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SIGNED GENERALIZED PETERSEN GRAPH AND ITS CHARACTERISTIC POLYNOMIAL

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Abstract. Let G^σ be a signed graph, where $G = (V, E)$ is the underlying simple graph and $\sigma : E(G) \to \{\pm 1\}$ is the sign function on $E(G)$. In this paper, we obtain k-th signed spectral moments and k-th signed Laplacian spectral moments of a signed graph G^σ, together with coefficients of their signed characteristic polynomial and signed Laplacian characteristic polynomial are calculated.

1. Introduction

Let $G = (V, E)$ be a simple graph and $\sigma : E(G) \to \{\pm 1\}$ a mapping on the edge set of G. The graph G together with the sign function σ is called a *signed graph* and is denoted by G^σ. If $\sigma(e) = +1$, then the edge e is called *positive*, and if $\sigma(e) = -1$, then the edge e is called *negative*. A *walk of length* k in a graph G is a sequence $v_1 e_1 \dots v_k e_k v_{k+1}$ with vertices $v_1, \dots, v_{k+1}$ and edges $e_1, \dots, e_k$ such that, we have $v_i \neq v_{i+1}$, $1 \leq i \leq k$ and e_i is an edge from v_i to v_{i+1}. In a signed graph G^σ, a walk is called *positive* (resp., *negative*) if the number of its negative edges is even (resp., odd). The number of positive (resp., negative) walks of length k from vertex v_i to vertex v_j is denoted by $w^+_{ij}(k)$ (resp., $w^-_{ij}(k)$). A *cycle* with $n \geq 3$ vertices is a simple graph whose vertices can be sorted as a sequence such that two vertices are adjacent if and only if they are subsequent members of the sequence. In a signed graph G^σ, a cycle is called *balanced* or *positive* (resp., *unbalanced* or *negative*) if the number of its negative edges is even (resp., odd). A signed graph

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is called *balanced* if all its cycles are balanced; otherwise, it is called *unbalanced* [1].

The *adjacency matrix* of a graph G, denoted by $A(G) = [a_{ij}]$, is an $n \times n$ matrix where $a_{ij} = 1$ if $v_i v_j$ is an edge of the graph and $a_{ij} = 0$ otherwise. The *adjacency matrix of a signed graph*, denoted by $A(G^\sigma) = [a^\sigma_{ij}]$, is an $n \times n$ matrix where $a^\sigma_{ij} = \sigma(ij)a_{ij}$ if $v_i v_j$ is an edge of the graph, and $a^\sigma_{ij} = 0$ otherwise. Thus, if $e = ij$ is an edge and $\sigma(ij) = +1$, then $a^\sigma_{ij} = 1$ and if $e = ij$ is an edge and $\sigma(ij) = -1$, then $a^\sigma_{ij} = -1$, and if $e = ij$ is not an edge, then $a^\sigma_{ij} = 0$. The adjacency matrix of a signed graph is symmetric (see [6] for some basic results on the adjacency spectrum of signed graphs). The signed Laplacian matrix of G^σ is the matrix $L(G^\sigma) = D(G) - A(G^\sigma)$, where $D(G)$ is the diagonal matrix of vertex degrees. Let G be a graph with adjacency matrix $A(G)$. We say that $\lambda(G)$ is an *eigenvalue of* G if there exists a non-zero vector X such that $A(G)X = \lambda(G)X$.

Throughout, t is the number of triangles, t^+ (resp., t^-) is the number of balanced (resp., unbalanced) triangles, q is the number of quadrangles, q^+ (resp., q^-) is the number of balanced (resp., unbalanced) quadrangles, t^+_i (resp., t^-_i) is the number of balanced (resp., unbalanced) triangles that are contain the i^{th} vertex, q^+_i (resp., q^-_i) is the number of balanced (resp., unbalanced) quadrangles that are contain the i^{th} vertex, t^+_{ij} (resp., t^-_{ij}) is the number of balanced (resp., unbalanced) triangles at edge ij, U^+_n (resp., U^-_n) is a balanced graph obtained from C^+_{n-1} (resp., C^-_{n-1}) by attaching a leaf to one of its vertices, and B_5, B_4 are two graphs obtained from two cycles C_3, C'_3 of length 3 by identifying one vertex of C_3 with one vertex of C'_3 and identifying one edge of C_3 with one edge of C'_3, respectively. $|S_n|$ is the number of star graphs of order n and $|P_n|$ is the number of paths of order n.

