

SEQUENTIALLY COMPACT S -ACTS

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ABSTRACT. The investigation of equational compactness was initiated by Banaschewski and Nelson. They proved that pure injectivity is equivalent to equational compactness. Here, we define the so-called sequentially compact acts over semigroups and study some of their categorical and homological properties. Some Baer conditions for injectivity are also presented.

1. INTRODUCTION AND PRELIMINARIES

An algebra A is called *equationally compact* if every system of polynomial equations (with constants from A) has a solution in A provided that every finite subsystem of it has a solution in A .

The notion of equational compactness of universal algebras has been studied by Banaschewski and Nelson [3]. They show that equational compactness is equivalent to pure injectivity, injectivity with respect to all pure monomorphisms. Moreover, Banaschewski [2] deals with this notion in the special case of G -sets for a group G . There are some weaker notions of equational compactness, such as, 1-equational compactness or f -equational compactness, where systems of equations contain, respectively, one variable or finite number of variables. Normak [13] shows that for a monoid S , all S -acts are f -equationally compact if and only if all S -acts are 1-equationally compact.

Giuli [10] has introduced the category of *Projection Algebras* A over a monoid S , by a system of equations $xs = a_s (s \in S, a_s \in A)$ as an algebraic version of ultrametric spaces. Computer scientists use this

MSC(2010): Primary: 08A60; Secondary: 08B30, 20M30, 20M50.
Keywords: Sequentially compact, f -pure injective, Injective S -act.
Received: 21 May 2016, Accepted: 06 October 2017.

notion as a convenient means for algebraic specification of process algebras. In this category, Ebrahimi and Mahmoudi [12] introduced the concept of sequential compactness and showed that sequential compactness coincides with injectivity and equational compactness.

Throughout this paper, S stands for a given semigroup and $\mathbf{Act} - \mathbf{S}$ will denote the category of all (right) S -acts, including \emptyset , and homomorphisms between them.

Based on the above mentioned points, we are persuaded to develop the notion of sequential compactness for a given semigroups S in the category $\mathbf{Act} - \mathbf{S}$.

In Section 3, we generalized the notion of sequential compactness for S -acts and study some basic and homological properties of it. Section 4 deals with the categorical properties of sequential compactness of S -acts, and Section 5 is devoted to the study of some Baer type Criteria for this notion.

Here we briefly recall the definition and the categorical and algebraic ingredients of the category $\mathbf{Act-S}$ of (right) S -acts over a semigroup S . For more information and the notions not mentioned here about this category, one may consult [11].

Recall that, for a semigroup S , a set A is a right S -act (or an S -set) if there is an S -action $\mu : A \times S \rightarrow A$, denoting $\mu(a, s) := as$, such that $a(st) = (as)t$ and if S is a monoid with 1, $a1 = a$.

A morphism $f : A \rightarrow B$ between S -acts A, B is called a *homomorphism* if, for each $a \in A, s \in S, f(as) = f(a)s$.

A subset A of an S -act B is called a subact of B and in this case, B is said to be an extension of A , if for each $a \in A$ and $s \in S, as \in A$. Also, an element $a \in A$ ($t \in S$) is said to be a *fixed element* (*left zero element*) if $as = a$ ($ts = t$) for all $s \in S$. The set of all fixed elements of an S -act A and the set of all left zero elements of a semigroup S are denoted, respectively, by $Fix(A)$ and $Z(S)$. The semigroup S is called a left zero semigroup if all of its elements are left zeros. The S -act $A \cup \{0\}$ with a fixed element 0 adjoined to A is denoted by A^0 . Notice that each semigroup S can be considered as an S -act with the action given by its multiplication, and adjoining an external left identity 1 to a semigroup S gives us an S -act $S^1 := S \cup \{1\}$. Also, here the empty set is allowed to be an S -act.

A subset I of S is a right ideal if for each $r \in I, s \in S, rs \in I$. A right ideal I of S is finitely generated if I has a finite subset $\{t_1, t_2, \dots, t_n\}$ such that $I = \bigcup_{i=1}^n t_i S^1$.

For a family $\{A_i\}$ of S -acts, their cartesian product $\prod_{i \in I} A_i$ with the S -action defined by $(a_i)s = (a_i s)$ is the product of the family $\{A_i\}$ in

Act-S. The coproduct of a family $\{A_i\}$ in **Act-S** is their disjoint union $\coprod_{i \in I} A_i = \cup_{i \in I} (A_i \times \{i\})$ with the action of S defined by $(a, i)s = (as, i)$ for $s \in S$, $a \in A_i$. Recall that for a family $\{A_i : i \in I\}$ of S -acts each with a unique fixed element 0 , the *direct sum* $\bigoplus_{i \in I} A_i$ is defined to be the subact of the product $\prod_{i \in I} A_i$ consisting of all $(a_i)_{i \in I}$ such that $a_i = 0$ except for a finite number of indices.

2. SEQUENTIAL COMPACTNESS

In this section, by means of the notion of the *sequential system of equations* on an S -act A we introduce the concept of so called *sequentially compact* or briefly *s-compact* S -acts. Then, we study some basic properties of *s-compactness*.

