_1	G_2	G_3	G_4
G/G_1	pq	0	0	0
G/G_2	p	1	0	0
G/G_3	q	0	q	0
G/G_4	1	1	1	1

2.1. **Computing the Table of Marks.** Suppose $\mathcal{G}(p,q,r)$ is the set of all groups of order pqr, where p, q and r are prime numbers. Hölder in [12] classified all groups of order pqr. It can be proved that up to isomorphism, all groups of order pqr are:

Case 1. $p=q=r$, there are five groups of order p^3 as follows:
- $P_1 \cong \mathbb{Z}_{p^3}$,
- $P_2 \cong \mathbb{Z}_p \times \mathbb{Z}_{p^2}$,
- $P_3 \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$,
- $P_4 \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{p^2}$,
- $P_5 \cong \mathbb{Z}_p \rtimes (\mathbb{Z}_p \times \mathbb{Z}_p)$.

Case 2. $p>q>r$, then all groups of order pqr are
- $G_1 \cong \mathbb{Z}_{pqr}$,
- $G_2 \cong \mathbb{Z}_r \times F_{p,q}(q|p-1)$,
- $G_3 \cong \mathbb{Z}_q \times F_{p,r}(r|p-1)$,
- $G_4 \cong \mathbb{Z}_p \times F_{q,r}(r|q-1)$,
- $G_5 \cong F_{p,qr}(qr|p-1)$,
- $G_{i+5} \cong \langle a,b,c : a^p = b^q = c^r = 1, ab = ba, c^{-1}bc = b^u, c^{-1}ac = a^{v^i}\rangle$, where $r|p-1, q-1, o(u) = r$ in $\mathbb{Z}_q^*$ and $o(v) = r$ in $\mathbb{Z}_p^*$ $(1 \leq i \leq r-1)$.

Case 3. $p<q$ and $r=p$, then all groups of order p^2q are
- $L_1 \cong \mathbb{Z}_{p^2q}$,
- $L_2 \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$,
- $L_3 \cong \mathbb{Z}_p \times F_{q,p}\ (p|q-1)$,
- $L_4 \cong F_{q,p^2}\ (p^2|q-1)$,
- $L_5 \cong \langle a,b : a^{p^2} = b^q = 1, a^{-1}ba = b^\alpha,\ \alpha^p \equiv 1\ (mod\ q)\rangle$.

Case 4. $q<p$ and $r=p$, then all groups of order p^2q are
- $Q_1 \cong \mathbb{Z}_{p^2q}$,
- $Q_2 \cong \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_p$,
- $Q_3 \cong \mathbb{Z}_p \times F_{p,q}\ (q|p-1)$,
- $Q_4 \cong F_{p^2,q}\ (q|p^2-1)$,
- $Q_5 \cong \langle a,b,c : a^p = b^q = c^p = 1, ac = ca, b^{-1}ab = a^\alpha, b^{-1}cb = c^{\alpha^x},\ \alpha^q \equiv 1\ (mod\ p), x = 1, ..., q-1\rangle$,
- $Q_6 \cong \langle a,b,c : a^p = b^q = c^p = 1, ac = ca, b^{-1}ab = a^\alpha c^{\beta D}, b^{-1}cb = a^\beta c^\alpha\rangle$, where $\alpha + \beta\sqrt{D} = \sigma^{p^2-1/q}$, σ is a

primitive element of $GF(p^2)$, $q \nmid p-1$, and $q \neq 2$ whereas $q|p+1$.

Suppose p is a prime number and $G = P_1$. Then, the group $\mathbb{Z}_{p^3} = \langle a \rangle$ has four non-conjugate subgroups such as $G_1 = \langle e \rangle$, $G_2 = \langle a^{p^2} \rangle$, $G_3 = \langle a^p \rangle$, and $G_4 = \langle a \rangle$. The table of marks of $\mathbb{Z}_{p^3}$ is as given in Table 3.

The subgroups of order p in group $G = \mathbb{Z}_p \times \mathbb{Z}_{p^2} = \langle a, b \rangle$ are $\{e\} \times \mathbb{Z}_p$ and $\langle a, b^{jp} \rangle$, where $(0 \leq j \leq p-1)$. We show them by $H_1, \dots, H_{p+1}$. On the other hand, there are $p+1$ non-conjugate subgroups of order p^2, namely $G_1 = \{e\} \times \mathbb{Z}_{p^2}$, $G_2 = \mathbb{Z}_p \times \mathbb{Z}_p$ and $G_{i,j} = \langle a^i, b^j \rangle$ $(1 \leq i, j \leq p-1)$. We show them by $G_1, \dots, G_{p+1}$. Since G is an abelian group, all subgroups are normal and then by Theorem 2.1 (e), all diagonal entries can be computed easily. Also, we note that $H_1 \subseteq G_i$ $(1 \leq i \leq p+1)$, and so $m_{i2} = p$ $(p+2 \leq i \leq 2p+3)$. We can easily see that $H_i \subseteq G_2$ $(1 \leq i \leq p+1)$, and so $m_{p+4,j} = p$ $(3 \leq j \leq p+2)$. The other entries of the table are zero. The table of marks of this group is given in Table 4.

In continuing, let H be a non-abelian group of order p^3 and exponent p^2. Then H has the following presentation:

$$\langle x, y : x^{p^2} = y^p = 1, y^{-1}xy = x^{p+1} \rangle.$$

Clearly, $|Z(H)| = p$ and H has two non-conjugate subgroups of order p, namely $H_2 = Z(H)$ and $H_3 = \langle y \rangle$. It is clear that $\langle y \rangle$ is not normal in H. Hence, $m_{21} = m_{22} = m_{31} = p^2$. Since $|N_H(H_2)| = p^2$, one can see that $m_{33} = p$. On the other hand, there are $p+1$ subgroups of order p^2 containing $Z(H)$, denoted by $H_4, \dots, H_{p+4}$. All of them are normal in H and therefore the table of marks of H is as reported in Table 6.

Finally, suppose G is a non-abelian group of order p^3 $(p \geq 3)$ with exponent p with the following presentation:

$$\langle x, y, z : x^p = y^p = z^p = 1, xy = yx, zy = yz, xz = zxy \rangle.$$

It is not difficult to see that all subgroups of order p of G are $\langle x^i y^j \rangle$, $\langle z^i y^j \rangle$, $\langle x^i z^j \rangle$, $\langle z^i x^j \rangle$ and $\langle x^i y^j z^k \rangle$ $(1 \leq i, j, k \leq p-1)$. But all non-conjugate subgroups of this form are $\langle y \rangle$, $\langle x \rangle$, $\langle z \rangle$, $\langle x^i z^j \rangle$ and the number of such subgroups is $p-1+3 = p+2$. Let us show them by $H_1, \dots, H_{p+2}$. For $2 \leq i \leq p+2$, $|N_G(H_i)| = p^2$, and $N_G(H_1) = G$. By using Theorem 2.1, $m_{ii} = p$ $(3 \leq i \leq p+3)$, and $m_{22} = p^2$. On the other hand, all non-conjugate subgroups of order p^2 are $\langle x^i, z^j \rangle (1 \leq i, j \leq p-1)$ and $\langle x, y \rangle$. Hence, there are $p-1+2 = p+1$ non-conjugate subgroups of this form. We denote them by $G_1, \dots, G_{p+1}$. For $1 \leq i \leq p+1$, we have $N_G(G_i) = G$

and by using Theorem 2.1, $m_{ii} = p$ $(p+4 \le i \le 2p+4)$. Since for $1 \le i \le p+1$, G_i is a normal subgroup of G and $H_1 \subseteq G_i$, by using Theorem 2.1, $m_{i2} = p$ $(p+4 \le i \le 2p+4)$ and the other entries are zero. The table of marks of this group is reported in Table 7. Thus, we proved the following theorem.