2. Main results

In this section, we would like to obtain k-th spectral moments and k-th Laplacian spectral moments of the signed generalized Petersen graphs. Taghvaee and Ashrafi in [3] and [4] determined the spectral moments and Laplacian spectral moments of generalized Petersen graph.

For the sake of completeness we mention below three important lemmas which are crucial in our main results in this paper (see, [1, 5]).

Lemma 2.1. *Let G^σ be a signed graph with signed adjacency matrix $A(G^\sigma)$. Then the (i, j)-entry of the matrix $A^k(G^\sigma)$ is equal to $w^+_{ij}(k) - w^-_{ij}(k)$.*

Suppose that G^σ is a signed graph with adjacency matrix $A(G^\sigma)$ and $\lambda_1(G^\sigma), \lambda_2(G^\sigma), \ldots, \lambda_n(G^\sigma)$ are eigenvalues of G^σ in non-increasing order. Then, $S_k(G^\sigma) = \sum_{i=1}^{n} \lambda_i^k(G^\sigma)$, $k \geqslant 0$ is the k-th signed spectral moment of G^σ.

Lemma 2.2. *The k-th signed spectral moments of G^σ is equal to the number of closed walks of length k.*

Lemma 2.3. *Let G^σ be a signed graph with n vertices and m edges. Then,*

$$S_2(G^\sigma) = 2m \quad and \;\; S_3(G^\sigma) = 6(t^+ - t^-).$$

It is also clear that $S_0(G^\sigma) = n$ and $S_1(G^\sigma) = 0$.

Theorem 2.4. *Let G^σ be a signed graph with n vertices and m edges. Then, we have:*

$$\begin{aligned}
S_4(G^\sigma) &= -2m + 2\sum_{i=1}^{n} d_i^2 + 8(q^+ - q^-),\\
S_5(G^\sigma) &= 30(t^+ - t^-) + 10(U_4^+ - U_4^-) + 10(C_5^+ - C_5^-),\\
S_6(G^\sigma) &= -10m + 6\sum_{i=1}^{n} d_i^2 + 6|P_4| + 12|S_4| + 12(U_5^+ - U_5^-)\\
&+ 36(B_4^+ - B_4^-) + 24(B_5^+ - B_5^-) + 24t(G) + 48(q^+ - q^-)\\
&+ 12(C_6^+ - C_6^-).
\end{aligned}$$

Proof. Vertices that belong to a closed walk of length 4 induce subgraph in G^σ are isomorphic to P_2, P_3 and q. Now, in the signed graph G^σ, the number of closed walks of length 4 which span these subgraphs is equal to $2|P_2|$, $4|P_3|$, $8(q^+ - q^-)$, respectively. Vertices that belong to a closed walk of length 5 induce subgraph in G^σ are isomorphic to t, U_4 and C_5. In the signed graph G^σ, the number of closed walks of length 5 which span these subgraphs is equal to $30(t^+ - t^-)$, $10(U_4^+ - U_4^-)$ and $10(C_5^+ - C_5^-)$, respectively. Now, we compute $S_6(G^\sigma)$. Vertices that belong to a closed walk of length 6 induce subgraph in G^σ are isomorphic to P_2, P_3, P_4, S_4, U_5, B_4, B_5, t, q, C_6. In the signed graph G^σ the number of closed walks of length 6 which span these subgraphs is equal to $2|P_2|$, $12|P_3|$, $6|P_4|$, $12|S_4|$, $12(U_5^+ - U_5^-)$, $36(B_4^+ - B_4^-)$, $24(B_5^+ - B_5^-)$, $24t$, $48(q^+ - q^-)$ and $12(C_6^+ - C_6^-)$, respectively. Note that the number of 3-path in connected graph G are equal to $-m + \frac{1}{2}\sum_{i=1}^{n} d_i^2$. □