Definition 2.1. [6] (1) Any set $\Sigma = \{xs = a_s : s \in S, a_s \in A\}$ is called a *sequential system of equations* with constants from an S -act A , or over A .

(2) We say that Σ is *solvable* in an extension B of A if there is some $b \in B$ such that for all $s \in S$, $bs = a_s$. The system Σ is said to be *consistent* if it has a solution in some extension B of A .

Note that there is a one to one correspondence between the set of all systems of equations Σ_A of the above form on an S -act A and the set of all maps $k : S \rightarrow A$. For any S -act B and $b \in B$, let us denote the left translation mapping $\lambda_b : S \rightarrow B$, defined by $\lambda_b(s) := bs$, by λ_b . In these notations, we have

Lemma 2.2. [6] *Let A be an S -act. A map $k : S \rightarrow A$ is a homomorphism if and only if there exists b in an extension B of A such that $k = \lambda_b$.*

Definition 2.3. An S -act A is said to be *sequentially compact* or simply, *s-compact* if every system of sequential equations $\Sigma = \{xs = a_s \mid s \in S, a_s \in A\}$ has a solution in A whenever this is the case for each finite subsystem of Σ .

In the following, by an easy observation, we obtain an equivalent condition to *s-compactness* which will be used in the proof of some results.

Lemma 2.4. *An S -act A is s-compact if and only if every map $f : S \rightarrow A$ is of the form λ_a for some $a \in A$ whenever for every finite subset T of S there is an element $a_T \in A$ such that $f|_T = \lambda_{a_T}$.*

In what follows, we are going to find some characterization for *s-compactness* based on purity, injectivity and retractness. To proceed, we listed some preliminaries.

Lemma 2.5. (i) Let A be an S -act. If $f : S \rightarrow A$ is a map whose restriction to each finite subset T of S is of the form λ_{a_T} for some $a_T \in A$, then f is a homomorphism.

(ii) Let Σ be a system of sequential equations on A such that each of its finite subsets has a solution in A . Then Σ has a solution in some extension B of A .

Proof. (i) Let $s, t \in S$. Considering $T = \{t, ts\}$, there is an element $a_T \in A$ such that $f|_T = \lambda_{a_T}$. Thus $f(t)s = (a_T t)s = a_T(ts) = f(ts)$ and hence f is a homomorphism.

(ii) Let $\Sigma = \{xs = a_s \mid s \in S, a_s \in A\}$ satisfying the property given in (ii). Consider the map $f : S \rightarrow A$ by $f(s) = a_s$. It is easy to check that the restriction of f to every finite subset T of S is of the form λ_{a_T} for some $a_T \in A$. So by part (i), f is a homomorphism. Let $E(A)$ be the injective hull of A , which exists by [8]. Thus $f : S \rightarrow A \hookrightarrow E(A)$ is extended to $\bar{f} : S^1 \rightarrow E(A)$, whence $\bar{f}(1)$ is a solution of Σ . \square

Definition 2.6. (i) An S -act A is said to be f -pure in an S -act B if every $\Sigma = \{xs = a_s \mid s \in S, a_s \in A\}$ has a solution in A whenever this is the case for B and each finite subset of Σ . An S -act A is called *absolutely f -pure* if it is f -pure in each of its extensions.

(ii)[9] An S -act A is called an s -dense subact of B or an S -act B is called an s -dense extension of A , if for every $b \in B$, $bS \subseteq A$.

(iii)[5] An S -act A is called a *strongly- s -dense* subact of B if for every $b \in B$, $bS \subseteq A$ and for every finite subset T of S there is an element $a_T \in A$ such that $a_T t = bt$ for all $t \in T$. In this case, an S -act B is said to be a *strongly- s -dense extension* of A .

Lemma 2.7. [5] In **Act-S**, pushouts transfer strongly- s -dense monomorphisms. That is, for the following pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow h' \\ C & \xrightarrow{h} & Q \end{array}$$

we have h is strongly- s -dense if so is f .

An S -act A is said to be *strongly- s -dense injective* if for any strongly- s -dense monomorphism $g : B \rightarrow C$, any homomorphism $f : B \rightarrow A$ can be lifted to a homomorphism $\bar{f} : C \rightarrow A$. That is, the following diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ f \downarrow & \swarrow \bar{f} & \\ A & & \end{array}$$

An S -act A is said to be a *strongly- s -dense retract* of an its strongly- s -dense extension B , if there exists a morphism $f : B \rightarrow A$ such that $f|_A = id_A$. In this case, f is called a *strongly- s -dense retraction*. An S -act A is called an *absolute strongly- s -dense retract* if it is an strongly- s -dense retract of each of its strongly- s -dense extensions.

The following theorem shows that the s -compactness is equivalent to a kind of injectivity.

Theorem 2.8. *For an S -act A , the following assertions are equivalent:*

- (i) A is an s -compact S -act.
- (ii) A is absolute strongly- s -dense retract.
- (iii) A is strongly- s -dense retract of $A \cup \{b\}$, for every strongly- s -dense extension B of A and $b \in B$.
- (iv) A is strongly- s -dense injective.
- (v) A is absolutely f -pure.

