**Journal of Algebraic Systems**
**Vol. 5, No. 1, (2017), pp 73-84**

# ON THE MAXIMAL SPECTRUM OF A MODULE

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ABSTRACT. Let $R$ be a commutative ring with identity. The purpose of this paper is to introduce and study two classes of modules over $R$, called Max-injective and Max-strongly top modules and explore some of their basic properties. Our concern is to extend some properties of $X$-injective and strongly top modules to these classes of modules and obtain some related results.

## 1. Introduction

Throughout this paper, $R$ is a commutative ring with non-zero identity and $M$ is a unitary $R$-module. For any ideal $\mathfrak{I}$ of $R$ containing $\mathrm{Ann}_R(M)$, $\bar{R}$ and $\bar{\mathfrak{I}}$ denote $R/\mathrm{Ann}_R(M)$ and $\mathfrak{I}/\mathrm{Ann}_R(M)$, respectively. Further, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ denote the set of positive integers, the ring of integers, and the field of rational numbers, respectively.

For $M$ as an $R$-module and $P$, $N$ its submodules, the colon ideal of $M$ into $N$ is defined as $(N : M) = \{r \in R | rM \subseteq N\} = \mathrm{Ann}_R(M/N)$.

A submodule $P$ of $M$ is said to be a *prime submodule* or $\mathfrak{p}$-*prime submodule* if $P \neq M$ and for $\mathfrak{p} = (P : M)$, whenever $re \in P$ for $r \in R$ and $e \in M$, we have $r \in \mathfrak{p}$ or $e \in P$. If $Q$ is a maximal submodule of $M$, then $Q$ is a prime submodule and $(Q : M) := \mathfrak{m}$ is a maximal ideal of $R$. In this case, we say $Q$ is an $\mathfrak{m}$-maximal submodule of $M$ [8, p. 61].

The *prime spectrum* (*or simply, the spectrum*) of $M$ is the set of all prime submodules of $M$ and denoted by $\mathrm{Spec}_R(M)$ or $X$.

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MSC(2010): Primary: 13C13; Secondary: 13C99

Keywords: Prime submodule, Maximal submodule, Max-injective module, Max-strongly top module.
Received: 19 December 2016, Accepted: 7 May 2017.

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The set of all maximal submodules of $M$ is denoted by $\mathrm{Max}_R(M)$. Moreover, if $\mathfrak{p} \in \mathrm{Spec}(R)$ (resp., $\mathfrak{m} \in \mathrm{Max}(R)$), then $\mathrm{Spec}_{\mathfrak{p}}(M)$ (resp., $\mathrm{Max}_{\mathfrak{m}}(M)$) is the set of all $\mathfrak{p}$-prime (resp., $\mathfrak{m}$-maximal) submodules of $M$.

If $\mathrm{Spec}_R(M) \neq \emptyset$ (resp., $\mathrm{Max}_R(M) \neq \emptyset$), the mapping $\psi : \mathrm{Spec}_R(M) \to \mathrm{Spec}(\bar{R})$ (resp., $\phi : \mathrm{Max}_R(M) \to \mathrm{Max}(\bar{R})$ such that $\psi(P) = \overline{(P : M)}$ (resp., $\phi(Q) = \overline{(Q : M)}$) for every $P \in \mathrm{Spec}_R(M)$ (resp., $Q \in \mathrm{Max}_R(M)$), is called the *natural map* of $\mathrm{Spec}_R(M)$ (resp., $\mathrm{Max}_R(M)$) [11, p. 417].

$M$ is said to be $X$*-injective* if either $X = \emptyset$ or $X \neq \emptyset$ and the natural map of $X$ is injective [2, Definition 3.2].

The *Zariski topology* on $X = \mathrm{Spec}_R(M)$ is the topology $\tau_M$ described by taking the set $Z(M) = \{V(N) | N$ is a submodule of $M\}$ as the set of closed sets of $X$, where $V(N) = \{P \in X | (P : M) \supseteq (N : M)\}$ [11, p. 417].

The *quasi-Zariski topology* on $X = \mathrm{Spec}_R(M)$ is described as follows: put $V^*(N) = \{P \in X | P \supseteq N\}$ and $Z^*(M) = \{V^*(N) | N$ is a submodule of $M\}$. Then there exists a topology $\tau^*_M$ on $X$ having $Z^*(M)$ as the set of closed subsets of $X$ if and only if $Z^*(M)$ is closed under the finite union. When this is the case, $\tau^*_M$ is called the *quasi-Zariski topology* on $X$ and $M$ is called a top $R$-module [13, p. 85].

There exists a topology on $\mathrm{Max}_R(M)$ having $Z^m(M) = \{V^m(N) | N$ is a submodule of $M\}$ as the set of closed sets of $\mathrm{Max}_R(M)$, where $V^m(N) = \{Q \in \mathrm{Max}_R(M) | (Q : M) \supseteq (N : M)\}$. We denote this topology by $\tau^m_M$. In fact, this topology is the same as the subspace topology induced by $\tau_M$ on $\mathrm{Max}_R(M)$.