Let $T_k(G^\sigma) = \sum_{i=1}^{n} \mu_i^k(G^\sigma)$, $k \geqslant 0$, be the k-th spectral moment for the Laplacian spectrum of a signed graph G^σ, where $\mu_1(G^\sigma), \mu_2(G^\sigma), \ldots, \mu_n(G^\sigma)$ are the Laplacian eigenvalues. Belardo [1] proved that:

Theorem 2.5. [1] *Let G^σ be a signed graph with n vertices, m edges and degree sequence $(d_1, \ldots, d_n)$. Then we have:*

$$\begin{aligned}
T_0(G^\sigma) &= n, \\
T_1(G^\sigma) &= \sum_{i=1}^{n} d_i = 2m, \\
T_2(G^\sigma) &= 2m + \sum_{i=1}^{n} d_i^2, \\
T_3(G^\sigma) &= 6(t^- - t^+) + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3.
\end{aligned}$$

Theorem 2.6. *Let G^σ be a signed graph with n vertices, m edges and degree sequence $(d_1, \ldots, d_n)$. Then we have:*

$$\begin{aligned}
T_4(G^\sigma) &= S_4(G^\sigma) + 4\sum_{i=1}^{n} d_i^3 + \sum_{i=1}^{n} d_i^4 + 4\sum_{v_i \sim v_j}^{m} d_i d_j - 8\sum_{i=1}^{n} d_i(t_i^+ - t_i^-), \\
T_5(G^\sigma) &= -S_5(G^\sigma) - 5\sum_{i=1}^{n} d_i^2 + 5\sum_{i=1}^{n} d_i^3 + 5\sum_{i=1}^{n} d_i^4 + \sum_{i=1}^{n} d_i^5 \\
&+ 10\Big(\sum_{v_i \sim v_j}^{m} d_i d_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{i \neq j}^{n} d_i^2 d_j a_{ij}^\sigma - \sum_{i=1}^{n} d_i^2(t_i^+ - t_i^-) \\
&+ \sum_{i=1}^{n} d_i(q_i^+ - q_i^-) - \sum_{v_i \sim v_j}^{m} d_i d_j(t_{ij}^+ - t_{ij}^-)\Big),
\end{aligned}$$

where d_i is the degree of i-th vertex.

Proof. Let $L(G^\sigma) = D(G) - A(G^\sigma)$ be the Laplacian matrix of a signed graph G^σ and $A(G^\sigma) = A^\sigma$. First, we obtain $T_4(G^\sigma)$. We have

$$\begin{aligned}
T_4(G^\sigma) &= \sum_{i=1}^{n} \mu_i^4(G^\sigma) = tr(D - A^\sigma)^4 \\
&= tr((D - A^\sigma)^2(D - A^\sigma)^2) \\
&= tr(D^4) - 4tr(D^3A^\sigma) - 4tr(D(A^\sigma)^3) + 4tr((A^\sigma)^2D^2) \\
&+ 2tr(DA^\sigma DA^\sigma) + tr((A^\sigma)^4).
\end{aligned}$$