Proof. (i) \Rightarrow (ii) Let A be a strongly- s -dense subact of B and $b \in B$. Then $bS \subseteq A$ and for every finite subset T of S there is an element $a_T \in A$ such that $a_T t = bt$ for all $t \in T$. Since A is s -compact, by Lemma 2.4, there exists $a_b \in A$ such that $\lambda_b = \lambda_{a_b}$. Now, for every $b \in B \setminus A$ choose and fix such an $a_b \in A$. Define $\pi : B \rightarrow A$ by

$$\pi(b) = \begin{cases} b, & \text{if } b \in A \\ a_b, & \text{if } b \notin A \end{cases}$$

Which shows that π is a retraction.

(ii) \Rightarrow (iii) This is trivial.

(iii) \Rightarrow (i) Let every finite subset of $\Sigma = \{xs = a_s \mid s \in S, a_s \in A\}$ have a solution in A . Then, by Lemma 2.5, there is an extension B of A and $b \in B$ which is a solution for Σ . Therefore, A is strongly- s -dense in $A \cup \{b\}$ which implies that there is a retraction $g : A \cup \{b\} \rightarrow A$. So for every $s \in S$, $g(b)s = g(bs) = g(a_s) = a_s$, which means that Σ has a solution $g(b)$ in A .

The equivalence of (ii) and (iv) flows from [7, Lemma 3.5(i)] and Lemma 2.7.

(i) \Rightarrow (v) Let B be an extension of A and $b \in B$ be a solution for a system $\Sigma = \{xs = a_s \mid s \in S, a_s \in A\}$ and also let Σ have a solution a_T for each finite subset T of S . Since A is s -compact, Σ has a solution in A . So A is f -pure in B .

(v) \Rightarrow (i) Let every finite subset Σ' of $\Sigma = \{xs = a_s \mid s \in S, a_s \in A\}$ have a solution in A . By Lemma 2.5, there is an extension B of A such that Σ has a solution $b \in B$. So, for every finite subset T of S there is an element $a_T \in A$ such that $a_T t = bt$ ($t \in T$) and $bS \subseteq A$. Thus Σ has a solution in A and hence it is an s -compact S -act. \square

Theorem 2.9. *A subact A of an s -compact S -act B is s -compact if and only if A is f -pure in B .*

Proof. Let B be an s -compact f -pure extension of A and each finite subsystem of $\Sigma = \{xs = a_s \mid s \in S, a_s \in A\}$ have a solution in A . Since B is s -compact, Σ has a solution $b \in B$ and since B is an f -pure extension of A , Σ has a solution in A .

Conversely, by Theorem 2.8, an s -compact S -act A is f -pure in B . \square

Corollary 2.10. *Every retract of an s -compact S -act is s -compact.*

Proof. Using Theorem 2.9, suffices to show that every retract of an S -act B is an f -pure subact of it B . Let A be a retract of B and $g : B \rightarrow A$, a homomorphism such that $g|_A = id_A$. Suppose that $\Sigma = \{xs = a_s \mid s \in S, a_s \in A\}$ has a solution b in B and each finite subset of Σ has a solution in A . So for every $s \in S$, $bs = g(bs) = g(b)s$. Then Σ has a solution $g(b)$ in A . \square

In the following, we give some classification results for semigroups based on s -compactness.

Lemma 2.11. *If S is finitely generated as an S -act, then every S -act is s -compact.*

Proof. Using Lemmas 2.4 and 2.5, the proof is clear. \square

The following definition motivated from the equivalency of (i) and (iv) in Theorem 2.8.

Definition 2.12. An S -act A is called *quasi s -compact* if every homomorphism $f : C \rightarrow A$ from a strongly- s -dense subact C of A can be extended to A .

Theorem 2.13. *Let S be a countably generated semigroup. Then the following assertions are equivalent:*

- (i) *All S -acts are s -compact.*
- (ii) *All direct sums of s -compact S -acts are s -compact.*
- (iii) *All countable direct sums of injective S -acts are s -compact.*
- (iv) *S is finitely generated as an S -act.*
- (v) *Each countable direct sum of each family of quasi s -compact S -acts is s -compact.*

Proof. The implications (i) \Rightarrow (ii), (ii) \Rightarrow (iii) and (i) \Rightarrow (v) are obvious. The assertion (iv) \Rightarrow (i) is obtained from Lemma 2.11. Let us show (iii) \Rightarrow (iv). Suppose that $S = \bigcup_{n \in \mathbb{N}} t_n S^1$. Take $I_n = t_1 S^1 \cup \dots \cup t_n S^1$, for $n \in \mathbb{N}$. Then we have an infinite chain of right ideals