The *quasi-Zariski topology* on $\mathrm{Max}_R(M)$ is described as follows: put $V^{*m}(N) = \{Q \in \mathrm{Max}_R(M) | Q \supseteq N\}$ and $Z^{*m}(M) = \{V^{*m}(N) | N$ is a submodule of $M\}$. Then there exists a topology $\tau^{*m}_M$ on $\mathrm{Max}_R(M)$ having $Z^{*m}(M)$ as the set of closed subsets of $\mathrm{Max}_R(M)$ if and only if $Z^{*m}(M)$ is closed under the finite union. When this is the case, $\tau^{*m}_M$ is called the quasi-Zariski topology on $\mathrm{Max}_R(M)$ and $M$ is called a Max-top (or $M$-top) $R$-module [7, Notation 1.1.7]. We recall that when $M$ is a top module, this topology is the same as the subspace topology induced by $\tau^*_M$ on $\mathrm{Max}_R(M)$.

The present authors introduced the concept of strongly top modules and investigated some important properties of this family of modules. A top $R$-module $M$ is called strongly top if $\tau^*_M = \tau_M$[4, Definition 3.1].

In this paper, we will introduce two classes of modules, called Max-injective and Max-strongly top modules (see Definitions 3.1 and 3.12). It is shown that the class of Max-injective (resp., Max-strongly top) modules contains the family of $X$-injective (resp., strongly top) modules properly (see Propositions 3.3 and 3.13).

## 2. Preliminaries

In this section, we review some properties of prime and maximal submodules.

*Remark* 2.1. Let $M$ be an $R$-module.

(a) Let $K$ be a submodule of $M$ such that $(K : M)$ is a maximal ideal of $R$. Then, $K$ is a prime submodule of $M$ [8, Proposition 2];

(b) If $N$ is a maximal submodule of $M$, then $N$ is a prime submodule of $M$ and $(N : M)$ is a maximal ideal of $R$ [8, Proposition 4];

(c) Let $N$ be a prime submodule of $M$ and $S$ be a multiplicatively closed subset of $R$. Then, $S^{-1}(N :_R M) = (S^{-1}N :_{S^{-1}R} S^{-1}M)$ [10, Corollary 1].

*Remark* 2.2. [1, Proposition 3.3]. Let $M$ be an $R$-module and $\mathfrak{p} \in \mathrm{Max}(R)$. Then every $\mathfrak{p}$-prime submodule of $M$ is contained in some $\mathfrak{p}$-maximal submodule of $M$.

*Remark* 2.3. [9, Lemma 2]. Let $N$ and $L$ be submodules of an $R$-module $M$, and $P$ a $\mathfrak{p}$-prime submodule of $M$ such that $N \cap L \subseteq P$. If $(N : M) \nsubseteq \mathfrak{p}$, then $L \subseteq P$.

*Remark* 2.4. [13, Lemma 1.6]. Let $\mathfrak{p}$ be a prime ideal of $R$ and let $M$ be an $R$-module. Let $N$ be any submodule of $M$ and let $K \in \mathrm{Spec}_{\mathfrak{p}}(M)$. Then, $K \cap N = N$ or $K \cap N \in \mathrm{Spec}_{\mathfrak{p}}(N)$.

## 3. Main results

**Definition 3.1.** Let $M$ be an $R$-module. We say that $M$ is a *Max-injective module* if $\mathrm{Max}_R(M) = \emptyset$ or $\mathrm{Max}_R(M) \neq \emptyset$ and the natural map of $\mathrm{Max}_R(M)$ is injective.

**Proposition 3.2.**

(a) *Every $X$-injective module is Max-injective;*

(b) *$\mathbb{Q} \oplus \mathbb{Q}$ is not Max-injective $\mathbb{Q}$-module.*

*Proof.* (a) This is clear by Remark 2.1 (b).
(b) $0 \oplus \mathbb{Q}$ and $\mathbb{Q} \oplus 0$ are maximal submodules of the $\mathbb{Q}$-module $\mathbb{Q} \oplus \mathbb{Q}$ with $(0 \oplus \mathbb{Q} : \mathbb{Q} \oplus \mathbb{Q}) = (\mathbb{Q} \oplus 0 : \mathbb{Q} \oplus \mathbb{Q})$, while $0 \oplus \mathbb{Q} \neq \mathbb{Q} \oplus 0$. □

The following proposition shows that the class of Max-injective modules contains $X$-injective modules properly.