By Theorem 2.4, we have $tr((A^\sigma)^4) = S_4(G^\sigma)$, where $S_4(G^\sigma)$ is 4-th spectral moments of G^σ. We have, $tr(D^3A^\sigma) = 0$, $tr(D^4) = \sum_{i=1}^{n} d_i^4$, $tr((A^\sigma)^2D^2) = \sum_{i=1}^{n} d_i^3$, $tr(D(A^\sigma)^3) = 2\sum_{i=1}^{n} d_i(t_i^+ - t_i^-)$. By direct computation, we can see that $tr(DA^\sigma DA^\sigma) = 2\sum_{v_i\sim v_j}^{m} d_id_j$, where $v_i \sim v_j$ is an edge of G^σ. This shows that,

$$T_4(G^\sigma) = \sum_{i=1}^{n} d_i^4 - 8\sum_{i=1}^{n} d_i(t_i^+ - t_i^-) + 4\sum_{i=1}^{n} d_i^3 + 4\sum_{v_i\sim v_j}^{m} d_id_j + S_4(G^\sigma).$$

In a similar way, we can compute $T_5(G^\sigma)$. We have,

$$\begin{aligned} T_5(G^\sigma) &= \sum_{i=1}^{n} \mu_i^5(G^\sigma) = tr(D - A^\sigma)^5 \\ &= tr(D^5) + 5tr(D^3(A^\sigma)^2) - 5tr(D^2(A^\sigma)^3) + 5tr(D^2A^\sigma DA^\sigma) \\ &- 5tr(D(A^\sigma)^2DA^\sigma) + 5tr(D(A^\sigma)^4) - tr((A^\sigma)^5). \end{aligned}$$

By Theorem 2.4, we get $tr((A^\sigma)^5) = S_5(G^\sigma)$, where $S_5(G^\sigma)$ is 5-th spectral moments of G^σ. We have, $tr(D^5) = \sum_{i=1}^{n} d_i^5$, $tr(D^3(A^\sigma)^2) = \sum_{i=1}^{n} d_i^4$, $tr(D^2(A^\sigma)^3) = 2\sum_{i=1}^{n} d_i^2(t_i^+ - t_i^-)$, $tr(D(A^\sigma)^4) = \sum_{i=1}^{n} d_i^3 + 2\sum_{v_i\sim v_j}^{m} d_id_j - \sum_{i=1}^{n} d_i^2 + 2\sum_{i=1}^{n} d_i(q_i^+ - q_i^-)$, where $v_i \sim v_j$ is an edge of G^σ, $tr(D(A^\sigma)^2DA^\sigma) = 2\sum_{v_i\sim v_j}^{m} d_id_j(t_{ij}^+ - t_{ij}^-)$ and $tr(D^2A^\sigma DA^\sigma) = \sum_{i=1}^{n}\sum_{i\neq j}^{n} d_i^2d_ja_{ij}^\sigma$, where a_{ij}^σ is an element of the signed adjacency matrix. This implies that,

$$\begin{aligned} T_5(G^\sigma) &= \sum_{i=1}^{n} d_i^5 + 5\sum_{i=1}^{n} d_i^4 - 10\sum_{i=1}^{n} d_i^2(t_i^+ - t_i^-) + 5\sum_{i=1}^{n}\sum_{i\neq j}^{n} d_i^2d_ja_{ij}^\sigma \\ &- 10\sum_{v_i\sim v_j}^{m} d_id_j(t_{ij}^+ - t_{ij}^-) + 5\sum_{i=1}^{n} d_i^3 + 10\sum_{v_i\sim v_j}^{m} d_id_j - 5\sum_{i=1}^{n} d_i^2 \\ &+ 10\sum_{i=1}^{n} d_i(q_i^+ - q_i^-) - S_5(G^\sigma). \end{aligned}$$

This completes the proof. $\square$

Corollary 2.7. *Suppose that $T_k(G^\sigma)$ and $T_k(G)$ are the k-th Laplacian spectral moment of a signed graph G^σ and a simple graph G, respectively. Then we have:*

$$\begin{aligned} T_3(G^\sigma) &\leq T_3(G), \\ T_4(G^\sigma) &\leq T_4(G), \\ T_5(G^\sigma) &\leq T_5(G), \end{aligned}$$

with equality if and only if G^σ is balanced signed graph.