$I_1 \subseteq \dots \subseteq I_n \subseteq \dots$, and $S = \bigcup_{n \in \mathbb{N}} I_n$. Consider the Rees factor acts S^1/I_n for $n \in \mathbb{N}$, and let $E_n(S^1/I_n)$ be the injective hull of S^1/I_n . Then $E = \bigoplus_{n \in \mathbb{N}} E_n(S^1/I_n)$ is s -compact by the hypothesis. Denote $a_n = \{t_n/I_i\}_{i \in \mathbb{N}} = (t_n/I_1, t_n/I_2, \dots, t_n/I_{n-1}, \theta, \theta, \theta, \dots) = (1/I_1, 1/I_2, \dots, 1/I_{n-1}, \theta, \theta, \theta, \dots)t_n = a'_n t_n$. The system $\Sigma = \{xt_i = a_i \mid i = 1, 2, \dots\}$ is finitely solvable in E . For this, let $\Sigma' = \{xt_i = a_i \mid i = 1, 2, \dots, k\}$ be a finite subset of Σ . For each $1 \leq i \leq k$, $a'_k t_i = (1/I_1, 1/I_2, \dots, 1/I_{k-1}, \theta, \theta, \dots)t_i = (t_i/I_1, t_i/I_2, \dots, t_i/I_{i-1}, \theta, \theta, \theta, \dots) = a_i$. So Σ' has a solution $a = (x_1, x_2, \dots, x_N, \theta, \theta, \dots) \in E$ for some $n \in \mathbb{N}$. Therefore, as every element in a direct sum has only finitely many non-zero coordinates, we have for every $i > N + 1$, $t_i \in I_{N+1}$. So, $S \subseteq I_{N+1}$, which gives $S = I_{N+1}$ is finitely generated as an S -act.

(v) \Rightarrow (iii) follows from the fact that each injective S -act is quasi s -compact. \square

Corollary 2.14. *For a semigroup S , if all S -acts are s -compact, then each countably generated right ideal of S as an S -act is finitely generated.*

By the following definition we have another homological classification of those S for which all S -acts are s -compact.

Definition 2.15. A semigroup S is said to has *local left identities* if for every finite subset T of S there is an element $y_T \in S$ such that for each $t \in T$, $y_T t = t$.

Example 2.16. Note that all monoids and all lattices as a semigroup (by the action given by $xy = x \wedge y$), in particular chains, have local left identities.

Also if each nonempty right ideal of S is generated by an idempotent element, then S has local left identities.

Theorem 2.17. *For a semigroup S , the following are equivalent:*

- (i) *Every S -act A is s -compact and S has local left identities.*
- (ii) *S is s -compact as an S -act and has local left identities.*
- (iii) *S has a left identity.*

Proof. (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (iii) Since S has local left identities, the restriction of the identity map $id : S \rightarrow S$ on any finite subset T is of the form λ_{a_T} . So, by Lemma 2.4, $id = \lambda_{s_0}$ for some $s_0 \in S$, which implies that s_0 is a left identity element of S .

(iii) \Rightarrow (i) Apply Lemma 2.11. \square

Corollary 2.18. *If S has local left identities, then the following are equivalent:*

- (i) S is an s -compact S -act.
- (ii) Every S -act A is s -compact.
- (iii) S has a left identity.

Proposition 2.19. *If S has local left identities, then every S -act A has an s -compact strongly- s -dense extension.*

Proof. Let B be an s -dense extension of A and T be a finite subset of S . Since S has local left identities, there is an element $y_T \in S$ such that $y_T t = t$ ($t \in T$). So for each $b \in B$, $bt = (by_T)t$ and $by_T \in A$. Then each s -dense extension is strongly- s -dense extension. Also it is clear that $S^2 = S$. hence, by [9, Theorem 3.10], each S -act has an s -dense injective s -dense extension. Since every s -dense injective is s -compact, the assertion holds. \square

3. CATEGORICAL PROPERTIES

In this section we consider the behaviour of s -compactness of S -acts with respect to products, coproducts, and direct sums. Lemma 2.4 will play a fundamental role to prove all the results of this section.

Theorem 3.1. *The product $\prod_{i \in I} A_i$ of S -acts is s -compact if and only if each A_i is s -compact.*

Proof. Necessity. Let $A = \prod_{i \in I} A_i$ be an s -compact S -act and for some $k \in I$, $f : S \rightarrow A_k$ be a map whose restriction to every finite subset T of S is of the form λ_a ($a \in A_k$). Consider the map $\bar{f} : S \rightarrow \prod_{i \in I} A_i$ defined by

$$\bar{f}(s)(i) = \begin{cases} f(s), & \text{if } i = k \\ a_i s, & \text{if } i \neq k \end{cases}$$

where for any $i \neq k$, $a_i \in A_i$ is chosen and fixed by the axiom of choice. Let T be a finite subset of S . For every $t \in T$,

$$\bar{f}(t)(i) = \begin{cases} f(t) = a_T t, & \text{if } i = k \\ a_i t, & \text{if } i \neq k \end{cases}$$

which means that $\bar{f}(t) = \lambda_{\{a_i\}}(t)$ ($a_k = a_T$). Since $\prod_{i \in I} A_i$ is s -compact, $\bar{f} = \lambda_{\{a_i\}}$ and thus $f = \lambda_{a_k}$.

