

## MAXIMAL PRYM VARIETY AND MAXIMAL MORPHISM

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ABSTRACT. We investigated maximal Prym varieties on finite fields by attaining their upper bounds on the number of rational points. This concept gave us a motivation for defining a generalized definition of maximal curves i.e., maximal morphisms. By MAGMA, we give some non-trivial examples of maximal morphisms that results in non-trivial examples of maximal Prym varieties.

### 1. INTRODUCTION

The problem of counting the number of rational points of algebraic varieties defined over finite fields absorbs the attention of many researchers, because of their various applications in other branches of science and technology, such as Information Theory, Coding Theory, Cryptography and Physics.

A. Lesfari in [7] had shown the application of Prym varieties. The Kirchhoff's equation that describes the motion of a solid body can be modeled by genus two hyperelliptic functions.

Many papers have published about this important thought of point counting [4, 5, 9, 10, 13, 14]. Our goal is to study the number of rational points of Prym varieties. This concept has studied recently by M. Perret in [8].

Let  $\pi : D \mapsto C$  be a covering of smooth algebraic irreducible projective curves defined over a field  $k$  of zero or odd character then the

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Jacobian  $J_C$  of  $C$  is isogenous to a sub-abelian variety of the Jacobian  $J_D$  of  $D$ . If, moreover, we suppose that  $\pi$  has degree 2 then the non-trivial involution  $\sigma$  of this covering  $\sigma$  induces an involution  $\sigma^*$  on  $J_D$ .

**Definition 1.1.** The Prym variety  $\text{Pr} = \text{Pr}_\pi$  associated to the unramified double covering  $\pi : D \mapsto C$  of a curve  $C$  of genus  $g \geq 2$  is defined as  $\text{Pr} := \text{Im}(\sigma^* - id)$ . It is an abelian subvariety of  $J_D$  of dimension  $g - 1$  isogenous to a direct factor of  $J_C$  in  $J_D$ .

Suppose henceforth that  $k$  is the finite field  $\mathbb{F}_q$  with  $q$  elements,  $\text{Pr}$  is an abelian variety of dimension  $g - 1$ . There is following important question:

*When does Prym variety defined over finite fields reach its upper bound on the number of rational points?*

This paper contains contributions of defining maximal morphisms and some results about this concept that is a generalization of maximal curves. After, we have found some non-trivial examples of maximal morphisms. This new concept is useful for obtaining examples of maximal Prym varieties.

The remainder of this paper is organized as follows: Section 2 describes the previous literature related to the problem. Suitable definitions to formulate the new concept, maximal morphisms, are presented in Section 3, which is followed in Section 4 by some non-trivial examples that calculated by MAGMA. Section 5 provides some final conclusions.

## 2. RELATED WORKS

The following result of Weil formulates the bounds for the number of rational points of an algebraic variety defined over a finite field (see, [14]).

**Theorem 2.1** (Weil, 1948). *Let  $A$  be an abelian variety of dimension  $d$  defined over  $\mathbb{F}_q$ . Then, there exists  $\theta_1, \dots, \theta_d \in \mathbb{R}/(2\pi\mathbb{Z})$  such that for any  $n \geq 1$  the number of rational points of  $A$  over  $\mathbb{F}_{q^n}$ , is given by*

$$(i) \text{ card } A(\mathbb{F}_{q^n}) = \prod_{i=1}^d (q^n + 1 - 2\sqrt{q^n} \cos \theta_i) \text{ in particular,}$$

$$(ii) (q + 1 - 2\sqrt{q})^d \leq \text{card } A(\mathbb{F}_q) \leq (q + 1 + 2\sqrt{q})^d,$$

(iii) *If in addition,  $A$  is the Jacobian of a curve  $C$  of genus  $g$ , then  $d = g$  and the  $\theta_i$  are also related to the  $J_C$ , then the number of*

rational points of  $C$  over  $\mathbb{F}_{q^n}$ , is given by

$$\text{card } C(\mathbb{F}_{q^n}) = q^n + 1 - 2\sqrt{q^n} \left( \sum_{i=1}^g \cos n\theta_i \right).$$

The second part of Theorem 2.1 for the Prym variety  $\text{Pr}_\pi$  of a double unramified cover  $\pi$  of a curve  $C$  of genus  $g$  reads

$$(g + 1 - 2\sqrt{q})^{g-1} \leq \text{card } \text{Pr}(\mathbb{F}_q) \leq (g + 1 + 2\sqrt{q})^{g-1}. \quad (2.1)$$