Theorem 2.3. *The tables of marks of a group of order p^3 up to isomorphism are given in Tables 3-7.*

Theorem 2.4. *Let p, q and r be prime numbers such that $p > q > r$ and $G \in \mathcal{G}(p,q,r)$. Then, the table of marks of G is isomorphic with one of Tables 8-11.*

Proof. If $G \cong G_1$, then the table of marks of G can be computed by Theorem 2.1 (see Table 8). If H is isomorphic to $G_2 = \langle c\rangle \times \langle a,b\rangle$, then all non-conjugate subgroups of H are $H_1 = \langle e\rangle$, $H_2 = \langle c\rangle$, $H_3 = \langle b\rangle$, $H_4 = \langle b,c\rangle$, $H_5 = \langle a\rangle$, $H_6 = \langle a,c\rangle$, $H_7 = \langle a,b\rangle$ and $H_8 = H$. Applying Theorem 2.1 yields the first column of the table. Also, we have $m_{22} = pq$, $m_{33} = r$, $m_{44} = 1$, $m_{55} = qr$, $m_{66} = q$, and $m_{77} = r$. Since $ac = ca$ and $bc = cb$, we conclude that $N_H(H_2) = H$. On the other hand, let $g = c^k b^j a^i \in G$ be an arbitrary element such that $g^{-1}H_3 g = H_3$. Then we can easily see that $i = 0$ and so $N_H(H_3) = \langle b,c\rangle$. By a similar argument, we get $N_H(H_4) = H_4$ and $N_H(H_5) = N_H(H_6) = N_H(H_7) = H$. It is clear that $m_{32} = \beta_{H/H_3}(H_2) = 0$, $m_{42} = \beta_{H/H_4}(H_2) = p$ and $m_{43} = \beta_{H/H_4}(H_3) = 1$. Since, the subgroups H_5, H_6, H_7 are normal, by using Theorem 2.1 (e), we can show that $m_{52} = m_{53} = m_{54} = 0$, $m_{62} = m_{65} = q$, $m_{63} = m_{64} = 0$, $m_{72} = m_{74} = m_{76} = 0$ and $m_{73} = m_{75} = r$. The table of marks of this group is reported in Table 9.

The table of marks of two groups $G_3 = \mathbb{Z}_q \times F_{p,r}(r|p-1)$ and $G_4 = \mathbb{Z}_p \times F_{q,r}(r|q-1)$ are isomorphic with Table 7. If K is isomorphic to G_5, then the table of marks of K can be resulted from Theorem 2.2 (see Table 10). It remains to compute the table of marks of group G_{i+5}. Let $P \cong G_{i+5}$ $(1 \le i \le r-1)$. Then it is easy to see that $\langle a^k\rangle = \langle a^l\rangle$, $\langle b^t\rangle = \langle b^s\rangle$, $\langle c^m\rangle = \langle c^n\rangle$, $\langle b^t a^k\rangle = \langle b^s a^l\rangle$, $\langle c^m a^k\rangle = \langle c^n a^l\rangle$ and $\langle b^t c^m\rangle = \langle b^s c^n\rangle$, where $1 \le k,l \le p-1$, $1 \le t,s \le q-1$ and $1 \le m,n \le r-1$. Therefore, all non-conjugate subgroups of P are $P_1 = \langle e\rangle$, $P_2 = \langle c\rangle$, $P_3 = \langle b\rangle$, $P_4 = \langle a\rangle$, $P_5 = \langle bc\rangle$, $P_6 = \langle ac\rangle$, $P_7 = \langle ab\rangle$, and $P_8 = P$. One can easily check that $N_P(P_2) = P_2$, $N_P(P_3) = N_P(P_4) = N_P(P_7) = P$, $N_P(P_5) = P_5$ and $N_P(P_6) = P_6$. So, by applying Theorem 2.1, the entries of the diagonal and the first column of the table of marks can be computed. Since p,q,r are distinct prime numbers, we have $m_{32} = m_{42} = m_{43} = m_{54} = m_{63} = m_{65} = m_{72} = m_{75} = m_{76} = 0$. Finally, $ab = ba$, $c^{-1}bc = b^u$, and $c^{-1}ac = a^{v^i}$ yield the subgroup P_7 is normal, and the proof is complete. □

Theorem 2.5. *Let p and q be two prime numbers such that $q > p$, $p|q-1$ and $G \in \mathcal{G}(p^2, q)$. Then, the table of marks of G is isomorphic with one of Tables 12 - 16.*

Proof. We can prove that the group $G = \mathbb{Z}_{p^2q}$ has five non-conjugate subgroups such as $G_1 = \langle e \rangle$, $G_2 = \langle b^p \rangle$, $G_3 = \langle b \rangle$, $G_4 = \langle a \rangle$, $G_5 = \langle a, b^p \rangle$, and $G_6 = G$. Hence, the table of marks of this group follows from Theorem 2.1 (see Table 12). All non-conjugate subgroups of order p of $G = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q$ are $H_{i,j} = \langle c^i, a^j \rangle$ $(1 \leq i, j \leq p-1)$. It is not difficult to see that there are exactly $(p-1)(p-1)/(p-1) = p-1$ non-conjugate subgroups of this form. In addition, two subgroups $\{e\} \times \mathbb{Z}_p$, and $\mathbb{Z}_p \times \{e\}$ are of order p. In general, there exist $p+1$ non-conjugate subgroups of order p. Let us show them by $H_1, \ldots, H_{p+1}$. For $1 \leq j \leq p+1$, we have $N_G(H_j) = G$ and by using Theorem 2.1, we get $m_{ii} = pq$ $(2 \leq i \leq p+2)$. On the other hand, the subgroups $G_{i,j} = \langle c^i, b^j \rangle$ $(1 \leq i \leq p-1)$, $(1 \leq j \leq q-1)$ are of order pq. In this case, one can find $(p-1)(q-1)/kp = p-1$ non-conjugate subgroups of this form. Moreover, $\{e\} \times \mathbb{Z}_{pq}$ and $\mathbb{Z}_p \times \mathbb{Z}_q$ are subgroups of order pq and hence G has exactly $p+1$ subgroups of order pq. We show them by $G_1, \ldots, G_{p+1}$. For $1 \leq j \leq p+1$, we have $N_G(G_j) = G$ and by using Theorem 2.1, we have $m_{ii} = p$ $(p+5 \leq i \leq 2p+5)$. The other entries of this table can be derived from Theorem 2.1 (e). It is not difficult to see that the Sylow q-subgroup Q and the Sylow p-subgroup P are normal subgroups of $\mathbb{Z}_p \times \mathbb{Z}_{pq}$ and by using Theorem 2.1, the $p+3$-th and $p+4$-th column and row of the table can be derived. For $p+5 \leq i \leq 2p+5$ and $2 \leq j \leq p+2$, since $H_1 \subseteq G_j$, we have $m_{i2} = p$ and the other entries are zero. The table of marks of this group is as reported in Table 13.

The subgroups $H_{i,j} = \langle c^i, b^j \rangle$ $(1 \leq i, j \leq p-1)$ of $G = \mathbb{Z}_p \times F_{q,p}$ are of order p, and so there are exactly $(p-1)(p-1)/(p-1) = p-1$ non-conjugate subgroups of this form. In addition, two subgroups $\{e\} \times \mathbb{Z}_p$ and $\mathbb{Z}_p \times \{e\}$ are of order p, and hence there exist $p+1$ non-conjugate subgroups of order p. Let us denote them by $H_1, \ldots, H_{p+1}$. We claim that for $i \in \{1, \ldots, p+1\}$, we have $N_G(H_i) = \langle c, b \rangle$. Set $H_i = \langle c^r, b^s \rangle$ and suppose the element $g = c^k b^j a^i$ is an arbitrary element such that $g^{-1} H_i g = H_i$. Hence, $g^{-1} H_i g = a^{-i} H_i a^i = \langle c^r, b^{2s} a^{-iu^s + i} \rangle$ and so $a^{-iu^s+i} = 1$. This leads us to conclude that $-iu^s + i \equiv 0 \ (mod\ q)$. Consequently, the following cases hold:

Case 1. $q \mid i$, then $i = 0$ and so $g = c^k b^j$. By using Theorem 2.1, we get $m_{ii} = p$ $(3 \leq i \leq p+2)$.