Sufficiency. Let $f : S \rightarrow \prod_{i \in I} A_i$ be a map such that for every finite subset T of S , $f|_T$ is of the form λ_{a_T} ($a_T \in \prod_{i \in I} A_i$). So for every projection map $\pi_i : \prod_{i \in I} A_i \rightarrow A_i$, $\pi_i \circ f|_T$ is of the form λ_{a_T} ($a_T \in A_i$). Then $\pi_i \circ f = \lambda_{a_i}$ for some $a_i \in A_i$ and hence $f = \lambda_{\{a_i\}}$. \square

Theorem 3.2. *The coproduct $\coprod_{i \in I} A_i$ of S -acts is an s -compact S -act if and only if so is each A_i , $i \in I$.*

Proof. Necessity. Suppose that $f : S \rightarrow A_j$ is a map such that for every finite subset T of S there is an element $a_T \in A_j$ such that $f(t) = a_T t$ for each $t \in T$. For the j -th injection map $\tau_j : A_j \rightarrow \coprod_{i \in I} A_i$, $\tau_j f$ has the same property as f . So it is of the form λ_a for some $a \in \coprod_{i \in I} A_i$, by s -compactness of $\coprod_{i \in I} A_i$. Now we get $a \in A_i$, and $f = \lambda_a$.

Sufficiency. Let $\{A_i \mid i \in I\}$ be a family of s -compact S -acts and $f : S \rightarrow \coprod_{i \in I} A_i$ be a map such that its restriction to each finite subset T of S is of the form λ_{a_T} . Consider $s_1, s_2 \in S$ such that $f(s_1) \in A_m$ and $f(s_2) \in A_n$ ($m \neq n$). It is easily seen that, for a set $T = \{s_1, s_2\}$, $f|_T$ is not of the form λ_{a_T} . So there exists $k \in I$ such that $f(S) \subseteq A_k$ and since A_k is s -compact, $f = \lambda_a$ ($a \in A_k$). Therefore, $\coprod_{i \in I} A_i$ is an s -compact S -act. \square

For an S -act A and $a \in \text{Fix}(A)$, the one element S -act $\{a\}$ is s -compact. So the following corollary is straightforward.

Corollary 3.3. *For each S -act A , $\text{Fix}(A)$ is s -compact.*

In Theorems 3.1 and 3.2, there are equivalent conditions for product and coproduct of S -acts. But this situation does not hold for direct sums. In the following, we have investigated the behavior of direct sum about the s -compactness of a family of S -acts.

Theorem 3.4. *For a family $\{A_i \mid i \in I\}$ of S -acts each with a unique fixed element, if $\bigoplus_{i \in I} A_i$ is s -compact, then each A_j ($j \in I$) is s -compact.*

Proof. Let $f : S \rightarrow A_j$ be a map with the property as in Lemma 2.4. Then for the injection map $\sigma_j : A_j \rightarrow \bigoplus_{i \in I} A_i$, the map $\sigma_j f : S \rightarrow \bigoplus_{i \in I} A_i$ has the same property as f . So $\sigma_j f = \lambda_a$ for some $a \in \bigoplus_{i \in I} A_i$ and thus $f = \lambda_{a_j}$. \square

Proposition 3.5. *The following assertions are equivalent:*

- (i) *Each direct sum of s -compact S -acts is s -compact.*
- (ii) *Each direct sum of s -compact S -acts is f -pure in the direct product of them.*

Proof. (i) \Rightarrow (ii) Apply Theorem 2.8.

(ii) \Rightarrow (i) Applying Lemma 2.9 and Theorem 3.1, we get the result. \square

Proposition 3.6. *Let $\{A_i : i \in I\}$ be a family of s -compact S -acts each with at least one fixed element. The direct sum $\bigoplus_{i \in I} A_i$ is s -compact if and only if every map $S \rightarrow \bigoplus_{i \in I} A_i$ which its restrictions to all finite*

subsets are of the form λ_x , factors through a direct sum of finitely many A_i .

Proof. Let $\bigoplus_{i \in I} A_i$ be s -compact. Take a homomorphism $f : S \rightarrow \bigoplus_{i \in I} A_i$ with such property. By Lemma 2.4, there exists $a = (a_i)_{i \in I} \in \bigoplus_{i \in I} A_i$ such that $f = \lambda_a$. Consider a finite subset $J \subseteq I$ such that $a_i = 0$ for all $i \notin J$. Then, since $f(s) = as$ for each $s \in S$, f factors through the homomorphism $f : S \rightarrow \bigoplus_{i \in J} A_i \rightarrow \bigoplus_{i \in I} A_i$. The converse also holds by applying Lemma 2.4 and Theorem 3.1. \square

The following result generalizes Theorem 3.4, in the case of quasi s -compactness.