**Proposition 3.3.** *In the following cases, the $\mathbb{Z}$-module $M$ is Max-injective, while it is not $X$-injective.*

(a) $M = \mathbb{Q} \oplus \mathbb{Q}$;
(b) $M = \mathbb{Q} \oplus \prod_{i \in \mathbb{N}} \mathbb{Z}/p_i\mathbb{Z}$, *where* $\{p_i\}_{i \in \mathbb{N}}$ *are prime integers.*

*Proof.* (a) See [3, Table of Example 3.1].
(b) It is not difficult to see that $\text{Max}_{\mathbb{Z}}(M) = \{p_i M | i \in \mathbb{N}\}$ and

$$\{0 \oplus (\prod_{i \in \mathbb{N}} \mathbb{Z}/p_i\mathbb{Z}), \mathbb{Q} \oplus (\oplus_{i \in \mathbb{N}} \mathbb{Z}/p_i\mathbb{Z})\},$$

is a set of prime submodules of $M$. Hence, by the above arguments, $M$ is a Max-injective module. But $M$ is not $X$-injective, because $(0 \oplus (\prod_{i \in \mathbb{N}} \mathbb{Z}/p_i\mathbb{Z}) : M) = (\mathbb{Q} \oplus (\oplus_{i \in \mathbb{N}} \mathbb{Z}/p_i\mathbb{Z}) : M)$, while $0 \oplus (\prod_{i \in \mathbb{N}} \mathbb{Z}/p_i\mathbb{Z}) \neq \mathbb{Q} \oplus (\oplus_{i \in \mathbb{N}} \mathbb{Z}/p_i\mathbb{Z})$. □

We recall that a topological space $(X, \tau)$ is a $T_0$ space if for each pair $x$, $y \in X$, there exists an open set $U$ such that $x \in U$ but $y \notin U$.

**Lemma 3.4.** *Let $M$ be an $R$-module. Then the following are equivalent:*

(a) *$M$ is Max-injective;*
(b) $(Max_R(M), \tau_M^m)$ *is a* $T_0$ *space;*
(c) *For every* $P$, $Q \in Max_R(M)$, $(P : M) = (Q : M)$ *implies that* $P = Q$;
(d) $|Max_{\mathfrak{p}}(M)| \leq 1$ *for every* $\mathfrak{p} \in Max(R)$.

*Proof.* The proof is straightforward. □

**Lemma 3.5.** *Let $(M_i)_{i \in I}$ be a family of $R$-modules and let $\mathfrak{p} \in Max(R)$. Set $M = \oplus_{i \in I} M_i$. Then for each $Q_j \in Max_{\mathfrak{p}}(M_j)$, we have $Q_j \oplus (\oplus_{j \neq i \in I} M_i) \in Max_{\mathfrak{p}}(M)$.*

*Proof.* It is enough to prove the lemma in the case that $M = M_1 \oplus M_2$. So, let $Q_1 \in \text{Max}_R(M_1)$. Then, $M/Q_1 \oplus M_2$ is isomorphic to $M_1/Q_1$ is a simple $R$-module so that $Q_1 \oplus M_2$ is a maximal submodule of $M$. We have similar argument for $M_1 \oplus Q_2$, where $Q_2 \in \text{Max}_R(M_2)$. Hence, the proof is complete. □

**Proposition 3.6.** *Let $M$ be an $R$-module and let $\mathfrak{p} \in Max(R)$. Then*

(a) *Every homomorphic image of Max-injective $R$-module is Max-injective;*
(b) *If $M$ is a finitely generated Max-injective module, then $M_{\mathfrak{p}}$ is a Max-injective $R_{\mathfrak{p}}$-module;*
(c) *Let $M$ be a free $R$-module. Then $M$ is Max-injective if and only if $M$ is cyclic.*

*Proof.* (a) This is straightforward by using the fact that if $N$ is a submodule of $M$, then $\mathrm{Max}_R(M/N) = \{Q/N : Q \in \mathrm{Max}_R(M), Q \supseteq N\}$.
(b) Let $W_1$ and $W_2$ be maximal submodules of $M_\mathfrak{p}$ and $(W_1 : M_\mathfrak{p}) = (W_2 : M_\mathfrak{p})$. Then $W_1 \cap M$ and $W_2 \cap M$ are $\mathfrak{p}$-maximal submodules of $M$, by [5, Lemma 2.7]. Hence by hypothesis, $W_1 \cap M = W_2 \cap M$. Therefore, $(W_1 \cap M)_\mathfrak{p} = (W_2 \cap M)_\mathfrak{p}$. This means $W_1 = W_2$, as desired.
(c) ($\Leftarrow$) This follows from Proposition 3.2 (a). ($\Rightarrow$) Since $M$ is a free module, we have $M = \oplus_{i \in I} R$. We claim that $|I| = 1$. Otherwise if $|I| > 1$, then we can choose $\alpha, \beta \in I$ such that $\alpha \neq \beta$. Suppose that $\mathfrak{m} \in \mathrm{Max}(R)$. Then, $\mathfrak{m} \oplus (\oplus_{\alpha \neq i \in I} R) \in \mathrm{Max}_\mathfrak{m}(M)$ and $\mathfrak{m} \oplus (\oplus_{\beta \neq i \in I} R) \in \mathrm{Max}_\mathfrak{m}(M)$, by Lemma 3.5. Since $M$ is Max-injective, then $\mathfrak{m} \oplus (\oplus_{\alpha \neq i \in I} R) = \mathfrak{m} \oplus (\oplus_{\beta \neq i \in I} R)$, a contradiction. Hence, $M$ is cyclic, as desired. □

**Definition 3.7.** A family $(M_i)_{i \in I}$ of $R$-modules is said to be *max-compatible* if for all $i \neq j$ in $I$, there does not exist a maximal ideal $\mathfrak{p}$ in $R$ with $\mathrm{Max}_\mathfrak{p}(M_i)$ and $\mathrm{Max}_\mathfrak{p}(M_j)$ both non-empty.