Case 2. $q \mid u^s - 1$, hence $s = p$ and so $H_1 = \langle c^r \rangle$. This implies that $m_{22} = pq$.

On the other hand, $G_{i,j} = \langle c^i, a^j \rangle$ $(1 \le i \le p-1)$ and $(1 \le j \le q-1)$ are of order pq. It is not difficult to prove that there are $(p-1)(q-1)/kp = p-1$ non-conjugate subgroups of this form. Moreover, $\{e\} \times F_{q,p}$ and $\mathbb{Z}_p \times \mathbb{Z}_q$ are subgroups of order pq and hence G has exactly $p+1$ subgroups of order pq, denoted by $G_1, \ldots, G_{p+1}$. For $i \in \{1, \ldots, p+1\}$. We have $[G : G_i] = p$ and so G_i is normal. According to Theorem 2.1, we have $m_{ii} = p$ $(p+5 \le i \le 2p+5)$ and the other entries of this table can be computed from Theorem 2.1 (e). But Sylow q-subgroup Q is a normal subgroup of G, where $Q \subseteq G_i$ and so by using Theorem 2.1, we get $m_{p+4,p+4} = p^2$, $m_{i,p+4} = p$ $(p+5 \le i \le 2p+5)$ and the other entries are zero. The Sylow p-subgroup of G is $P = \langle b, c \rangle$, so if $g = c^k b^j a^i \in G$ is an arbitrary element, then $g^{-1}Pg = a^{-i}Pa^i = \langle b^2 a^{-iu+i}, c \rangle$, and hence $a^{-iu+i} = 1$. This leads us to conclude that $-iu^s + i \equiv 0 \ (mod\ q)$. Since $q \nmid u-1$, we have $q \mid i$ and thus $i = 0$. This implies that $g = c^k b^j$. By using Theorem 2.1 and above discussion, the $p+3$-th column and row of the table can be computed easily. For $p+5 \le i \le 2p+5$ and $2 \le j \le p+2$, since $H_1 \subseteq G_j$, we can verify that $m_{i2} = p$ and the other entries of this row are zero. The table of marks of this group is given in Table 14.

One can verify that the non-conjugate subgroups of $H = F_{q,p^2}$ are $H_1 = \langle e \rangle$, $H_i = \langle\ b^k \ | k = p \ or \ 1 \rangle$, $(2 \le i \le 3)$, $H_4 = \langle a \rangle$, $H_5 = \langle a, b^p \rangle$ and $H_6 = H$. Consider the table of marks $M = M(H) = (m_{ij})$, the first column of this table can be computed from Theorem 2.1 (c). The normalizer of H_2 is equal to $\langle b \rangle$. For an arbitrary element $g = b^s a^r \in H$, we have $g^{-1}H_2 g = a^{-r}H_2 a^r = \langle b^{2p} a^{-ru^p+r} \rangle$ and so $a^{-ru^p+r} = 1$, which yields that $-ru^p + r \equiv 0 \ (mod\ q)$. Thus q divides r and then $r = 0$ or $g = b^s$. By using Theorem 2.1, we have $m_{22} = p$ and the normalizer of H_3 is equal to $\langle b \rangle$. Hence, we have $m_{33} = 1$. According to Sylow Theorem, H_4 is normal subgroup of F_{q,p^2} and by using Theorem 2.1, we get $m_{44} = p^2$ and $m_{4j} = 0$ $(2 \le j \le 3)$. Since $[H : H_5] = p$ while p is the smallest prime number which divides the order of group, clearly H_5 is a normal subgroup of H, and so by using Theorem 2.1 (e), we get $m_{52} = m_{54} = m_{55} = p$. The other entries of this row are zero and the table of marks of F_{q,p^2} is as reported in Table 15.

In continuing, consider group G with the following presentation:

$$G = \langle a, b : a^{p^2} = b^q = 1, a^{-1}ba = b^{\alpha},\ \alpha^p \equiv 1 \ (mod\ q) \rangle.$$

It is not difficult to see that all non-conjugate subgroups of G are $K_1 = \langle e \rangle$, $K_2 = \langle a^p \rangle$, $K_3 = \langle a \rangle$, $K_4 = \langle b \rangle$, $K_5 = \langle a^p, b \rangle$ and $K_6 = G$. The first column of this table can be derived from Theorem 2.1 (c). We have $N_G(K_2) = G$ and so $m_{22} = pq$. On the other hand, $N_G(K_3) = K_3$ yields that $m_{33} = 1$. Since $K_2 \subseteq K_3$, we conclude that $m_{32} = q$. By

Sylow Theorem, K_4 is a normal subgroup of G and by using Theorem 2.1, we have $m_{44} = p^2$ and $m_{4j} = 0$ $(2 \leq j \leq 3)$. Since $[G : K_5] = p$ and p is the smallest prime number that divides the order of group, hence K_5 is normal subgroup of G. Therefore, by Theorem 2.1 (e), we conclude that $m_{52} = m_{54} = m_{55} = p$ and the other entries of this row are zero (see Table 16). □

Theorem 2.6. *Let p and q be two prime numbers such that $p > q$, $q|p-1$ and $G \in \mathcal{G}(p^2, q)$. Then, the table of marks of G is isomorphic with one of the Tables 17-21.*

Proof. The table of marks of groups Q_1 and Q_2 can be derived from Theorem 2.5 (see Tables 12, 13). Suppose that $G \cong Q_3$. Then one can easily check that $H_{i,j} = \langle c^i, a^j \rangle$ $(1 \leq i, j \leq p-1)$ are subgroups of order p. It is not difficult to see that there are exactly $(p-1)(p-1)/(p-1)q = k$ non-conjugate subgroups of this form. In addition, two subgroups $\{e\} \times \mathbb{Z}_p$ and $\mathbb{Z}_p \times \{e\}$ are of order p. In general, there are $k+2$ non-conjugate subgroups of order p. Let us show them by $H_1 = \mathbb{Z}_p \times \{e\}, H_2 = \{e\} \times \mathbb{Z}_p$ and $H_3, \ldots, H_{k+2}$. We claim that for $i \in \{3, \ldots, k+2\}$, we have $N_G(H_i) = \langle c, a \rangle$. To do this, suppose $H_i = \langle c^r, a^s \rangle$ and $g = c^k b^j a^i$ is an element of G such that $g^{-1} H_i g = H_i$. Hence, $g^{-1} H_i g = b^{-j} H_i b^j = \langle c^r, a^{su^j} \rangle$, thus by using Theorem 2.1, we have $m_{ii} = p$ $(5 \leq i \leq k+4)$. Since H_1 and H_2 are normal subgroups of G, hence $m_{ii} = pq$ $(i = 3, 4)$. On the other hand, all subgroups of order pq are $G_1 = \{e\} \times F_{p,q}$ and $G_2 = \langle c, b \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_q$. Now, $N_G(G_1) = G$, since for $g = c^k b^j a^i \in G$, we have $g^{-1} G_1 g = \langle b^{-j} a b^j, a^{-i} b a^i \rangle = \langle a^{-ju}, b^2 a^{-iu+i} \rangle$. Similarly, we can prove that $N_G(G_2) = G_2$. Hence, according to Theorem 2.1, we get $m_{k+5,k+5} = p$, $m_{k+6,k+6} = 1$ and the other entries of this table can be derived directly. The Sylow p-subgroup P is a normal subgroup of $\mathbb{Z}_p \times F_{p,q}$ and the latest column and row of the table can be computed from Theorem 2.1. The Sylow q-subgroup of Q_3 is $Q = \langle b \rangle$ and we have $N_G(Q) = \langle c, b \rangle$, which yields the second column and row of the table. For $i = k+5$ and $j = 4$, since $H_2 \subseteq G_1$, we conclude $m_{k+5,4} = p$. For $i = k+6$ and $j = 3$, since $H_1 \subseteq G_2$, we have $m_{k+6,3} = p$. It then follows that all non-conjugate subgroups of $H = F_{p^2,q}$ are $H_1 = \langle e \rangle$, $H_2 = \langle b \rangle$, $H_3 = \langle a^p \rangle$, $H_4 = \langle a^p, b \rangle$, $H_5 = \langle a \rangle$ and $H_6 = H$. The first column of $M(F_{p^2 q})$ can be derived from Theorem 2.1 (c). On the other hand, for $g = b^j a^i \in H$, we have $g^{-1} H_2 g = \langle a^{-i} b a^i \rangle = \langle b^2 a^{-iu+i} \rangle$ and so $N_H(H_2) = H_2$. This yields that $m_{22} = 1$. Also, for subgroup H_3, we have $g^{-1} H_3 g = \langle b^{-j} a^p b^j \rangle = \langle a^{ju^p} \rangle$, thus $N_H(H_3) = H$ and so $m_{33} = pq$. On the other hand, $g^{-1} H_4 g = \langle b^{-j} a^p b^j, a^{-i} b a^i \rangle = \langle a^{ju^p}, b^2 a^{-iu+i} \rangle$ and we conclude $N_H(H_4) = H_4$ or $m_{44} = 1$. Since p and q are prime numbers, by Theorem 2.1, we have