These upper and lower bounds in (2.1) are the “best possible”, in the sense that both can be reached. Indeed, it is known that an elliptic curve is a Prym variety. Now, suppose that  $E$  is so that it reaches the upper (resp., lower) bound of Weil’s inequality, such an elliptic curve exists if  $q$  is square [8], then  $E$  reaches the upper (resp., lower) bound of (2.1). The existence of such an elliptic curve  $E$ , proves also the second part of (Theorem 2.1) for the Jacobian variety  $J_C$  of a curve  $C$ ,

$$(q + 1 - 2\sqrt{q})^g \leq \text{card } J_C(\mathbb{F}_q) \leq (q + 1 + 2\sqrt{q})^g, \quad (2.2)$$

is also best possible at least for  $g = 1$ . Several sharper lower and upper bounds for Jacobian were also given in the literature, for instance:

**Theorem 2.2** (G. Lachaud, M. Martin-Dechamps [6]). *Let  $J_C$  be the Jacobian variety of a genus  $g$  of a curve  $C$  defined over  $\mathbb{F}_q$  and  $\text{card } C(\mathbb{F}_q)$  be the number of rational points of  $C$ . Then*

$$(\sqrt{q} - 1)^2 \frac{(q^{g-1} - 1)(\text{card } C(\mathbb{F}_q) + q - 1)}{g(q - 1)} \leq \text{card } J_C(\mathbb{F}_q). \quad (2.3)$$

If  $C$  admits a map of degree  $n$  onto the projective line, then one has also

$$\text{card } J_C(\mathbb{F}_q) \leq \frac{e}{q} (2g\sqrt{e})^{n-1} q^g. \quad (2.4)$$

In [8], Marc Perret presented some bounds for Prym variety. Let  $C$  be an algebraic curve and the number of  $\mathbb{F}_q$ -rational points of  $C$  denoted by  $\text{card } C(\mathbb{F}_q)$ .

**Theorem 2.3.** *Let  $C$  be an absolutely irreducible projective smooth algebraic curve defined over the finite field  $k$  of the odd characteristic with  $q$  elements. Let also  $g$  be the genus of  $C$  and  $\pi : D \rightarrow C$  be an unramified covering of degree 2. Then,*

$$(i) \quad \left( \frac{\sqrt{q} + 1}{\sqrt{q} - 1} \right) \left( \frac{\text{card } D(\mathbb{F}_q) - \text{card } C(\mathbb{F}_q)}{(2\sqrt{q})} - 2\delta \right) (q - 1)^{g-1} \leq \text{card } \text{Pr}(\mathbb{F}_q)$$

with

$$\delta = \begin{cases} 1 & \text{if } \frac{\text{card } D(\mathbb{F}_q) - \text{card } C(\mathbb{F}_q)}{2\sqrt{q}} \in \mathbb{Z}; \\ 0 & \text{otherwise.} \end{cases}$$

(ii)

$$\text{card Pr}(\mathbb{F}_q) \leq \left( q + 1 \frac{\text{card } D(\mathbb{F}_q) - \text{card } C(\mathbb{F}_q)}{g-1} \right)^{g-1}.$$

*Remark 2.4.* Let  $X$  be a curve with genus  $g$  defined over  $\mathbb{F}_q$ . There is a formal series over  $X$  relative to  $\mathbb{F}_q$  called *Zêta Function*:

$$Z_{X,q}(t) := \exp \left( \sum_{i=1}^{\infty} \frac{\text{card } X(\mathbb{F}_{q^i})}{i} t^i \right). \quad (2.5)$$

There exists a polynomial of degree  $2g$  with integer coefficients, such that

$$Z_{X,q}(t) = \frac{P(t)}{(1-t)(1-qt)}. \quad (2.6)$$

*Remark 2.5.* [12]

(i) Let

$$P(t) = \sum_{i=0}^{2g} a_i t^i, \quad (2.7)$$

then  $a_0 = 1$ ,  $a_{2g} = q^g$  and  $a_{2g-i} = q^{g-i} a_i$  for  $i = 0, \dots, g$ .

(ii) Let

$$h(t) = h_{X,q}(t) := t^{2g} P(t^{-1}). \quad (2.8)$$

Then, the  $2g$  roots (counted with multiplicity)  $\alpha_1, \dots, \alpha_{2g}$  of  $h(t)$  can be arranged such that  $\alpha_j \alpha_{g+j} = q$  for  $j = 1, \dots, g$ . Note that,  $a_1 = -\sum_{j=1}^{2g} \alpha_j$ .