**Theorem 3.8.** *Let $(M_i)_{i \in I}$ be a family of $R$-modules and let $M = \oplus_{i \in I} M_i$. Assume that $M$ is a* Max*-injective $R$-module. Then*

(a) *$(M_i)_{i \in I}$ is a family of max-compatible* Max*-injective modules;*
(b) $\mathrm{Max}_R(M) = \{Q_j \oplus (\oplus_{j \neq i \in I} M_i) | Q_j \in \mathrm{Max}_R(M_j), j \in I\}$.

*Proof.* (a) Let $M = \oplus_{i \in I} M_i$ be a Max-injective $R$-module. Then for each $i \in I$, $M_i$ is Max-injective, by Proposition 3.6 (a). Now, let $k, j \in I$ with $k \neq j$ and $\mathfrak{p} \in \mathrm{Max}(R)$. We will prove that $\mathrm{Max}_\mathfrak{p}(M_k) = \emptyset$ or $\mathrm{Max}_\mathfrak{p}(M_j) = \emptyset$. If both are non-empty, we can find $Q_k \in \mathrm{Max}_\mathfrak{p}(M_k)$ (resp., $Q_j \in \mathrm{Max}_\mathfrak{p}(M_j)$). Hence, $Q_k \oplus (\oplus_{k \neq i \in I} M_i) \in \mathrm{Max}_\mathfrak{p}(M)$ (resp., $Q_j \oplus (\oplus_{j \neq i \in I} M_i) \in \mathrm{Max}_\mathfrak{p}(M)$), by Lemma 3.5. Since $M$ is Max-injective, it follows that $Q_k \oplus (\oplus_{k \neq i \in I} M_i) = Q_j \oplus (\oplus_{j \neq i \in I} M_i)$, a contradiction. (b) Let $Q \in \mathrm{Max}_R(M)$ so that $(Q : M) = \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Max}(R)$. Since $Q \neq M$, there exists $j \in I$ such that $Q \cap M_j \neq M_j$. Then we have $Q \cap M_j \in \mathrm{Spec}_\mathfrak{p}(M_j)$, by Remark 2.4. Hence, there exists $W \in \mathrm{Max}_\mathfrak{p}(M_j)$ such that $Q \cap M_j \subseteq W$, by Remark 2.2. Thus, $W \oplus (\oplus_{j \neq i \in I} M_i) \in \mathrm{Max}_\mathfrak{p}(M)$, by Lemma 3.5. Since $M$ is Max-injective, it follows that $Q = W \oplus (\oplus_{j \neq i \in I} M_i)$. Therefore,

$$\mathrm{Max}_R(M) \subseteq \{Q_j \oplus (\oplus_{j \neq i \in I} M_i) | Q_j \in \mathrm{Max}_R(M_j), j \in I\}.$$

The reverse inclusion is obvious by Lemma 3.5, and we are done. □

A submodule $N$ of an $R$-module $M$ is semi-maximal if $N$ is an intersection of maximal submodules. Also, by $\mathrm{Rad}(N)$ we mean the intersection of all maximal submodules of $M$ containing $N$, and in case

$N$ is not contained in any maximal submodule, $\mathrm{Rad}(N)$ is defined to be $M$.

We need the following proposition.

**Proposition 3.9.** *Let $M$ be an $R$-module. Then the following statements are equivalent:*

(a) *$M$ is Max-top;*
(b) *For every maximal submodule $Q$ of $M$, whenever $N$ and $L$ are semi-maximal submodules of $M$ with $N \cap L \subseteq Q$, then $N \subseteq Q$ or $L \subseteq Q$.*

*Proof.* (a)⇒(b) Let $Q \in \mathrm{Max}_R(M)$ and also $N$, $L$ be semi-maximal submodules of $M$ such that $N \cap L \subseteq Q$. Since $N$ and $L$ are semi-maximal submodules of $M$, we have $N = \cap_{i\in\Lambda_1} N_i$ and $L = \cap_{t\in\Lambda_2} L_t$, where $N_i$, $L_t \in \mathrm{Max}_R(M)$ for all $i \in \Lambda_1$ and $t \in \Lambda_2$. Since $M$ is Max-top, there exists submodule $J$ of $M$ such that $V^{*m}(N) \cup V^{*m}(L) = V^{*m}(J)$. It is easy to see that $J \subseteq N \cap L$. Hence, $V^{*m}(N \cap L) \subseteq V^{*m}(J)$. Now, we have

$$V^{*m}(N) \cup V^{*m}(L) \subseteq V^{*m}(N \cap L) \subseteq V^{*m}(J) \subseteq V^{*m}(N) \cup V^{*m}(L).$$

Therefore, $V^{*m}(N \cap L) = V^{*m}(N) \cup V^{*m}(L)$. Now, $N \cap L \subseteq Q$ implies that $Q \in V^{*m}(N \cap L)$, so that $Q \in V^{*m}(N)$ or $Q \in V^{*m}(L)$. Therefore, $N \subseteq Q$ or $L \subseteq Q$, as required.