$m_{42} = 1$ and $m_{43} = p$. By using Sylow Theorem, we can show that H_5 is normal subgroup of $F_{p^2,q}$, and so the fifth row of this table can be resulted from Theorem 2.1 (e). The subgroups of order p in Q_5 are $H_{i,j} = \langle c^i, a^j\rangle$ $(1 \leq i, j \leq p-1)$. It is not difficult to see that there are exactly $(p-1)(p-1)/(p-1)q = k$ non-conjugate subgroups of this form. In addition, two subgroups $\{e\} \times \mathbb{Z}_p$ and $\mathbb{Z}_p \times \{e\}$ are of order p. In general, there are $k+2$ non-conjugate subgroups of order p denoted by $H_1 = \{e\} \times \mathbb{Z}_p, H_2 = \mathbb{Z}_p \times \{e\}, H_3, \ldots, H_{k+2}$. We claim that for $i \in \{3, \ldots, k+2\}$, we have $N_{Q_5}(H_i) = \langle c, a\rangle$. To do this, suppose that $H_i = \langle c^r, a^s\rangle$ and $g = c^k b^j a^i \in Q_5$ is an arbitrary element such that $g^{-1}H_i g = H_i$. Hence, $g^{-1}H_i g = b^{-j}H_i b^j = \langle c^r, a^{su^j}\rangle$, thus by using Theorem 2.1, we get $m_{ii} = p$ $(5 \leq i \leq k+4)$. Since H_1 and H_2 are normal subgroups of Q_5, hence $m_{ii} = pq$ $(i = 3, 4)$. On the other hand, all subgroups of order pq are $G_1 = \langle c, b\rangle$ and $G_2 = \langle a, b\rangle$. For $g = c^k b^j a^r \in Q_5$, we have $g^{-1}G_i g = G_i$, thus $N_{Q_5}(G_i) = G_i$ $(i = 1, 2)$ and so according to Theorem 2.1, we have $m_{k+5,k+5} = m_{k+6,k+6} = 1$. The other entries of this table can be derived from Theorem 2.1. But the Sylow p-subgroup P is a normal subgroup of Q_5 and thus by using Theorem 2.1, the latest column and row of the table can be computed. The Sylow q-subgroup Q_5 is $Q = \langle b\rangle$ and for $g = c^k b^j a^i \in Q_5$ we have $g^{-1}Qg = Q$, so $m_{22} = 1$. Since $H_2 \subseteq G_1$, it follows that $m_{k+5,4} = p$ and since $H_1 \subseteq G_2$, we have $m_{k+6,3} = p$. Also, the other entries are zero, and so $M(Q_5)$ is as given in Table 19.

The subgroups $H_{i,j} = \langle c^i, a^j\rangle$ $(1 \leq i, j \leq p-1)$ of group $G = Q_6$ are of order p. The number of non-conjugate subgroups of this form is exactly $(p-1)(p-1)/(p-1)q = k$. We denote these subgroups by $H_1, \ldots, H_k$. Let $H_i = \langle c^r, a^s\rangle$ and suppose $g = c^k b^j a^i \in Q_6$ is an arbitrary element such that $g^{-1}H_i g = H_i$. Hence, $g^{-1}H_i g = b^{-j}H_i b^j = \langle c^r, a^{su^j}\rangle$ and so $N_{Q_6}(H_i) = \langle c, a\rangle$. By using Theorem 2.1, we can verify that $m_{ii} = p$ $(3 \leq i \leq k+2)$. On the other hand, G has no subgroup of order pq and the Sylow p-subgroup P of Q_6 is normal. Now, Theorem 2.1 yields the latest row and column of Table 21. The Sylow q-subgroup of Q_6 is $Q = \langle b\rangle$ and we can prove that $N_{Q_6}(Q) = Q$. Hence, the second column and row of Table 21 can be derived. □

2.2. **Computing the Markaracter Table.** The matrix $MC(G)$ obtained from the table of marks $M(G)$ of group G in which we select rows and columns corresponding to cyclic subgroups of G is called the *markaracter table* of G. It is merit to mention here that the markaracter table of a finite group was firstly introduced by Shinsaku Fujita to discuss marks and characters of a finite group in a common basis, see [4, 5]. Fujita originally developed his theory to be the foundation for enumeration of molecules [4]. We encourage the interested readers to consult papers [5, 6, 7, 8, 9] as well as [2, 11], for more information on this topic.

Suppose A and B are $m \times n$ and $p \times q$ matrices, respectively. The tensor product $A \otimes B$ of matrices A and B is the $mp \times nq$ block matrix:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

Theorem 2.7. [15] *Let p be a prime number, q be a positive integer such that $q|p-1$ and $q = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_s^{\alpha_s}$ be its decomposition into distinct primes $q_1 < q_2 < \cdots < q_s$. Suppose $\tau(n)$ denotes the number of divisors of n and $d_1, \ldots, d_{\tau(q)}$ are positive divisors of q. Then, the markaracter table of the Frobenius group $F_{p,q}$ can be computed as reported in Table 22.*

Lemma 2.8. *Suppose G_1 and G_2 are two finite groups with co-prime orders. Then, the markaracter table of $G_1 \times G_2$ is tensor product of $MC(G_1)$ and $MC(G_2)$.*

Proof. Let A, A_1 and A_2 be the set of all non-conjugate cyclic subgroups of $G_1 \times G_2$, G_1 and G_2, respectively. Suppose that $U = \langle u \rangle \in A_1$ and $V = \langle v \rangle \in A_2$. Then $U \times V$ is a cyclic group generated by (u, v). So, $U \times V$ is conjugate with a cyclic subgroup in A. On the other hand, if $H = \langle h \rangle \in A$, then $h = (u, v)$ such that $u \in G_1$, $v \in G_2$ and $gcd(o(u), o(v)) = 1$. Then, there are $U \in A_1$ and $V \in A_2$ conjugate with $\langle u \rangle$ and $\langle v \rangle$, respectively, such that $H = U \times V$. Therefore, $MC(G_1 \times G_2) = MC(G_1) \otimes MC(G_2)$. □

Theorem 2.9. *Suppose G is a group of order p^3. Then, the markaracter table of G is given in Tables 23-25.*

Proof. If $G = \mathbb{Z}_{p^3}$, then clearly $MC(G) = M(G)$. When $G = \mathbb{Z}_p \times \mathbb{Z}_{p^2}$, by using Theorem 2.3, all non-conjugate subgroups are cyclic. So, $MC(G) = M(G)$. In this case, we have $G = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, all non-conjugate subgroups of order p are cyclic and since these subgroups are normal, the markaracter table of G can be computed from Theorem 2.3. The markaracter tables of two non-abelian groups of order p^3 can be derived from Tables 6,7, respectively. □