Now, let  $J_X$  be the Jacobian of  $X$ . According to Theorem 2.1, we know that  $h(t)$  is exactly the characteristic polynomial of  $\text{Fr } J_X$  on the Tate module, where  $\text{Fr}$  is the Frobenius endomorphism (relative to  $\mathbb{F}_q$ ).

Now, let  $C$  and  $D$  be two curves and  $\pi : D \mapsto C$  a unramified double covering. We know that the Prym variety  $\text{Pr}_\pi$  is a direct factor of  $J_C$  in  $J_D$ .

**Theorem 2.6** (Y. Aubry, M. Perret [1]). *Let  $\pi : D \mapsto C$  be a finite morphism between two reduced algebraic smooth absolutely irreducible projective curves  $D$  and  $C$  defined over a finite field  $k$ . Then, the numerator of the zeta function of  $C$  divides that of  $D$  in  $\mathbb{Z}[t]$ .*

*Proof.* For any prime  $\ell$  different to the characteristic of  $\mathbb{F}_q$ , we consider the  $\mathbb{Q}_\ell$ -vector space  $T_\ell(J_C) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$  of dimension  $2g_C$ , where  $T_\ell(J_X)$  is the Tate module of Jacobian  $J_C$  of  $C$ . The numerator  $h_{C,q}(t)$  of the zeta function of  $C$  is the reciprocal characteristic polynomial of Frobenius endomorphism on  $T_\ell(J_C) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ . The function

$$\pi^* : J_C \longrightarrow J_D,$$

which is induced by  $\pi$  on Jacobian, is of the finite kernel, and sends every point of  $\ell^n$ -torsion of  $J_C$  on points of  $\ell^n$ -torsion of  $J_D$ . So, we have a morphism injective of  $\mathbb{Q}_\ell$ -vector spaces

$$\pi^* \otimes 1 : T_\ell(J_C) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \longrightarrow T_\ell(J_D) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

The Frobenius morphism in  $T_\ell(J_D) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$  is a stable subspace of the Frobenius morphism in  $T_\ell(J_C) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ . Therefore, the characteristic polynomial of  $T_\ell(J_C)$  divides the characteristic polynomial of  $T_\ell(J_D)$  in  $\mathbb{Q}[t]$ , and hence in  $\mathbb{Z}[t]$ , since  $h_C, h_D \in \mathbb{Z}[t]$ . Thus, we conclude that  $h_C(t)$  divides  $h_D(t)$ .  $\square$

### 3. RELATIVE MAXIMAL MORPHISMS

In this section, we give the definition of maximal morphism, which generalizes the concept of famous maximal curve.

A. Weil [14], had shown that the number of  $\mathbb{F}_q$ -rational points of the curve  $C$  of genus  $g$  satisfies

$$q + 1 - 2g\sqrt{q} \leq \text{card } C(\mathbb{F}_q) \leq 1 + q + 2g\sqrt{q}.$$

J. P. Serre [9, 10], improved Weil's upper bound. If  $q$  is not square, then

$$\text{card } C(\mathbb{F}_q) \leq q + 1 + g[2\sqrt{q}].$$

A curve is called *Weil (Serre) maximal* if it gets the Weil (Serre) upper bound of  $\mathbb{F}_q$ -rational points. Many works have published about maximal curves and related concepts [4, 5, 9, 10, 13, 14].

**Corollary 3.1.** *If  $\pi : D \mapsto C$  is a surjective morphism between irreducible smooth projective algebraic curves on a finite field  $\mathbb{F}_q$ , then*

$$|\text{card } D(\mathbb{F}_q) - \text{card } C(\mathbb{F}_q)| \leq 2(g_D - g_C)\sqrt{q}.$$

*Proof.* According to Theorem 2.6, all eigenvalues of Frobenius endomorphism on  $T_\ell(J_D)$  with multiplicity, contains that of Frobenius endomorphism on  $T_\ell(J_C)$ . We then apply Theorem 2.1 (iii), and the result follows.  $\square$

**Proposition 3.2.** *If  $\pi : D \mapsto C$  is a surjective morphism between irreducible smooth projective algebraic curves on a finite field, then*

$$|\text{card } D(\mathbb{F}_q) - \text{card } C(\mathbb{F}_q)| \leq (g_D - g_C) [2\sqrt{q}].$$