Theorem 3.7. *If a direct sum $\bigoplus_{i \in I} A_i$ of S -acts A_i is quasi s -compact, then so is each $A_i, i \in I$.*

Proof. Let $k \in I$, $h : B \hookrightarrow A_k$ be a strongly- s -dense monomorphism and $f : B \rightarrow A_k$ be a homomorphism. Then $h \bigoplus (\bigoplus_{i \neq k} id_{A_i}) : B \bigoplus (\bigoplus_{i \neq k} A_i) \rightarrow \bigoplus_{i \in I} A_i$ is a strongly- s -dense monomorphism, and $f \bigoplus (\bigoplus_{i \neq k} id_{A_i}) : B \bigoplus (\bigoplus_{i \neq k} A_i) \rightarrow \bigoplus_{i \in I} A_i$ is a homomorphism. So, by the assumption, there exists a homomorphism $g : \bigoplus_{i \in I} A_i \rightarrow \bigoplus_{i \in I} A_i$ extending $f \bigoplus (\bigoplus_{i \neq k} id_{A_i})$. Now, the homomorphism $p_k g \tau_k : A_k \rightarrow A_k$ extends f which $\tau_k : A_k \rightarrow \bigoplus_{i \in I} A_i$, is the k -th injection map and $p_k : \bigoplus_{i \in I} A_i \rightarrow A_k$ is the k -th projection map. \square

The converse of Theorem 3.4 does not hold in general, but it holds for finitely s -compact S -acts as follows.

Definition 3.8. An S -act A is said to be *finitely s -compact* if it is strongly- s -dense injective with respect to finitely generated strongly- s -dense subacts.

Theorem 3.9. *For a family $\{A_i \mid i \in I\}$ of S -acts each with a unique fixed element, $\bigoplus_{i \in I} A_i$ is finitely s -compact if and only if each $A_j (j \in I)$ is finitely s -compact.*

Proof. Sufficiency. Using the same argument as in the proof of Theorem 3.4, the assertion holds.

Necessity. It is not difficult to show that each homomorphism $F \rightarrow \bigoplus_{i \in I} A_i$ from a finitely generated S -act F , factors through a direct sum of finitely many A_i . Now since the direct sum of a finitely many S -acts is the direct product of them, by using [1, Proposition 10.40], the proof is complete. \square

Proposition 3.10. *For a semigroup S , all S -acts are finitely s -compact if and only if all finitely generated S -acts are s -compact.*

Proof. (\Rightarrow) Let A be a finitely generated S -act. By hypothesis, A is finitely s -compact. Since A is a finitely generated S -act, it is easy to check that A is an absolute strongly- s -dense retract. Now, by Theorem 2.8, A is an s -compact S -act.

(\Leftarrow) Let A be an S -act and $h : F \hookrightarrow B$ be a strongly- s -dense monomorphism for a finitely generated S -act F and $f : F \rightarrow A$ be a homomorphism. By hypothesis, F is s -compact and so a strongly- s -dense retract, by Theorem 2.8. The reminder is straightforward. \square

As a consequence of Theorems 3.1 and 3.4, we conclude the following theorem. First we recall the following definition.

Definition 3.11. An s -compact S -act A is called *countably \sum - s -compact* if every countable direct sum of A is s -compact.

Theorem 3.12. For countably generated semigroup S , the following assertions are equivalent:

- (i) All S -acts are s -compact.
- (ii) Each s -compact S -act is countably \sum - s -compact.

Proof. (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (i) Suppose that $\{E_n \mid n \in \mathbb{N}\}$ is a family of injective S -acts. By Theorem 2.8(iv), every injective S -act is s -compact and by Theorem 3.1, $A = \prod_{n \in \mathbb{N}} E_n$ is an s -compact S -act, and so by (ii), $\bigoplus_{n \in \mathbb{N}} A$ is also an s -compact S -act. Moreover, $A = \prod_{n \in \mathbb{N}} E_n = E_m \oplus \prod_{n \neq m} E_n$, for each $m \in \mathbb{N}$ and $\bigoplus_{m \in \mathbb{N}} A = \bigoplus_{m \in \mathbb{N}} (E_m \oplus \prod_{n \neq m} E_n) = (\bigoplus_{m \in \mathbb{N}} E_m) \oplus (\bigoplus_{m \in \mathbb{N}} \prod_{n \neq m} E_n)$. This facts together with Theorem 3.4 imply that $\bigoplus_{m \in \mathbb{N}} E_m$ is s -compact. Now Theorem 2.13(iii) completes the proof. \square

4. SOME BAER TYPE CRITERIAS FOR INJECTIVITY OF S -ACTS

Injectivity is one of the central notions in many branches of mathematics. One usually takes a subclass \mathcal{M} of monomorphisms in a category \mathcal{A} , members of which may be called \mathcal{M} -morphisms, and give the following definition.

An object A of \mathcal{A} is said to be \mathcal{M} -*injective* if for any \mathcal{M} -morphism $g : B \rightarrow C$, any morphism $f : B \rightarrow A$ can be lifted to a morphism $\bar{f} : C \rightarrow A$ of \mathcal{A} . That is, the following diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{g} & C \\ f \downarrow & \swarrow \bar{f} & \\ A & & \end{array}$$

One line of study in this regard is to investigate the relation between injectivity with respect to a subclass \mathcal{M}_1 of monomorphisms, which is

called \mathcal{M}_1 -injectivity, and injectivity with respect to another subclass \mathcal{M}_2 of monomorphisms, the results of which may be called the **Baer type criteria**. Note that if $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then \mathcal{M}_2 -injectivity implies \mathcal{M}_1 -injectivity. The Baer type problem is about the converse of this fact.