Let G be a cyclic group of order $n = p_1^{\alpha_1} \ldots a_r^{\alpha_r}$. Then, Lemma 2.8 shows that

$$MC(\mathbb{Z}_n) = MC(\mathbb{Z}_{p_1^{\alpha_1}}) \otimes \ldots \otimes MC(\mathbb{Z}_{p_r^{\alpha_r}}).$$

Theorem 2.10. *The markaracter table of a group of order pqr ($p > q > r$) is equal with one of the following matrices:*

i) $MC(G_1) = MC(\mathbb{Z}_p) \otimes MC(\mathbb{Z}_q) \otimes MC(\mathbb{Z}_r)$,
ii) $MC(G_2) = MC(F_{p,q}) \otimes MC(\mathbb{Z}_r)(q|p-1)$,
iii) $MC(G_3) = MC(F_{p,r}) \otimes MC(\mathbb{Z}_q)(r|p-1)$,
iv) $MC(G_4) = MC(F_{q,r}) \otimes MC(\mathbb{Z}_p)(r|q-1)$,
v) *If* $qr|p-1$ *then* $MC(G_5) = MC(F_{p,qr})$,
vi) *If* $r|p-1, q-1$, *then the markaracter of* G_{i+5} *is as reported in Table 24.*

Proof. Let G be a group of order pqr. If G is isomorphic to one of groups $G_1, \ldots, G_4$, then by applying Lemma 2.8, the proof is clear. If G is isomorphic to G_5, then the markaracter of G can be computed from Theorem 2.7. It remains to compute the markaracter table of groups G_{i+5} ($1 \leq i \leq r-1$). Letting $G = G_6$, it is easy to see that $\langle a^\alpha \rangle = \langle a^\beta \rangle$, $\langle b^\delta \rangle = \langle b^\eta \rangle$, $\langle c^\theta \rangle = \langle c^\lambda \rangle$ and $\langle b^\mu a^\nu \rangle = \langle b^\rho a^\varphi \rangle$, where $1 \leq \alpha, \beta, \nu, \varphi \leq p-1$, $1 \leq \delta, \eta, \mu, \rho \leq q-1$ and $1 \leq \theta, \lambda \leq r-1$. Therefore, all non-conjugate cyclic subgroups of G are $\langle e \rangle$, $\langle a \rangle$, $\langle b \rangle$, $\langle ab \rangle$, $\langle c \rangle$. Let $H_1 = \langle e \rangle$, $H_2 = \langle c \rangle$, $H_3 = \langle b \rangle$, $H_4 = \langle a \rangle$ and $H_5 = \langle ab \rangle$. One can easily check that $N_G(H_2) = H_2$ and $N_G(H_3) = N_G(H_4) = N_G(H_5) = G$. Hence, by Theorem 2.1, all entries of the diagonal and the first column of markaracter table can be derived. Since p, q, r are distinct prime numbers, according to Theorem 2.1, we have $m_{32} = m_{42} = m_{43} = m_{52} = 0$. Finally, the relations $ab = ba$, $c^{-1}bc = b^u$ and $c^{-1}ac = a^{v^i}$ yield that the subgroup H_5 is normal. This completes the proof. □

In continuing, we determine the markaracter table of groups of order p^2q.

Theorem 2.11. *Let p and q be two prime numbers such that $q > p$, $p|q-1$ and $G \in \mathcal{G}(p^2, q)$. Then, the markaracter table of G is isomorphic with one of Tables 26 - 30.*

Proof. Let $G \cong L_1$. Since L_1 is cyclic, then clearly $MC(L_1) = M(\mathbb{Z}_{p^2q})$ (see Table 26). All cyclic subgroups of L_2 are $H_{i,j} = \langle (c^i, a^j) \rangle$ ($1 \leq i, j \leq p-1$), where $c^p = a^p = 1, ac = ca$ and $\{e\} \times \mathbb{Z}_p$, $\mathbb{Z}_p \times \{e\}$. We show them by $H_1, \ldots H_{p+1}$. On the other hand, all cyclic subgroups of order pq of G are $G_{i,j} = \langle (c^i, b^j) \rangle$ $(1 \leq i \leq p-1), (1 \leq j \leq q-1)$, where $c^p = b^q = 1, bc = cb$ and $\{e\} \times \mathbb{Z}_{pq}$, $\mathbb{Z}_p \times \mathbb{Z}_q$. We show them by $G_1, \ldots, G_{p+1}$. Also, the Sylow q-subgroup Q is cyclic. So, by using Theorem 2.5, the markaracter table of L_2 is as given in Tasble 27.

All cyclic subgroups of order p in L_3 are $L_{i,j} = \langle (c^i, b^j) \rangle$ ($1 \leq i, j \leq p-1$), where $c^p = b^p = 1, bc = cb$ and $\{e\} \times \mathbb{Z}_p$, $\mathbb{Z}_p \times \{e\}$, denoted by $G_1, \ldots, G_{p+1}$. The other cyclic subgroups of L_3 are $G_{p+2} = Q$, where

$Q = \langle a \rangle$ is Sylow q-subgroup and $G_{p+3} = \langle c, b \rangle$. By using Theorem 2.5, $MC(L_3)$ is isomorphic with Table 28.

All cyclic subgroups of L_4 are $G_1 = \langle e \rangle$, $G_i = \langle\ b^k\ | k = p\ or\ 1 \rangle$, $(2 \leq i \leq 3)$ and $G_4 = \langle a \rangle$. So, the markaracter table can be derived from Theorem 2.5 (see Table 29).

Finally, all cyclic subgroups of L_5 are $H_1 = \langle e \rangle$, $H_2 = \langle a^p \rangle$, $H_3 = \langle a \rangle$, $H_4 = \langle b \rangle$ and $H_5 = \langle (a^p, b) \rangle$. The markaracter table of L_5 can be derived from Theorem 2.5 (see Table 30). □

Theorem 2.12. *Let p and q be two prime numbers such that $p > q$, $q|p-1$ and $G \in \mathcal{G}(p^2, q)$. Then, $MC(Q_1) = M(Q_1)$ and the markaracter table of groups $Q_2, \ldots, Q_6$ are as reported in Tables 31-33.*

Proof. All cyclic subgroups of $F_{p^2,q}$ are $G_1 = \langle e \rangle$, $G_2 = \langle b \rangle$, $G_3 = \langle a^p \rangle$ and $G_4 = \langle a \rangle$. So, the markaracter table is as given in Table 31. All cyclic subgroups of order p of Q_5 are $H_1, H_2, H_3, \ldots, H_{k+2}$, as defined in Theorem 2.6. On the other hand, the Sylow q-subgroup $Q = \langle b \rangle$ and $\langle e \rangle$ which are cyclic subgroups of Q_5. The markaracter table of $F_{p^2,q}$ can be derived from Theorem 2.6 (see Table 32). Finally, in group Q_6, the cyclic subgroups are $\langle e \rangle$, $H_1, \ldots, H_k$, as introduced in Theorem 2.6 together with Sylow q-subgroup $Q = \langle b \rangle$. So, the markaracter table is as given in Table 33. □

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Appendix.
The Table of Marks and Markaracter Table of Groups

Table 3. The Table of Marks of the Cyclic Group of Order p^3.

$M(\mathbb{Z}_{p^3})$	G_1	G_2	G_3	$\mathbb{Z}_{p^3}$
$\mathbb{Z}_{p^3}/G_1$	p^3	0	0	0
$\mathbb{Z}_{p^3}/G_2$	p^2	p^2	0	0
$\mathbb{Z}_{p^3}/G_3$	p	p	p	0
$\mathbb{Z}_{p^3}/\mathbb{Z}_{p^3}$	1	1	1	1

Table 4. The Table of Marks of Group $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$.