*Proof.* Let  $A \subset \mathbb{C}$  be the set of algebraic integers, i.e., a complex number  $\alpha$  is in  $A$  if and only if  $\alpha$  satisfies in equation  $\alpha^m + b_{m-1}\alpha^{m-1} + \dots + b_1\alpha + b_0 = 0$  for coefficients  $b_i \in \mathbb{Z}$ . It is an elementary fact of algebraic number theory that

$$A \text{ is subring of } \mathbb{C}, \text{ and } A \cap \mathbb{Q} = \mathbb{Z}. \quad (3.1)$$

We consider the  $L$ -polynomial  $L_D(t) = \prod_{i=1}^{2g_D} (1 - \alpha_i t)$  and  $L_C(t) = \prod_{i=1}^{2g_C} (1 - \beta_i t)$ . Complex numbers  $\alpha_1, \dots, \alpha_{2g_D}$  are the algebraic integers with  $|\alpha_i| = q^{1/2}$  (Theorems V.2.1 and V.1.15 of [12]). They can be arranged so that  $(\alpha_1, \dots, \alpha_{2g_C}, \alpha_{2g_C+1}, \dots, \alpha_{2g_D}) = (\beta_1, \dots, \beta_{2g_C}, \beta_{2g_C+1}, \dots, \beta_{2g_D})$  and  $\alpha_i \alpha_{g_D+i} = q$ . Therefore,

$$\bar{\alpha}_i = \alpha_{g_D+i} = q/\alpha_i \text{ for } 1 \leq i \leq g_D.$$

(We denote by  $\bar{\alpha}$  the complex conjugate of  $\alpha$ .) Let

$$\gamma_i := \alpha_i + \bar{\alpha}_i + [2q^{1/2}] + 1$$

and

$$\delta_i := -(\alpha_i + \bar{\alpha}_i) + [2q^{1/2}] + 1.$$

According to (3.1),  $\gamma_i$  and  $\delta_i$  are algebraic integers and since  $|\alpha_i| = q^{1/2}$ , they satisfy

$$\gamma_i > 0, \delta_i > 0. \quad (3.2)$$

Let  $\prod_{i=1}^{2g_D} (t - \alpha_i) = L_D^\perp(t) \in Z(t)$ ,  $\prod_{i=1}^{2g_C} (t - \beta_i) = L_C^\perp(t) \in Z(t)$  and  $L_D^\perp(t) = L_C^\perp(t) \prod_{i=1}^{2g_D-2g_C} (t - \alpha_i)$ . Any extension

$$\sigma : \mathbb{Q}(\alpha_1, \dots, \alpha_{2g_D-2g_C}) \rightarrow \mathbb{C},$$

permute  $\alpha_1, \dots, \alpha_{2g_D-2g_C}$ , since  $\prod_{i=1}^{2g_D-2g_C} (t - \alpha_i) \in Z(t)$ . On the other hand, if  $\sigma(\alpha_i) = \alpha_j$  then,

$$\sigma(\bar{\alpha}_i) = \sigma(q/\alpha_i) = q/\sigma(\alpha_i) = \overline{\sigma(\alpha_i)} = \bar{\alpha}_j.$$

Therefore,  $\sigma$  acts as a permutation over the sets  $\{\gamma_1, \dots, \gamma_{g_D-g_C}\}$  and  $\{\delta_1, \dots, \delta_{g_D-g_C}\}$ . Define

$$\gamma := \prod_{i=1}^{g_D-g_C} \gamma_i \text{ and } \delta := \prod_{i=1}^{g_D-g_C} \delta_i.$$

Then,  $\gamma$  and  $\Delta$  are algebraic integers invariants in all extensions of  $Q(\alpha_1, \dots, \alpha_{2g_D-2g_C})$  in  $C$ . Therefore,  $\gamma, \delta \in \mathbb{Q} \cap A = \mathbb{Z}$ . With (3.2),  $\gamma > 0$  and  $\delta > 0$ , and hence

$$\prod_{i=1}^{g_D-g_C} \gamma_i \geq 1 \text{ and } \prod_{i=1}^{g_D-g_C} \delta_i \geq 1.$$

The well-known inequality between arithmetic and geometry gives

$$\frac{1}{(g_D - g_C)} \sum_{i=1}^{g_D-g_C} \gamma_i \geq \left( \prod_{i=1}^{g_D-g_C} \gamma_i \right)^{1/(g_D-g_C)} \geq 1.$$