(b)⇒ (a) Let $S$ and $T$ be submodules of $M$. We will show that

$$V^{*m}(S) \cup V^{*m}(T) = V^{*m}(\mathrm{Rad}(S) \cap \mathrm{Rad}(T)).$$

Clearly, for every submodule $K$ of $M$, we have $V^{*m}(K) = V^{*m}(\mathrm{Rad}(K))$. Hence,

$$V^{*m}(S) \cup V^{*m}(T) \subseteq V^{*m}(\mathrm{Rad}(S) \cap \mathrm{Rad}(T)).$$

To see the reverse inclusion, let $P \in V^{*m}(\mathrm{Rad}(S) \cap \mathrm{Rad}(T))$, so that $\mathrm{Rad}(S) \cap \mathrm{Rad}(T) \subseteq P$. It then follows that $\mathrm{Rad}(S) \subseteq P$ or $\mathrm{Rad}(T) \subseteq P$, by hypothesis. In either case, we have $P \in V^{*m}(S) \cup V^{*m}(T)$, and the proof is complete. □

*Remark* 3.10. We recall that every top module is $X$-injective, by [2, Proposition 3.3]. The following example shows that this property is not true for Max-top and Max-injective modules, in general.

**Example 3.11.** Consider $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ as a $\mathbb{Z}_2$-module. Then, $M$ is Max-top but is not Max-injective. ( See [7, Example 1.1.16].)

**Definition 3.12.** Let $M$ be a Max-top $R$-module. We say that $M$ is a Max-*strongly top* module if $\tau_M^{*m} = \tau_M^m$.

It is easy to check that every strongly top module is Max-strongly top. The following proposition shows that this containment is proper in general.

**Proposition 3.13.** *Let $M = \mathbb{Q} \oplus \mathbb{Q}$. Then $M$ is a Max-strongly top $\mathbb{Z}$-module, while it is not strongly top.*

*Proof.* This follows by [3, Table of Example 3.1] and the fact that every strongly top module is a top module. □

*Remark* 3.14. Let $(X, \tau)$ be a topological space and $Y \subseteq X$. We write $(cl(Y))_{(X,\tau)}$ to denote the topological closure of $Y$ in $(X, \tau)$.

**Lemma 3.15.** *Let $M$ be an $R$-module. Then the following statements are equivalent:*

(a) *$M$ is an Max-strongly top module;*
(b) *For every submodule $N$ of $M$, there exists submodule $K$ of $M$ such that $V^{*m}(N) = V^m(K)$;*
(c) *$V^{*m}(N) = V^m(Rad(N))$, for every submodule $N$ of $M$.*

*Proof.* (a) ⇔ (b) This follows from the fact that we have always $\tau^m_M \subseteq \tau^{*m}_M$.

(a) ⇔ (c) Let $M$ be an Max-strongly top $R$-module and $N$ a submodule of $M$. By hypothesis, there exists submodule $K$ of $M$ such that $V^{*m}(N) = V^m(K)$. But, $V^m(K)$ is a closed subset of $(\text{Max}_R(M), \tau^m_M)$, hence

$$(cl(V^m(K)))_{(\text{Max}_R(M),\tau^m_M)} = V^m(K).$$

On the other hand, it is well known that

$$(cl(V^m(K)))_{(\text{Max}_R(M),\tau^m_M)} = (cl(V^m(K)))_{(\text{Spec}_R(M),\tau_M)} \cap \text{Max}_R(M).$$

Now, by [11, Proposition 5.1], we have

$$(cl(V^m(K)))_{(\text{Max}_R(M),\tau^m_M)} = V(\cap_{Q\in V^m(K)}Q) \cap \text{Max}_R(M).$$

We claim that

$$V(\cap_{Q\in V^m(K)}Q) \cap \text{Max}_R(M) = V^m(\text{Rad}(N)).$$

To see this, Let $P \in V(\cap_{Q\in V^m(K)}Q) \cap \text{Max}_R(M)$. Then,

$$(P : M) \supseteq (\cap_{Q\in V^m(K)}Q : M) \supseteq \cap_{Q\in V^m(K)}(Q : M) \supseteq (K : M).$$

Hence, $P \in V^m(K)$. But, $V^m(K) = V^{*m}(N) \subseteq V^m(\text{Rad}(N))$. Therefore,

$$V(\cap_{Q\in V^m(K)}Q) \cap \text{Max}_R(M) \subseteq V^m(\text{Rad}(N)).$$

To see the reverse inclusion, let $W \in V^m(\text{Rad}(N))$. Then, we have

$$(W : M) \supseteq (\text{Rad}(N) : M) \supseteq (\cap_{Q\in V^m(K)}Q : M).$$

This implies that, $W \in V(\cap_{Q\in V^m(K)} Q) \cap \mathrm{Max}_R(M)$ and

$$V^m(\mathrm{Rad}(N)) \subseteq V(\cap_{Q\in V^m(K)} Q) \cap \mathrm{Max}_R(M).$$