$M(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$	$\langle\rangle$	H_1	H_2	$\ldots$	H_{p+1}	G_1	G_2	$\ldots$	G_{p+1}	G
$G/\langle\rangle$	p^3	0	0	$\ldots$	0	0	0	$\ldots$	0	0
G/H_1	p^2	p^2	0	$\ldots$	0	0	0	$\ldots$	0	0
G/H_2	p^2	0	p^2	$\ldots$	0	0	0	$\ldots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
G/H_{p+1}	p^2	0	0	$\ldots$	p^2	0	0	$\ldots$	0	0
G/G_1	p	p	0	$\ldots$	0	p	0	$\ldots$	0	0
G/G_2	p	p	p	$\ldots$	p	0	p	$\ldots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
G/G_{p+1}	p	p	0	$\ldots$	0	0	0	$\ldots$	p	0
G/G	1	1	1	$\ldots$	1	1	1	$\ldots$	1	1

Table 5. The Table of Marks of Group $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

$M(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$	$\langle\rangle$	H_1	H_2	$\ldots$	H_t	G_1	G_2	$\ldots$	G_t	G
$G/\langle\rangle$	p^3	0	0	$\ldots$	0	0	0	$\ldots$	0	0
G/H_1	p^2	p^2	0	$\ldots$	0	0	0	$\ldots$	0	0
G/H_2	p^2	0	p^2	$\ldots$	0	0	0	$\ldots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
G/H_t	p^2	0	0	$\ldots$	p^2	0	0	$\ldots$	0	0
G/G_1	p	p	0	$\ldots$	0	p	0	$\ldots$	0	0
G/G_2	p	p	p	$\ldots$	p	0	p	$\ldots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
G/G_t	p	p	0	$\ldots$	0	0	0	$\ldots$	p	0
G/G	1	1	1	$\ldots$	1	1	1	$\ldots$	1	1

Table 6. The Table of Marks of Group $\mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}$.

$M(H)$	H_1	H_2	H_3	H_4	H_5	$\dots$	H_{p+4}	H
H/H_1	p^3	0	0	0	0	$\dots$	0	0
H/H_2	p^2	p^2	0	0	0	$\dots$	0	0
H/H_3	p^2	0	p	0	0	$\dots$	0	0
H/H_4	p	p	0	p	0	$\dots$	0	0
H/H_5	p	p	p	0	p	$\dots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
H/H_{p+4}	p	p	0	0	0	$\dots$	p	0
H/H	1	1	1	1	1	$\dots$	1	1

Table 7. The Table of Marks of Group $\mathbb{Z}_p \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$.

$M(G)$	$\langle\rangle$	H_1	H_2	H_3	$\dots$	H_{p+2}	G_1	G_2	$\dots$	G_{p+1}	G
$G/\langle\rangle$	p^3	0	0	0	$\dots$	0	0	0	$\dots$	0	0
G/H_1	p^2	p^2	0	0	$\dots$	0	0	0	$\dots$	0	0
G/H_2	p^2	0	p	0	$\dots$	0	0	0	$\dots$	0	0
G/H_3	p^2	0	0	p	$\dots$	0	0	0	$\dots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
G/H_{p+2}	p^2	0	0	0	$\dots$	p	0	0	$\dots$	0	0
G/G_1	p	p	p	0	$\dots$	0	p	0	$\dots$	0	0
G/G_2	p	p	0	p	$\dots$	0	0	p	$\dots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
G/G_{p+1}	p	p	0	0	$\dots$	p	0	0	$\dots$	p	0
G/G	1	1	1	1	$\dots$	1	1	1	$\dots$	1	1

Table 8. The Table of Marks of Group $\mathbb{Z}_{pqr}$.

$M(\mathbb{Z}_{pqr})$	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
G/G_1	pqr	0	0	0	0	0	0	0
G/G_2	pq	pq	0	0	0	0	0	0
G/G_3	pr	0	pr	0	0	0	0	0
G/G_4	qr	0	0	qr	0	0	0	0
G/G_5	p	p	p	0	p	0	0	0
G/G_6	q	q	0	q	0	q	0	0
G/G_7	r	0	r	r	0	0	r	0
G/G_8	1	1	1	1	1	1	1	1

Table 9. The Table of Marks of Group $\mathbb{Z}_r \times F_{p,q}$ $(q|p-1)$.

$M(\mathbb{Z}_r \times F_{p,q})$	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
H/H_1	pqr	0	0	0	0	0	0	0
H/H_2	pq	pq	0	0	0	0	0	0
H/H_3	pr	0	r	0	0	0	0	0
H/H_4	p	p	1	1	0	0	0	0
H/H_5	qr	0	0	0	qr	0	0	0
H/H_6	q	q	0	0	q	q	0	0
H/H_7	r	0	r	0	r	0	r	0
H/H_8	1	1	1	1	1	1	1	1

Table 10. The Table of Marks of Group $F_{p,qr}$ $(qr|p-1)$.

$M(F_{p,qr})$	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8
K/K_1	pqr	0	0	0	0	0	0	0
K/K_2	pq	q	0	0	0	0	0	0
K/K_3	pr	0	r	0	0	0	0	0
K/K_4	p	1	1	1	0	0	0	0
K/K_5	qr	0	0	0	qr	0	0	0
K/K_6	q	q	0	0	q	q	0	0
K/K_7	r	0	r	0	r	0	r	0
K/K_8	1	1	1	1	1	1	1	1

Table 11. The Table of Marks of Group G_{i+5} $(1 \leq i \leq r-1)$.

$M(G_{i+5})$	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
P/P_1	pqr	0	0	0	0	0	0	0
P/P_2	pq	1	0	0	0	0	0	0
P/P_3	pr	0	pr	0	0	0	0	0
P/P_4	pq	0	0	pq	0	0	0	0
P/P_5	p	1	p	0	1	0	0	0
P/P_6	q	1	0	q	0	q	0	0
P/P_7	r	0	r	r	0	0	r	0
P/P_8	1	1	1	1	1	1	1	1

Table 12. The Table of Marks of Group $\mathbb{Z}_{p^2q}$.

$M(\mathbb{Z}_{p^2q})$	G_1	G_2	G_3	G_4	G_5	G_6
G/G_1	p^2q	0	0	0	0	0
G/G_2	pq	pq	0	0	0	0
G/G_3	q	q	q	0	0	0
G/G_4	p^2	0	0	p^2	0	0
G/G_5	p	p	0	p	p	0
G/G_6	1	1	1	1	1	1

Table 13. The Table of Marks of Group $\mathbb{Z}_p \times \mathbb{Z}_{pq}$.

$M(\mathbb{Z}_p \times \mathbb{Z}_{pq})$	$\langle\rangle$	H_1	H_2	$\ldots$	H_{p+1}	P	Q	G_1	G_2	$\ldots$	G_{p+1}	G
$G/\langle\rangle$	p^2q	0	0	$\ldots$	0	0	0	0	0	$\ldots$	0	0
G/H_1	pq	pq	0	$\ldots$	0	0	0	0	0	$\ldots$	0	0
G/H_2	pq	0	pq	$\ldots$	0	0	0	0	0	$\ldots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
G/H_{p+1}	pq	0	0	$\ldots$	pq	0	0	0	0	$\ldots$	0	0
G/P	q	q	q	$\ldots$	q	q	0	0	0	$\ldots$	0	0
G/Q	p^2	0	0	$\ldots$	0	0	p^2	0	0	$\ldots$	0	0
G/G_1	p	p	0	$\ldots$	0	0	p	p	0	$\ldots$	0	0
G/G_2	p	p	0	$\ldots$	0	0	p	0	p	$\ldots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
G/G_{p+1}	p	p	0	$\ldots$	0	0	p	0	0	$\ldots$	p	0
G/G	1	1	1	$\ldots$	1	1	1	1	1	$\ldots$	1	1

Table 14. The Table of Marks of Group $\mathbb{Z}_p \times F_{q,p}$.