Definition 2.8. The signed generalized Petersen graph, denoted by $GP^\sigma(n,k)$, is a graph with vertices and edges given by

$$V\big(GP^\sigma(n,k)\big) = \{a_i, b_i | 1 \leq i \leq n\},$$

and

$$E\big(GP^\sigma(n,k)\big) = \{a_ib_i, a_ia_{i+1}, b_ib_{i+k} | 1 \leq i \leq n\},$$

respectively. Here, $i+k$ are integers modulo n, where $n > 6$. Since $GP(n,k) \cong GP(n,n-k)$, suppose that $k \leqslant \lfloor \frac{n-1}{2} \rfloor$. Let $A(n,k)$ and $B(n,k)$ be the induced subgraphs of $GP(n,k)$ with the set of vertices $\{a_1,\ldots,a_n\}$ and $\{b_1,\ldots,b_n\}$, respectively. The subgraphs $A(n,k)$ and $B(n,k)$ are the outer and inner subgraphs of $GP(n,k)$, respectively.

Theorem 2.9. *If $S_i(GP^\sigma(n,k))$, $2 \leq i \leq 6$, is a signed spectral moment of $GP^\sigma(n,k)$, then the following holds:*

$$\begin{aligned} S_2(GP^\sigma(n,k)) &= 6n, \\ S_3(GP^\sigma(n,k)) &= \begin{cases} 6(t^+ - t^-) & \textit{if } 3|n,\ k = \frac{n}{3} \\ 0 & \textit{otherwise}, \end{cases} \\ S_4(GP^\sigma(n,k)) &= \begin{cases} 30n + 8(q^+ - q^-) & \textit{if } (4|n,\ k = \frac{n}{4}) \\ & \quad \textit{or } (k=1) \\ 30n & \textit{otherwise}, \end{cases} \\ S_5(GP^\sigma(n,k)) &= \begin{cases} 10(C_5^+ - C_5^-) & \textit{if } (5|n,\ k = \frac{n}{5}) \\ & \textit{or } (k = \frac{2n}{5},\ n \neq 10) \\ & \textit{or } \ (k=2) \\ 60(t^+ - t^-) & \textit{if } 3|n,\ k = \frac{n}{3} \\ 0 & \textit{otherwise}, \end{cases} \end{aligned}$$

$$S_6(GP^\sigma(n,k)) = \begin{cases} 174n+12(C_6^+-C_6^-) & \text{if } (6|n,\ k=\frac{n}{6},\ n\neq 18) \text{ or } \\ & (2|n,\ k=\frac{n}{2}-1,\ n\neq 8) \text{ or } \\ & (k=3,\ n\neq\{8,9,10,12,18\}) \\ 174n+96(q^+-q^-) & \text{if } 4|n,\ k=\frac{n}{4}, n\neq 12 \\ 174n+96(q^+-q^-) & \text{if } k=1 \\ +12(C_6^+-C_6^-) & \\ 176n & \text{if } 3|n,\ k=\frac{n}{3} \\ 174n & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2.3, we get $S_2(GP^\sigma(n,k)) = 6n$. Suppose that $3|n$ and $k=\dfrac{n}{3}$. Then, $b_ib_{i+\frac{n}{3}}b_{i+\frac{2n}{3}}b_i$ is a triangle of $GP^\sigma(n,k)$. By Theorem 2.3, we have $S_3(GP^\sigma(n,\dfrac{n}{3})) = 6(t^+-t^-)$. Otherwise, there are no triangle in $GP^\sigma(n,k)$ and also $S_3(GP^\sigma(n,k)) = 0$. Assume that $4|n$ and $k=\dfrac{n}{4}$, then $b_ib_{i+\frac{n}{4}}b_{i+\frac{2n}{4}}b_{i+\frac{3n}{4}}$ is a quadrangle in $GP^\sigma(n,k)$ and if $k=1$, then $a_ib_ib_{i+1}a_{i+1}a_i$ is a quadrangle in $GP^\sigma(n,k)$. Then, by Theorem 2.4, we have

$$\begin{aligned} S_4(GP^\sigma(n,\frac{n}{4})) &= -6n+36n+8(q^+-q^-) \\ &= 30n+8(q^+-q^-). \end{aligned}$$