By the above arguments, we have $V^{*m}(N) = V^m(\mathrm{Rad}(N))$. The reverse implication follows from the fact that $\tau^m_M \subseteq \tau^{*m}_M$. □

*Remark* 3.16. The ring $R$ is a *perfect ring* if it is satisfies DCC condition on principal ideals. Clearly, every Artinian ring is perfect. One can easily see that if $R$ is a perfect ring, then every prime ideal of $R$ is a maximal ideal. Furthermore, every perfect ring is a semilocal ring [6, Theorem P or Example 3(6)].

**Proposition 3.17.** *Let $M$ be a Max-injective $R$-module. Then $M$ is Max-strongly top in the following cases:*

(a) *$M$ is non-faithful and $R$ is PID;*
(b) *$|Max(R)| < \infty$;*
(c) *$R$ is a perfect ring.*

*Proof.* (a) Let $N$ be a submodule of $M$. To prove $M$ is Max-strongly top module, it is enough to show that $V^{*m}(N) = V^m(\mathrm{Rad}(N))$, by Proposition 3.15. Clearly, $V^{*m}(N) \subseteq V^m(\mathrm{Rad}(N))$. To see the reverse inclusion, let

$$\Lambda = \{W | W \in \mathrm{Max}_R(M), W \supseteq N\}.$$

Obviously, $\Lambda$ is a finite set because $R$ is PID and each $W \in \Lambda$ is a maximal submodule and $M$ is non-faithful. Now, let $Q \in V^m(\mathrm{Rad}(N))$. Then, $Q \in \mathrm{Max}_R(M)$ and we have

$$(Q : M) \supseteq (\mathrm{Rad}(N) : M) \supseteq \cap_{W\in\Lambda}(W : M).$$

This implies that $(Q : M) = (K : M)$, for some $K \in \Lambda$. So, $Q = K$ by hypothesis. Therefore, $Q \supseteq N$ so that $Q \in V^{*m}(N)$, as desired.

(b) and (c) We have similar argument as in part (a). □

**Corollary 3.18.** *Let $M$ be a Max-injective $R$-module. Then $M$ is Max-top in each case listed in Proposition 3.17.*

**Proposition 3.19.** *In the following, in each case, the $R$-module $M$ is Max-strongly top:*

(a) *$M = \mathbb{Z}_{(p)} = S^{-1}\mathbb{Z}$, where $p$ is a prime integer, $S = \mathbb{Z} \setminus (p)$ and $R = \mathbb{Z}$;*
(b) *$|Max(R)| < \infty$ and for every $Q \in Max_R(M)$, there exists $\mathfrak{p} \in Max(R)$ such that $Q = \mathfrak{p}M$;*
(c) *$M = \oplus_{i\in I} M_i$, where $(M_i)_{i\in I}$ is a family of prime compatible $X$-injective $R$-modules and $R$ is a perfect ring;*
(d) *$M = \oplus_{\lambda\in\Lambda} R/I_\lambda$, where $\Lambda$ is a finite index set and $I_\lambda$ $(\lambda \in \Lambda)$ are comaximal ideals of $R$.*

*Proof.* (a) This follows from [3, Table of Example 3.1], Proposition 3.2 (a), and Proposition 3.17 (a).
(b) Follows from Proposition 3.17 (b).
(c) Follows from [2, Proposition 3.7 (c)], Proposition 3.2 (a), and Proposition 3.17 (c).
(d) Follows from [13, Corollary 5.5], Theorem 3.8 (b), and Lemma 3.15. □

**Proposition 3.20.** *Let $M$ be an $R$-module and also $\mathfrak{p} \in Max(R)$. Then,*

(a) *Every homomorphic image of Max-strongly top $R$-module is Max-strongly top;*
(b) *If $M$ is a finitely generated Max-strongly top module, then $M_\mathfrak{p}$ is Max-strongly top $R_\mathfrak{p}$-module.*

*Proof.* (a) Let $M$ be a Max-strongly top $R$-module and $N$ a submodule of $M$. Let $K/N$ be a submodule of $M/N$. By Lemma 3.15, it is enough to prove that $V^m(\mathrm{Rad}(K/N)) = V^{*m}(K/N)$. To see this, let $L \in V^m(\mathrm{Rad}(K/N))$. Then, $L = Q/N$, where $N \subseteq Q \in \mathrm{Max}_R(M)$. This implies that

$$\begin{aligned}(Q/N : M/N) &\supseteq (\cap_{N\subseteq P\in V^{*m}(K)} P/N : M/N)\\ &= \cap_{N\subseteq P\in V^{*m}(K)}(P/N : M/N).\end{aligned}$$

Therefore, we have $(Q : M) \supseteq (\bigcap_{N\subseteq P\in V^{*m}(K)} P : M)$, by [12, Result 1]. It then follows that $Q \in V^m(\mathrm{Rad}(K))$. Since $M$ is a Max-strongly top $R$-module, we have $V^{*m}(K) = V^m(\mathrm{Rad}(K))$, by Lemma 3.15 so that $Q \in V^{*m}(K)$. Hence, $V^m(\mathrm{Rad}(K/N)) \subseteq V^{*m}(K/N)$. The reverse inclusion is clear, and the proof is complete.