$M(\mathbb{Z}_p \times F_{q,p})$	$\langle\rangle$	H_1	H_2	$\dots$	H_{p+1}	P	Q	G_1	G_2	$\dots$	G_{p+1}	G
$G/\langle\rangle$	p^2q	0	0	$\dots$	0	0	0	0	0	$\dots$	0	0
G/H_1	pq	pq	0	$\dots$	0	0	0	0	0	$\dots$	0	0
G/H_2	pq	0	p	$\dots$	0	0	0	0	0	$\dots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
G/H_{p+1}	pq	0	0	$\dots$	p	0	0	0	0	$\dots$	0	0
G/P	q	q	1	$\dots$	1	1	0	0	0	$\dots$	0	0
G/Q	p^2	0	0	$\dots$	0	0	p^2	0	0	$\dots$	0	0
G/G_1	p	p	0	$\dots$	0	0	p	p	0	$\dots$	0	0
G/G_2	p	p	0	$\dots$	0	0	p	0	p	$\dots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
G/G_{p+1}	p	p	0	$\dots$	0	0	p	0	0	$\dots$	p	0
G/G	1	1	1	$\dots$	1	1	1	1	1	$\dots$	1	1

Table 15. The Table of Marks of Frobenius Group F_{q,p^2}.

$M(F_{q,p^2})$	H_1	H_2	H_3	H_4	H_5	H_6
H/H_1	p^2q	0	0	0	0	0
H/H_2	pq	p	0	0	0	0
H/H_3	q	1	1	0	0	0
H/H_4	p^2	0	0	p^2	0	0
H/H_5	p	p	0	p	p	0
H/H_6	1	1	1	1	1	1

Table 16. The Table of Marks of Group L_5.

$M(L_5)$	K_1	K_2	K_3	K_4	K_5	K_6
L_5/K_1	p^2q	0	0	0	0	0
L_5/K_2	pq	pq	0	0	0	0
L_5/K_3	q	q	1	0	0	0
L_5/K_4	p^2	0	0	p^2	0	0
L_5/K_5	p	p	0	p	p	0
L_5/K_6	1	1	1	1	1	1

Table 17. The Table of Marks of Group $\mathbb{Z}_p \times F_{p,q}$, $(k = p - 1/q)$.

$M(\mathbb{Z}_p \times F_{p,q})$	$\langle\rangle$	Q	H_1	H_2	H_3	$\ldots$	H_{k+2}	G_1	G_2	P	G
$G/\langle\rangle$	p^2q	0	0	0	0	$\ldots$	0	0	0	0	0
G/Q	p^2	p	0	0	0	$\ldots$	0	0	0	0	0
G/H_1	pq	0	pq	0	0	$\ldots$	0	0	0	0	0
G/H_2	pq	0	0	pq	0	$\ldots$	0	0	0	0	0
G/H_3	pq	0	0	0	p	$\ldots$	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
G/H_{k+2}	pq	0	0	0	0	$\ldots$	p	0	0	0	0
G/G_1	p	p	0	p	0	$\ldots$	0	p	0	0	0
G/G_2	p	1	p	0	0	$\ldots$	0	0	1	0	0
G/P	q	0	q	q	q	$\ldots$	q	0	0	q	0
G/G	1	1	1	1	1	$\ldots$	1	1	1	1	1

Table 18. The Table of Marks of Group $F_{p^2,q}$.

$M(F_{p^2,q})$	H_1	H_2	H_3	H_4	H_5	H_6
H/H_1	p^2q	0	0	0	0	0
H/H_2	p^2	1	0	0	0	0
H/H_3	pq	0	pq	0	0	0
H/H_4	p	1	p	1	0	0
H/H_5	q	0	q	0	q	0
H/H_6	1	1	1	1	1	1

Table 19. The Table of Marks of Group Q_5, $(k = p - 1/q)$.

$M(Q_5)$	$\langle\rangle$	Q	H_1	H_2	H_3	$\ldots$	H_{k+2}	G_1	G_2	P	Q_5
$Q_5/\langle\rangle$	p^2q	0	0	0	0	$\ldots$	0	0	0	0	0
Q_5/Q	p^2	1	0	0	0	$\ldots$	0	0	0	0	0
Q_5/H_1	pq	0	pq	0	0	$\ldots$	0	0	0	0	0
Q_5/H_2	pq	0	0	pq	0	$\ldots$	0	0	0	0	0
Q_5/H_3	pq	0	0	0	p	$\ldots$	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Q_5/H_{k+2}	pq	0	0	0	0	$\ldots$	p	0	0	0	0
Q_5/G_1	p	1	0	p	0	$\ldots$	0	1	0	0	0
Q_5/G_2	p	1	p	0	0	$\ldots$	0	0	1	0	0
Q_5/P	q	0	q	q	q	$\ldots$	q	0	0	q	0
Q_5/Q_5	1	1	1	1	1	$\ldots$	1	1	1	1	1

Table 20. The Table of Marks of Group Q_6, $(k = p - 1/q)$.

$M(Q_6)$	$\langle\rangle$	Q	H_1	H_2	$\dots$	H_{k+2}	P	Q_6
$Q_6/\langle\rangle$	p^2q	0	0	0	$\dots$	0	0	0
Q_6/Q	p^2	1	0	0	$\dots$	0	0	0
Q_6/H_1	pq	0	p	0	$\dots$	0	0	0
Q_6/H_2	pq	0	0	p	$\dots$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
Q_6/H_{k+2}	pq	0	0	0	$\dots$	p	0	0
Q_6/P	q	0	q	q	$\dots$	q	q	0
Q_6/Q_6	1	1	1	1	$\dots$	1	1	1

Table 21. The Markaracter Table of the Frobenius Group $F_{p,q}$.

$MC(F_{p,q})$	G_1	G_2	G_3	$\dots$	G_i	$\dots$	$G_{\tau(q)}$	$G_{\tau(q)+1}$
G/G_1	pq	0	0	$\dots$	0	$\dots$	0	0
G/G_2	$\frac{pq}{d_2}$	$d_{\tau(q)-1}$	0	$\dots$	0	$\dots$	0	0
G/G_3	$\frac{pq}{d_3}$	0	$d_{\tau(q)-2}$	$\dots$	0	$\dots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
G/G_i	$\frac{pq}{d_i}$	$m_{i,3}$	$m_{i,4}$	$\dots$	$d_{\tau(q)-i+1}$	$\dots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$G/G_{\tau(q)}$	p	1	1	$\dots$	1	$\dots$	1	0
$G/G_{\tau(q)+1}$	q	0	0	$\dots$	0	$\dots$	0	q

where $m_{i,j} = \begin{cases} \frac{q}{d_i}, & d_j|d_i \\ 0, & o.w. \end{cases}$.

Table 22. The Markaracter Table of Group $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, $t = p^2 + p + 1$.

$MC(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$	$\langle\rangle$	H_1	H_2	$\dots$	H_t
$G/\langle\rangle$	p^3	0	0	$\dots$	0
G/H_1	p^2	p^2	0	$\dots$	0
G/H_2	p^2	0	p^2	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
G/H_t	p^2	0	0	$\dots$	0

Table 23. The Markaracter Table of Group $\mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}$, $t = p + 1$.

$MC(\mathbb{Z}_p \rtimes \mathbb{Z}_{p^2})$	$\langle\rangle$	H_1	H_2	$\ldots$	H_t
$G/\langle\rangle$	p^3	0	0	$\ldots$	0
G/H_1	p^2	p^2	0	$\ldots$	0
G/H_2	p^2	0	p^2	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
G/H_t	p^2	0	0	$\ldots$	0

Table 24. The Markaracter Table of Group $\mathbb{Z}_p \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$.

$MC(\mathbb{Z}_p \rtimes (\mathbb{Z}_p \times \mathbb{Z}_p))$	$\langle\rangle$	H_1	H_2	H_3	$\ldots$	H_{p+2}
$G/\langle\rangle$	p^3	0	0	0	$\ldots$	0
G/H_1	p^2	p^2	0	0	$\ldots$	0
G/H_2	p^2	0	p	0	$\ldots$	0
G/H_3	p^2	0	0	p	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
G/H_{p+2}	p^2	0	0	0	$\ldots$	p

Table 25. The Markaracter Table of Group $G \cong G_{i+5}$ of order pqr.