Otherwise, $q^+-q^-=0$. So, $S_4(GP^\sigma(n,k)) = 30n$. Assume that $5|n$, $k=\dfrac{n}{5}$ or $k=\dfrac{2n}{5}$ and $n\neq 10$. If $k=\dfrac{n}{5}$, then $b_ib_{i+\frac{n}{5}}b_{i+\frac{2n}{5}}b_{i+\frac{3n}{5}}b_{i+\frac{4n}{5}}b_i$ is a pentagon in $GP^\sigma(n,k)$, if $k=\frac{2n}{5}$, then $b_ib_{i+\frac{n}{5}}b_{i+\frac{2n}{5}}b_{i+\frac{4n}{5}}b_{i+\frac{6n}{5}}b_{i+\frac{8n}{5}}b_i$ is another pentagon of $GP^\sigma(n,k)$ and if $k=2$ and $n\neq 10$, then $a_ib_ib_{i+2}a_{i+2}a_{i+1}a_i$ is a pentagon in $GP^\sigma(n,k)$. On the other hand, $U_4^+-U_4^-=0=t^+-t^-$. Thus,

$$S_5(GP^\sigma(n,\frac{n}{5})) = S_5(GP^\sigma(n,\frac{2n}{5})) = S_5(GP^\sigma(n,2)) = 10(C_5^+-C_5^-).$$

In other cases, $S_5(GP^\sigma(n,k)) = 0$. If $3|n$, $k=\dfrac{n}{3}$, then we have $U_4^+-U_4^-=3(t^+-t^-)$ and $C_5^+-C_5^-=0$. Hence,

$$S_5(GP^\sigma(n,\frac{n}{3})) = 60(t^+-t^-).$$

Now, we obtain $S_6(GP^\sigma(n,k))$. By Theorem 2.4, we have

$$\begin{aligned} S_6(G^\sigma) &= -10m + 6\sum_{i=1}^{n} d_i^2 + 6|P_4| + 12|S_4| + 12(U_5^+ - U_5^-) \\ &+ 36(B_4^+ - B_4^-) + 24(B_5^+ - B_5^-) + 24t(G) + 48(q^+ - q^-) \\ &+ 12(C_6^+ - C_6^-). \end{aligned}$$

Moreover, $|S_4| = 2n$ and $B_4^+ - B_4^- = B_5^+ - B_5^- = 0$. Now, assume that $6|n$, $k = \frac{n}{6}$ and $n \neq 18$. Then, $b_i b_{i+\frac{n}{6}} b_{i+\frac{n}{3}} b_{i+\frac{n}{2}} b_{i+\frac{4n}{6}} b_{i+\frac{5n}{6}}$ is a hexagon in $GP^\sigma(n,k)$. If $2|n$ and $k = \frac{n}{2} - 1$, then $a_i b_i b_{i+\frac{n}{2}-1} b_{i+n-2} a_{i+n-2} a_{i+n-1} a_i$ is a hexagon of $GP^\sigma(n,k)$ and if $k = 3$ and $n \neq \{8, 9, 10, 12, 18\}$, then $a_i b_i b_{i+3} a_{i+3} a_{i+2} a_{i+1} a_i$ is a hexagon in $GP^\sigma(n,k)$. Therefore, $|P_4| = 12n$ and $q^+ - q^- = 0 = U_5^+ - U_5^-$. Thus,

$$\begin{aligned} S_6(GP^\sigma(n,\frac{n}{6})) &= S_6(GP^\sigma(n,3)) = S_6(GP^\sigma(n,\frac{n}{2}-1)) \\ &= -30n + 108n + 72n + 24n + 12(C_6^+ - C_6^-) \\ &= 174n + 12(C_6^+ - C_6^-). \end{aligned}$$