(b) Let $N_\mathfrak{p}$ a submodule of $M_\mathfrak{p}$ for some submodule $N$ of $M$. By Lemma 3.15, it is enough to prove that $V^{*m}(N_\mathfrak{p}) = V^m(\mathrm{Rad}(N_\mathfrak{p}))$. It is clear that $V^{*m}(N_\mathfrak{p}) \subseteq V^m(\mathrm{Rad}(N_\mathfrak{p}))$. Conversely, assume that $W \in V^m(\mathrm{Rad}(N_\mathfrak{p}))$. Then, there exists $Q \in \mathrm{Max}_R(M)$ such that $W = Q_\mathfrak{p}$ and $(Q :_R M) = \mathfrak{p}$, by [5, Lemma 2.7]. It then follows that

$$\begin{aligned}(Q_\mathfrak{p} :_{R_\mathfrak{p}} M_\mathfrak{p}) &\supseteq (\mathrm{Rad}(N_\mathfrak{p}) :_{R_\mathfrak{p}} M_\mathfrak{p})\\ &\supseteq ((\mathrm{Rad}(N))_\mathfrak{p} :_{R_\mathfrak{p}} M_\mathfrak{p})\\ &\supseteq (\mathrm{Rad}(N) :_R M)_\mathfrak{p}.\end{aligned}$$

But $(Q_\mathfrak{p} :_{R_\mathfrak{p}} M_\mathfrak{p}) = (Q :_R M)_\mathfrak{p}$, by Remark 2.1 (c). Therefore, $Q \in V^m(\mathrm{Rad}(N))$, so that $Q \in V^{*m}(N)$, by Lemma 3.15. This implies that $W \in V^{*m}(N_\mathfrak{p})$. Hence the proof is complete. □

**Theorem 3.21.** *Let $(M_i)_{i\in I}$ be a family of $R$-modules and let $M = \oplus_{i\in I} M_i$. Suppose that there exists $t \in I$ such that $M_t$ is simple and faithful. Then,*

(a) *If $M$ is* Max*-strongly top, then for every $j \in I$ with $j \neq t$, we have*$\mathrm{Max}_R(M_j) = \emptyset$*;*

(b) *If $M$ is* Max*-injective and $(M_i)_{i\in I}$ is a family of $X$-injective modules, then $M$ is* Max *strongly top if and only if* $|\mathrm{Max}_R(M)| = 1$*;*

(c) *If $M$ is $X$-injective, then $M$ is* Max *strongly top if and only if* $|\mathrm{Max}_R(M)| = 1$*;*

(d) *If $M$ is $X$-injective and* $\mathrm{Max}_R(M_t) = \{0\}$*, then $M$ is* Max*-strongly top if and only if for every $j \in I$ with $j \neq t$,* $\mathrm{Max}_R(M_j) = \emptyset$*.*

*Proof.* (a) Let $j \in I$ with $j \neq t$. We will show that $\mathrm{Max}_R(M_j) = \emptyset$. Otherwise, choose $Q_j \in \mathrm{Max}_R(M_j)$. Set $M^j := \oplus_{j\neq i\in I} M_i$. Then, by Lemma 3.5, $K_j := Q_j \oplus M^j \in \mathrm{Max}_R(M)$. Clearly, $0 \in \mathrm{Max}_R(M_t)$ so that $K_t = 0 \oplus M^t \in \mathrm{Max}_R(M)$, by Lemma 3.5. Clearly, $(K_t : M) = 0$ and hence $K_j \in V^m(K_t)$. Now, by Lemma 3.15,

$$V^{*m}(K_t) = V^m(\mathrm{Rad}(K_t)) = V^m(K_t).$$

Therefore, $K_j \in V^{*m}(K_t)$ so that $K_j \supseteq K_t$. This implies that $Q_j \supseteq M_j$, a contradiction.

(b) ($\Leftarrow$) This is clear by Lemma 3.15. Conversely, by Theorem 3.8 (b),

$$\mathrm{Max}_R(M) = \{Q_j \oplus (\oplus_{j\neq i\in I} M_i) | Q_j \in \mathrm{Max}_R(M_j), j \in I\}.$$

Now the result follows from part (a).

(c) and (d) Follows by Lemma 3.15, Theorem 3.8 (c), and part (a). □

We need the following simple lemma.