The object A is said to be an \mathcal{M} -retract of its \mathcal{M} -extension B if there exists a morphism $f : B \rightarrow A$ such that $f|_A = id_A$, in this case f is said to be an \mathcal{M} -retraction. An object A is called an *absolute \mathcal{M} -retract* if it is an \mathcal{M} -retract of each of its \mathcal{M} -extensions. When \mathcal{M} is the class of all monomorphisms, we do not mention \mathcal{M} and get the ordinary notions of injectivity, retract, and absolute retract.

Here we consider some classes of monomorphisms such as s -dense or strongly- s -dense monomorphisms instead of \mathcal{M} .

Consider \mathcal{M}_1 , the class of all strongly- s -dense monomorphisms and \mathcal{M}_2 , the class of all s -dense monomorphisms and \mathcal{M} , the class of all monomorphism. By Theorem 2.8, every \mathcal{M}_1 -injective S -act is s -compact. It is clear that $\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}$, thus every injective S -act is s -dense injective and every s -dense injective S -act is s -compact.

Here we give an example which is an s -compact S -act but it is not s -dense injective. By Corollary 3.3, the left zero semigroup S , as an S -act, is s -compact. We know that if S has more than two elements, then it has no left identity. Let S be an s -dense injective S -act. So the identity map $id_s : S \rightarrow S$ can be extended to $f : S^1 \rightarrow S$. Thus $f(1)$ is a left identity of S , which is a contradiction.

From now on, we use injectivity with respect to these classes of monomorphisms to give some Baer type results about injectivity. Note that each injective S -act A has a *fixed* element and injective hulls of S -acts always exist (see [11]).

The following results are the Baer type ones for $\mathcal{M}_1 \subseteq \mathcal{M}_2$.

Theorem 4.1. *If S has local left identities, then every s -compact S -act is s -dense injective.*

Proof. Let A be an s -compact S -act and $f : S \rightarrow A$ be a homomorphism. By Lemma 2.2, the set $\Sigma = \{xs = f(s) \mid s \in S\}$ has a solution b in an extension B of A . Since S has local left identities, for every finite subset T of S there is an element $y_T \in S$ such that $y_T t = t (t \in T)$. So for every $t \in T$, $(by_T)t = f(t)$ and $by_T = f(y_T) \in A$. Thus every finite subset of Σ has a solution in A . Since A is an s -compact S -act, Σ has a solution $a \in A$. Therefore, $f = \lambda_a$ for some $a \in A$ and by [9, Theorem 3.3], A is s -dense injective. \square

Proposition 4.2. *If every nonempty proper right ideal of S is generated by a central idempotent element, then every s -compact S -act with at least one fixed element is injective.*

Proof. By Example 2.16, S has local left identities. Now Theorems 4.1 and [4, Theorem 11] give the result. \square

An important semigroup satisfying proposition 4.2 is $S = (\mathbb{N}, \min)$. The category of all S -acts over this semigroup, so called *projection algebras*, has been studied by Giuli [10] and Ebrahimi and Mahmoud [12]. These types of S -acts are mostly used in computer science.

A semigroup S is called a *Clifford semigroup* (see, [11]) if each $\alpha \in S$ has a pseudoinverse element (there exists $\beta \in S$ such that $\alpha\beta\alpha = \alpha$) and the set of all idempotents is a subset of the set of all central elements.

It is easy to check that if β is a pseudoinverse of $\alpha \in S$, $\alpha\beta$ is an idempotent element and $\alpha S^1 = \alpha\beta S^1$. So the following corollary is straightforward.

Corollary 4.3. *Let S be a Clifford semigroup. If each proper nonempty right ideals of S is principal, then every s -compact S -act with at least one fixed element is injective.*

The following two classes of semigroups are special cases of Clifford semigroups.

- (i) Commutative chains with the order relation ($x \leq y \Leftrightarrow xy = x$) or ($x \leq y \Leftrightarrow xy = y$).
- (ii) Commutative bands (all elements of S are idempotents).

From now on, for a subact A of an S -act B and $b \in B$, consider $\bar{A} = \{b \in B \mid bS \subseteq A\}$ and $I_b = \{s \in S \mid bs \in A\}$. The subset \bar{A} is a subact of B and I_b is a right ideal of S . It is clear that $A \subseteq \bar{A}$ and if $A_1 \subseteq A_2$, then $\bar{A}_1 \subseteq \bar{A}_2$. Also for each homomorphism $f : B \rightarrow C$ and $A \subseteq B$, $f(\bar{A}) \subseteq \overline{f(A)}$. These properties show that the map $C : Sub(B) \rightarrow Sub(B)$ defined by $C(A) = \bar{A}$ is a closure operator.