Therefore,

$$\begin{aligned} g_D - g_C &\leq \left( \sum_{i=1}^{g_D-g_C} (\alpha_i + \bar{\alpha}_i) \right) + (g_D - g_C)[2q^{1/2}] + g_D - g_C \\ &= \sum_{i=1}^{2g_D-2g_C} \alpha_i + (g_D - g_C)[2q^{1/2}] + g_D - g_C. \end{aligned}$$

By observing that  $\sum_{i=1}^{2g_D-2g_C} \alpha_i = N$ , we get

$$N \leq (g_D - g_C)[2q^{1/2}].$$

In the same way, inequality

$$\frac{1}{(g_D - g_C)} \sum_{i=1}^{g_D-g_C} \delta_i \geq \left( \prod_{i=1}^{g_D-g_C} \delta_i \right)^{1/(g_D-g_C)} \geq 1,$$

gives

$$N \geq -(g_D - g_C)[2q^{1/2}].$$

□

*Remark 3.3.* Since  $\text{Pr}$  has dimension  $g - 1$ , we already saw in the Theorem 2.1 that

$$(q + 1 - 2\sqrt{q})^{g-1} \leq \text{card } \text{Pr}(\mathbb{F}_q) \leq (q + 1 + 2\sqrt{q})^{g-1}.$$

On the other hand, we know  $\text{card } J_C(\mathbb{F}_q) = \prod_{\omega \in \text{Spec } \mathbb{F}(C)} (1 - \omega)$  and  $|\omega| = 2\sqrt{q}$  for all  $\omega \in \text{Spec } \mathbb{F}(C)$ . So,

$$-2(g-1)\sqrt{q} \leq \text{card } D(\mathbb{F}_q) - \text{card } C(\mathbb{F}_q) \leq 2(g-1)\sqrt{q}.$$

Hence, our bounds in theorem are dependent to  $\text{card } D(\mathbb{F}_q) - \text{card } C(\mathbb{F}_q)$  and are always “better” than Weil’s one (in the sense that, for instance, our upper bound is smaller than Weil’s one).

Hence, if  $\text{card } D(\mathbb{F}_q) - \text{card } C(\mathbb{F}_q)$  reaches the upper bound, then  $\text{Pr}(\mathbb{F}_q)$  will have the maximal number of rational points if Prym variety is in the Jacobian form.

**Definition 3.4.** A Prym variety  $\text{Pr}_\pi$  is maximal if it attains the upper bound of the number of rational points, where  $\pi$  is a double unramified covering between two curves.

*Remark 3.5.* We know  $\text{Pr}_\pi$  is an abelian subvariety of  $J_D$ . Historically, Prym varieties were considered interesting exclusively, because they give examples of principally polarized abelian varieties that are not Jacobian varieties. However, if  $\dim(\text{Pr}_\pi) \leq 2$ , then  $\text{Pr}_\pi$  generally is a Jacobian variety (see, [3]).

**Definition 3.6.** Let  $\pi : D \mapsto C$  is a double unramified covering and  $C, D$  are two smooth, irreducible curves. We say  $\pi$  is a Weil maximal morphism on  $\mathbb{F}_q$  if  $\text{card } D(\mathbb{F}_q) - \text{card } C(\mathbb{F}_q) = 2(g-1)\sqrt{q}$ . Similarly, we can define Serre maximal morphism (optimal maximal morphism).

**Definition 3.7.** Let  $\pi : D \mapsto C$  be a unbranched double covering between two smooth, projective, irreducible curves. We say that  $\pi$  is a *Serre-maximal morphism* over  $\mathbb{F}_q$ , if

$$\text{card } D(\mathbb{F}_q) - \text{card } C(\mathbb{F}_q) = (g-1)[2\sqrt{q}].$$

**Proposition 3.8.** *Let  $\text{Pr}_\pi \cong \text{Jac}(E)$ . If  $\pi$  is a Weil maximal morphism, then  $\text{Pr}_\pi$  is also Weil maximal.*

*Proof.* We know that  $\text{Pr}_\pi$  is a direct factor of  $J_C$  in  $J_D$  and the number of eigenvalues of  $\text{Fr}_{\text{Pr}_\pi}$  is exactly  $2(g-1)$ . Hence,  $\text{Pr}_\pi$  is maximal if and only if each proper value  $\omega_i$  verifies  $\omega_i = -\sqrt{q}$ . This property is equivalent to the maximality of  $\pi$ .  $\square$

**Theorem 3.9.** *Suppose that  $\text{Pr}_\pi \cong \text{Jac}(F)$  for a curve  $F$ . Then, the following assertions are equivalent:*

- (i)  $\pi$  is Weil-maximal;
- (ii)  $\text{Pr}_\pi$  is Weil-maximal;
- (iii)  $F$  is Weil-maximal.