**Lemma 3.22.** *Let $M$ be an $R$-module and $\phi : Max_R(M) \to Max(\bar{R})$ be the natural map of $Max_R(M)$. Then, $\phi^{-1}(V^m(\bar{\mathfrak{I}})) = V^m(\mathfrak{I}M)$, for every ideal $\mathfrak{I}$ of $R$ containing $Ann(M)$.*

*Proof.* Straightforward. □

An $R$-module $M$ is said to be Max-*surjective* if either $M = (0)$ or $M \neq (0)$ and the natural map of $\mathrm{Max}_R(M)$ is surjective [1, Definition 3.1].

**Theorem 3.23.** *Let $M$ be* Max*-surjective,* Max*-injective, and a* Max*-strongly top $R$-module. Then* $(\mathrm{Max}_R(M), \tau^m_M)$ *and* $(\mathrm{Max}_R(M), \tau^{*m}_M)$ *are homeomorphic with* $\mathrm{Max}(\bar{R})$ *with its topology induced by the Zariski topology of* $\mathrm{Spec}(\bar{R})$*.*

*Proof.* Let $\phi : \mathrm{Max}_R(M) \to \mathrm{Max}(\bar{R})$ be the natural map of $\mathrm{Max}_R(M)$. As $M$ is a Max-surjective, a Max-injective module, $\phi$ is a bijective map. Now, let $\mathfrak{I}$ be an ideal of $R$ such that $\mathrm{Ann}_R(M) \subseteq \mathfrak{I}$. By Lemma 3.22 and [11, Result 3],we have

$$\phi^{-1}(V^m(\bar{\mathfrak{I}})) = V^m(\mathfrak{I}M) = \mathrm{Max}_R(M) \cap V(\mathfrak{I}M)$$
$$= \mathrm{Max}_R(M) \cap V^*(\mathfrak{I}M) = V^{*m}(\mathfrak{I}M).$$

So, $\phi : (\mathrm{Max}_R(M), \tau_M^{*m}) \to \mathrm{Max}(\bar{R})$ is continuous. Now, let $N$ be a non-zero submodule of $M$. Then, by Lemma 3.22 and Lemma 3.15, we get

$$\begin{aligned}\phi^{-1}(V^m(\overline{(\mathrm{Rad}(N) : M)})) &= V^m((\mathrm{Rad}(N) : M)M)\\ &= V^m(\mathrm{Rad}(N)) = V^{*m}(N).\end{aligned}$$

Since $\phi$ is surjective, then

$$\phi(V^{*m}(N)) = V^m(\overline{(\mathrm{Rad}(N) : M)}).$$

Hence, $\phi : (\mathrm{Max}_R(M), \tau_M^{*m}) \to \mathrm{Max}(\bar{R})$ is a closed map. Therefore, $(\mathrm{Max}_R(M), \tau_M^{*m})$ is homeomorphic with $\mathrm{Max}(\bar{R})$. Now, since $M$ is Max-strongly top, we have $\tau_M^m = \tau_M^{*m}$. Hence, $(\mathrm{Max}_R(M), \tau_M^m)$ is homeomorphic with $\mathrm{Max}(\bar{R})$, as required. □

**Example 3.24.** Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3$. Then $(\mathrm{Max}_{\mathbb{Z}}(M), \tau_M^m)$ and $(\mathrm{Max}_{\mathbb{Z}}(M), \tau_M^{*m})$ are homeomorphic with $\mathrm{Max}(\mathbb{Z}/\mathrm{Ann}_{\mathbb{Z}}(M))$, by [3, Table of Example 3.1], Proposition 3.17 (a), and Theorem 3.23.

An $R$-module $M$ is a *multiplication module* if for every submodule $N$ of $M$, there exists an ideal $\mathfrak{I}$ of $R$ such that $N = \mathfrak{I}M$ [13, p. 91].

**Corollary 3.25.** *Let $M$ be a finitely generated multiplication $R$-module. Then $(Max_R(M), \tau_M^m)$ and $(Max_R(M), \tau_M^{*m})$ are homeomorphic to $Max(\bar{R})$.*

*Proof.* $M$ is both Max-surjective and Max-injective by [1, Example 3.2], [2, Proposition 3.3], and Proposition 3.2 (a). Now, the result follows by [4, Example 3.1 (a)] and Theorem 3.23 and the fact that every strongly top module is Max-strongly top. □

## Acknowledgments

The authors would like to thank the referee for the careful reading of the manuscript and valuable comments.

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Journal of Algebraic Systems

# ON THE MAXIMAL SPECTRUM OF A MODULE

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# تحقیقی روی طیف ماکزیمال یک مدول

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فرض کنید $R$ یک حلقه جابجایی با عنصرهای ناصفر و $M$ یک $R-$مدول یکانی باشد. هدف از این مقاله معرفی و مطالعه پاره‌ای از خواص اساسی دو رده از مدول‌ها به نام‌های $Max$-اِنژکتیو و $Max$-قویاً تاپ و تعمیم بعضی از خواص مدول‌های $X$-انژکتیو و قویاً تاپ به این دو رده از مدول‌ها و به دست آوردن برخی از نتایج مرتبط است.

کلمات کلیدی: زیرمدول اول، زیرمدول ماکزیمال، مدول $Max$-اِنژکتیو، مدول $Max$-قویاً تاپ.