$MC(G)$	H_1	H_2	H_3	H_4	H_5
G/H_1	pqr	0	0	0	0
G/H_2	pq	1	0	0	0
G/H_3	pr	0	pr	0	0
G/H_4	qr	0	0	qr	0
G/H_5	r	0	r	r	r

Table 26. The Markaracter Table of Group $\mathbb{Z}_{p^2q}$.

$MC(\mathbb{Z}_{p^2q})$	G_1	G_2	G_3	G_4	G_5	G_6
G/G_1	p^2q	0	0	0	0	0
G/G_2	p^2	p^2	0	0	0	0
G/G_3	pq	0	pq	0	0	0
G/G_4	p	p	p	p	0	0
G/G_5	q	0	q	0	q	0
G/G_6	1	1	1	1	1	1

Table 27. The Markaracter Table of Group $\mathbb{Z}_p \times \mathbb{Z}_{pq}$.

$MC(\mathbb{Z}_p \times \mathbb{Z}_{pq})$	$\langle\rangle$	H_1	H_2	$\dots$	H_{p+1}	Q	G_1	G_2	$\dots$	G_{p+1}
$G/\langle\rangle$	p^2q	0	0	$\dots$	0	0	0	0	$\dots$	0
G/H_1	pq	pq	0	$\dots$	0	0	0	0	$\dots$	0
G/H_2	pq	0	pq	$\dots$	0	0	0	0	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	0	0	$\dots$	0
G/H_{p+1}	pq	0	0	$\dots$	pq	0	0	0	$\dots$	0
G/Q	p^2	0	0	$\dots$	0	p^2	0	0	$\dots$	0
G/G_1	p	p	0	$\dots$	0	p	p	0	$\dots$	0
G/G_2	p	p	0	$\dots$	0	p	0	p	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	0	0	$\dots$	0
G/G_{p+1}	p	p	0	$\dots$	0	p	0	0	$\dots$	p

Table 28. The Markaracter Table of Group $\mathbb{Z}_p \times F_{q,p} = \langle c\rangle \times \langle a, b\rangle$.

$MC(\mathbb{Z}_p \times F_{q,p})$	$\langle\rangle$	G_1	G_2	$\dots$	G_{p+1}	G_{p+2}	G_{p+3}
$G/\langle\rangle$	p^2q	0	0	$\dots$	0	0	0
G/G_1	pq	pq	0	$\dots$	0	0	0
G/G_2	pq	0	p	$\dots$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
G/G_{p+1}	pq	0	0	$\dots$	p	0	0
G/G_{p+2}	p^2	0	0	$\dots$	0	p^2	0
G/G_{p+3}	p	p	0	$\dots$	0	p	p

Table 29. The Markaracter Table of Frobenius Group F_{q,p^2}.

$MC(F_{q,p^2})$	G_1	G_2	G_3	G_4
G/G_1	p^2q	0	0	0
G/G_2	pq	p	0	0
G/G_3	q	1	1	0
G/G_4	p^2	0	0	p^2

Table 30. The Markaracter Table of Group L_5.

$MC(L_5)$	H_1	H_2	H_3	H_4	H_5
L_5/H_1	p^2q	0	0	0	0
L_5/H_2	pq	pq	0	0	0
L_5/H_3	q	q	1	0	0
L_5/H_4	p^2	0	0	p^2	0
L_5/H_5	p	p	0	p	p

Table 31. The Markaracter Table of Group $\mathbb{Z}_p \times \mathbb{Z}_{qp}$.

$MC(\mathbb{Z}_p \times \mathbb{Z}_{qp})$	$\langle\rangle$	H_1	H_2	$\ldots$	H_{p+1}	Q	G_1	G_2	$\ldots$	G_{p+1}
$G/\langle\rangle$	p^2q	0	0	$\ldots$	0	0	0	0	$\ldots$	0
G/H_1	pq	pq	0	$\ldots$	0	0	0	0	$\ldots$	0
G/H_2	pq	0	pq	$\ldots$	0	0	0	0	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	0	0	$\ldots$	0
G/H_{p+1}	pq	0	0	$\ldots$	pq	0	0	0	$\ldots$	0
G/Q	p^2	0	0	$\ldots$	0	p^2	0	0	$\ldots$	0
G/G_1	p	p	0	$\ldots$	0	p	p	0	$\ldots$	0
G/G_2	p	p	0	$\ldots$	0	p	0	p	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	0	0	$\ldots$	0
G/G_{p+1}	p	p	0	$\ldots$	0	p	0	0	$\ldots$	p

Table 32. The Markaracter Table of Group $\mathbb{Z}_p \times F_{p,q}$.

$MC(\mathbb{Z}_p \times F_{q,p})$	$\langle\rangle$	G_1	G_2	$\ldots$	G_{p+1}	G_{p+2}	G_{p+3}
$G/\langle\rangle$	p^2q	0	0	$\ldots$	0	0	0
G/G_1	pq	pq	0	$\ldots$	0	0	0
G/G_2	pq	0	p	$\ldots$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
G/G_{p+1}	pq	0	0	$\ldots$	p	0	0
G/G_{p+2}	p^2	0	0	$\ldots$	0	p^2	0
G/G_{p+3}	p	p	0	$\ldots$	0	p	p

Table 33. The Markaracter Table of Group $F_{p^2,q}$.

$MC(F_{p^2,q})$	G_1	G_2	G_3	G_4
G/G_1	p^2q	0	0	0
G/G_2	p^2	1	0	0
G/G_3	pq	0	pq	0
G/G_4	q	0	q	q

Table 34. The Markaracter Table of Group Q_5, $(k = p - 1/q)$.

$MC(Q_5)$	$\langle\rangle$	Q	H_1	H_2	H_3	$\ldots$	H_{k+2}
$Q_5/\langle\rangle$	p^2q	0	0	0	0	$\ldots$	0
Q_5/Q	p^2	1	0	0	0	$\ldots$	0
Q_5/H_1	pq	0	pq	0	0	$\ldots$	0
Q_5/H_2	pq	0	0	pq	0	$\ldots$	0
Q_5/H_3	pq	0	0	0	p	$\ldots$	0
$\vdots$	$\vdots$	0	0	0	0	$\ddots$	$\vdots$
Q_5/H_{k+2}	pq	0	0	0	0	$\ldots$	p

Table 35. The Markaracter Table of Group Q_6, $(k = p - 1/q)$.

$MC(Q_6)$	$\langle\rangle$	Q	H_1	H_2	$\ldots$	H_{k+2}
$Q_6/\langle\rangle$	p^2q	0	0	0	$\ldots$	0
Q_6/Q	p^2	1	0	0	$\ldots$	0
Q_6/H_1	pq	0	p	0	$\ldots$	0
Q_6/H_2	pq	0	0	p	$\ldots$	0
$\vdots$	$\vdots$	0	0	0	$\ddots$	0
Q_6/H_{k+2}	pq	0	0	0	$\ldots$	p

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TABLE OF MARKS OF FINITE GROUPS

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فرض کنید G یک گروه متناهی و $\mathcal{C}(G)$ یک خانواده از زیرگروه‌های دوبه‌دو غیر مزدوج G باشد. ماتریسی که درایه $HK-$ام آن تعداد نقاط ثابت مجموعه G/K تحت عمل H باشد را جدول نمره G می‌نامیم، که در آن H و K در میان عناصر $\mathcal{C}(G)$ تغییر می‌کنند. در این مقاله، جدول‌های نمره و نمرشت گروه‌های از مرتبه pqr را که در آن p، q و r اعداد اول متمایز هستند را محاسبه می‌کنیم.

کلمات کلیدی: گروه فروبنیوس، جدول نمره، زیرگروه‌های دوبه‌دو غیر مزدوج یک گروه.