Definition 4.4. An S -act B is an *essential extension* of A , if each homomorphism $f : B \rightarrow C$ is a monomorphism, whenever $f|_A$ is a monomorphism.

Theorem 4.5. *If S has local left identities and for every nontrivial right ideal I of S , $\bar{I} \neq I$, then every s -compact S -act is injective.*

Proof. Let A be an s -compact S -act, B be an essential extension of A and $b \in \bar{A} \setminus A$. It is clear that $A \cup \{b\}$ is an essential extension of

A . Since S has local left identities, for every finite subset T of S there exists $y_T \in S$ such that $y_T t = t$. So $bt = (by_T)t$ which $by_T \in A$ and hence $A \cup \{b\}$ is a strongly- s -dense extension of A . It follows from Theorem 2.8 that there is a homomorphism $g : A \cup \{b\} \rightarrow A$ with $g|_A = id_A$, which implies that g is an isomorphism. Thus $b \in A$ and hence $\bar{A} = A$.

Now let $b \in B \setminus A$. If $I_b = S$, then $b \in \bar{A} = A$ which is a contradiction. Thus $I_b \neq S$ and by hypothesis $\bar{I}_b \neq I_b$. For each $s \in \bar{I}_b \setminus I_b$, it is not difficult to show that $bs \in \bar{A} = A$ and hence $s \in I_b$ which is a contradiction. So $B = A$ and [11, Proposition III.1.20] deduced that A is an injective S -act. \square

Recall that an S -act A is called *quasi injective* if every homomorphism $f : B \rightarrow A$ from a subact B of A can be extended to A .

Theorem 4.6. *Every s -compact S -act is injective if and only if every s -compact S -act is quasi injective and has a fixed element.*

Proof. We show only the unclear direction. Let A be an s -compact S -act and $E(A)$ be the injective hull of A . Choose and fix elements $0 \in Fix(A)$ and $\theta \in Fix(E(A))$. It is clear that two S -acts A and $\{0\} \times A$ are isomorphic. Consider $\{0\} \times A \xrightarrow{\tau_1} \{0\} \times E(A) \xrightarrow{\tau_2} A \times E(A)$, where τ_1 and τ_2 are inclusions and define the map $\tau_A : \{0\} \times A \rightarrow A \times E(A)$, by $\tau_A(0, a) = (a, \theta)$. Since every injective S -act is s -compact, by Theorem 3.1, the S -act $A \times E(A)$ is s -compact and hence it is quasi injective. So there exists a homomorphism $g : A \times E(A) \rightarrow A \times E(A)$ such that $g\tau_2\tau_1 = \tau_A$. For the homomorphism $p_A : A \times E(A) \rightarrow \{0\} \times A$, defined by $p_A(a, b) = (0, a)$, we have $p_A g \tau_2 \tau_1(0, a) = p_A \tau_A(0, a) = (0, a)$ and hence $\{0\} \times A$ is a retract of the injective S -act $\{0\} \times E(A)$. Thus A is injective. \square

Theorem 4.7. *An S -act A is injective if and only if it is s -compact as well as f -pure injective.*

Proof. Let A be an s -compact and f -pure injective S -act and $E(A)$ be the injective hull of A . Since A is s -compact, by Theorem 2.8, it is f -pure subact of $E(A)$ and hence it is a retract of $E(A)$, which implies that A is injective. \square

Theorem 4.8. *If every essential extension of any S -act A is strongly s -dense extension, then every s -compact S -act is injective.*

Proof. Let A be an s -compact S -act and $\tau : A \rightarrow E(A)$ be the injective hull of A . Since τ is essential, it is strongly s -dense extension and since A is s -compact, by Theorem 2.8, it is strongly s -dense injective. So

there exists a homomorphism $g : E(A) \rightarrow A$ such that $g\tau = id_A$, which is a monomorphism by essentiality of τ . furthermore, $g\tau g = id_{Ag} = g = gid_{E(A)}$ which implies that $\tau g = id_{E(A)}$. Thus τ is an isomorphism and hence A is injective. \square

Acknowledgments

I would like to express my appreciation to the referee for carefully reading and usefull suggestions on the paper.

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SEQUENTIALLY COMPACT S -ACT

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فشردگی معادله‌ای در رسته S - سیستم‌ها

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مطالعات در مورد فشردگی معادله‌ای یک جبر را در ابتدا بناشفسکی و نلسون آغاز کردند. آنها نشان دادند که در جبرهای جامع مفاهیم خلوص انژکتیوی و فشردگی معادله‌ای هم ارز هستند. در اینجا ما مفهوم فشردگی دنباله‌ای روی یک نیمگروه S را معرفی کرده و برخی از خواص رسته‌ای و همولوژیکی آن را در رسته S سیستم‌ها مورد مطالعه قرار می‌دهیم. در نهایت تعدادی محک بئر برای انژکتیوی نیز بیان شده است.

کلمات کلیدی: فشردگی معادله‌ای، f -خالص انژکتیوی، سیستم انژکتیو.