*Proof.* (i) $\Leftrightarrow$ (ii): We know that  $\text{Pr}_\pi$  is a direct factor of  $J_C$  in  $J_D$  and the number of eigenvalues of  $\text{Fr}_{\text{Pr}_\pi}$  with multiplicity is exactly  $2(g-1)$ . Therefore,  $\text{Pr}_\pi$  is maximal if and only if each proper value  $\omega_i$  verifies  $\omega_i = -\sqrt{q}$ . This property is equivalent to the maximality of  $\pi$ .

(ii) $\Leftrightarrow$ (iii): In fact, we must prove also  $\text{Jac}(F)$  is maximal if and only if  $F$  is maximal. Thanks to Theorem (i), (iii) we have  $\theta_i = \pi$  for all  $i$ .  $\square$

**Theorem 3.10.** *Suppose that  $\text{Pr}_\pi \cong \text{Jac}(F)$  for a curve  $F$ . Then, the following assertions are equivalent:*

- (i)  $\pi$  is Serre-maximal;
- (ii)  $F$  is Serre-maximal.

*Proof.* (i) $\Rightarrow$ (ii):  $\pi$  is Serre maximal if  $|\text{Tr}_{\text{Fr}_{\text{Pr}_\pi}}| = (g-1)[2\sqrt{q}]$ . But  $\text{Pr}_\pi \cong \text{Jac}(F)$ , so  $|\text{Tr}_{\text{Fr}_{\text{Jac}_F}}| = (g-1)[2\sqrt{q}]$ . With [9, Theorem 1], we have equality if and only if the characteristic polynomial of  $\text{Fr}_{\text{Pr}_\pi}$  is equal to  $(X^2 \pm mX + q)^{g-1}$ , where  $m = [2\sqrt{q}]$ . Then every eigenvalue of Tate matrix  $\text{Fr}_{\text{Jac}_F}$  is equal to  $[\sqrt{q}]$ .

(ii) $\Rightarrow$ (i): If  $F$  is Serre maximal, then  $|\text{Tr}_{\text{Fr}_{\text{Jac}_F}}| = (g-1)[2\sqrt{q}]$ . But we know that  $\text{Pr}_\pi \cong \text{Jac}(F)$  and  $|\text{Tr}_{\text{Fr}_{\text{Pr}_\pi}}| = (g-1)[2\sqrt{q}]$ , then  $\pi$  is Serre maximale.  $\square$

**Proposition 3.11.** *If  $D$  is a Weil maximal curve, then  $C$  is Weil maximal and every double covering  $\pi : D \mapsto C$  is Weil maximal.*

*Proof.* With the maximality of  $D$ , we know that all eigenvalues of  $T_\ell(J_D)$  are equal to  $-\sqrt{q}$ . Then, using Theorem 3.9, we get the desired result.  $\square$

*Remark 3.12.* The Proposition 3.11 will be correct if we replace Weil maximal morphism with *Serre maximal morphism* or *optimal morphism*.

*Remark 3.13.* If  $C$  or  $D$  is a maximal curve, then any unramified double covering  $\pi : D \mapsto C$  is maximal morphism. Hence, this case gives the trivial examples of maximal morphisms.

#### 4. SOME NON-TRIVIAL EXAMPLES

Are there some non-trivial maximal morphisms or equivalently maximal Prym varieties in Jacobian form?

In this section, we try to give some non-trivial explicit examples by the result of [3], and by MAGMA [2] (a software package designed to solve the computationally hard problem in algebra, ...).

In fact, with [3] in characteristic zero, we have some cases that Prym varieties are isomorph with Jacobian varieties. In fact, [3] gives examples of isomorphisms of Prym varieties that are Jacobian varieties.

Let  $C$  be the double covering of  $\mathbb{P}^1$ . Then,  $C$  has an affine model

$$C : y^2 = f(x),$$

where  $f \in k[x]$  is a square-free polynomial of degree  $2g_C + 2$ . According to *Kummer's theory*, for any factorization  $f = F_1 F_2$ , with  $F_1, F_2 \in k[x]$

of an even degree, we have a curve  $D$  which is given by the affine model:

$$\begin{array}{ccccc}
 & & D & & \\
 & \swarrow \pi_1 & \downarrow \pi & \searrow \pi_2 & \\
 F_1 & & C & & F_2 \\
 & \searrow x & \downarrow x & \swarrow x & \\
 & & \mathbb{P}^1 & & 
 \end{array}$$

$$\begin{cases} y_1^2 = F_1(x) \\ y_2^2 = F_2(x), \end{cases} \quad (4.1)$$

and a unramified morphism of degree 2

$$\pi : D \mapsto C$$

$$(x, y_1, y_2) \mapsto (x, y_1 y_2) = (x, y).$$

Consider two curves

$$F_1 : y_1^2 = F_1(x)$$

and

$$F_2 : y_2^2 = F_2(x),$$

with the obvious projections  $\pi_1 : D \rightarrow F_1$  and  $\pi_2 : D \rightarrow F_2$ .

**Proposition 4.1.** [3] *Let  $C, D, F_1, F_2, \pi, \pi_1, \pi_2$  be as defined above. Then,*

$$\pi_1^* \times \pi_2^* : \text{Jac}(F_1) \times \text{Jac}(F_2) \rightarrow \text{Prym}(D/C),$$

*is an isomorphism of abelian varieties.*

*Remark 4.2.* By letting  $\deg F_1(x) = 2$  in Proposition 4.1, we have  $F_1 : y_1^2 = F_1(x)$  is of genus 0. Then,  $F_1 \cong P_1$ , and therefore

$$\text{Jac}(F_1) \times \text{Jac}(F_2) \cong \text{Jac}(F_2) \cong \text{Prym}(D/C).$$

*Remark 4.3.* Consider the affine forms of curves  $C, D$  and  $F$ . Suppose that  $C : y^2 = Q(x)R(x)$ , where  $\deg(Q) = 2, \deg(R) = 6$  and  $D : y_1^2 = Q(x)$  and  $y_2^2 = R(x), F : y^2 = R(x)$ . Consider  $\pi : D \mapsto C$  is double unramified covering then  $\text{Pr}_\pi \cong \text{Jac}(F)$ , so for obtaining nontrivial examples of maximal morphisms we have to find some examples such that  $D, C$  are not maximal curves. However,  $F$  is a maximal curve with MAGMA, we can attain some examples of nontrivial morphisms. Now, we give an example that we calculated with MAGMA.

Consider the field  $\mathbb{F}_{81}$  and let  $R(x) = x^6 + x^5 + x^4 + x^3 + 2x^2 + 2x + \alpha^{50}$  and  $Q(x) = x^2 + x + \alpha$ , where  $\alpha$  is a primitive element of  $\mathbb{F}_{81}$ . The following table, prepared with MAGMA, give us some nontrivial examples of maximal morphism (and hence maximal Prym variety).

$q$	81
$C$	$y^2 = x^8 + 2x^7 + \alpha^{77}x^6 + \alpha^{77}x^5 + \alpha x^4 + \alpha^{28}x^3 + \alpha^{12}x^2 + \alpha^{53}x + \alpha^{51}$
$D$	$y_1^2 = x^2 + x + \alpha, y_2^2 = x^6 + x^5 + x^4 + x^3 + 2x^2 + 2x + \alpha^{50}$
$F$	$y^2 = x^6 + x^5 + x^4 + x^3 + 2x^2 + 2x + \alpha^{50}$
$q$	81
$C$	$y^2 = x^8 + 2x^7 + \alpha^{70}x^6 + \alpha^{70}x^5 + \alpha^{10}x^4 + \alpha^{20}x^3 + \alpha^{70}x^2 + \alpha^{10}x + \alpha^{60}$
$D$	$y_1^2 = x^2 + x + \alpha, y_2^2 = x^6 + x^5 + x^4 + x^3 + 2x^2 + 2x + \alpha^{50}$
$F$	$y^2 = x^6 + x^5 + x^4 + x^3 + 2x^2 + 2x + \alpha^{50}$

## 5. CONCLUSION

Finding non-trivial examples of maximal Prym variety, grants us the motivation for defining a new concept which we call maximal morphism, that is a generalization of maximal curves.

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