If $3|n$ and $k = \frac{n}{3}$, then $|P_4| = 11n$ and $C_6^+ - C_6^- = 0$. Hence,

$$\begin{aligned} S_6(GP^\sigma(n,\frac{n}{3})) &= -30n + 108n + 66n + 24n + 8n \\ &= 176n. \end{aligned}$$

Suppose that $4|n$, $k = \frac{n}{4}$ and $n \neq 12$. So $U_5^+ - U_5^- = 4(q^+ - q^-)$. Therefore,

$$\begin{aligned} S_6(GP^\sigma(n,\frac{n}{4})) &= -30n + 108n + 72n + 24n + 96(q^+ - q^-) \\ &= 174n + 96(q^+ - q^-). \end{aligned}$$

Finally if $k = 1$, then $U_5^+ - U_5^- = 4(q^+ - q^-)$. Since $a_i a_{i+1} a_{i+2} b_{i+2} b_{i+4} b_i a_i$ is a hexagon in $GP^\sigma(n,k)$, we get

$$\begin{aligned} S_6(GP^\sigma(n,1)) &= -30n + 108n + 72n + 24n + 96(q^+ - q^-) + 12(C_6^+ - C_6^-) \\ &= 174n + 96(q^+ - q^-) + 12(C_6^+ - C_6^-). \end{aligned}$$

Otherwise, $C_6^+ - C_6^- = 0$ and $|P_4| = 12n$. Thus,

$$S_6(GP^\sigma(n,k)) = -30n + 108n + 72n + 24n = 174n.$$

This completes the proof. □

Theorem 2.10. *If $T_i(GP^\sigma(n,k))$, $0 \leq i \leq 5$, is a signed Laplacian spectral moments of $GP^\sigma(n,k)$, then*

$$T_0(GP^\sigma(n,k)) = 2n, \quad T_1(GP^\sigma(n,k)) = 6n, \quad T_2(GP^\sigma(n,k)) = 24n,$$

$$T_3(GP^\sigma(n,k)) = \begin{cases} 108n - 6(t^+ - t^-) & \textit{if } 3|n,\ k = \frac{n}{3} \\ 108n & \textit{otherwise,} \end{cases}$$

$$T_4(GP^\sigma(n,k)) = \begin{cases} 516n - 72(t^+ - t^-) & \textit{if } 3|n, k = \frac{n}{3} \\ 516n + 8(q^+ - q^-) & \textit{if } (4|n, k = \frac{n}{4}) \\ & \textit{or } (k = 1) \\ 516n & \textit{otherwise,} \end{cases}$$

$$T_5(GP^\sigma(n,k)) = \begin{cases} 2556n - 10(C_5^+ - C_5^-) & \textit{if } (5|n,\ k = \frac{n}{5} \textit{ or } k = \frac{2n}{5}) \\ & \textit{or } (k = 2) \\ 2556 + 120(q^+ - q^-) & \textit{if } 4|n,\ k = \frac{n}{4} \\ 2556n - 600(t^+ - t^-) & \textit{if } 3|n,\ k = \frac{n}{3} \\ 2556n + 30\sum_{i=1}^{2n}(q^+ - q^-) & \textit{if } k = 1 \\ 2556n & \textit{otherwise.} \end{cases}$$

Proof. Since $|V(GP^\sigma(n,k))| = 2n$ and $|E(GP^\sigma(n,k))| = 3n$, we get $T_0(GP^\sigma(n,k)) = 2n$ and $T_1(GP^\sigma(n,k)) = 6n$. By Theorem 2.5, we have $T_2(GP^\sigma(n,k)) = 6n + 18n = 24n$. Now, consider $T_3(GP^\sigma(n,k))$. If $3|n$ and $k = \dfrac{n}{3}$, then $b_i b_{i+\frac{n}{3}} b_{i+\frac{2n}{3}} b_i$ is a triangle of $GP^\sigma(n,k)$. So, the number of triangles in $GP^\sigma(n,k)$ is either equal to $t^+ - t^-$ or equal to zero. In the first case, we have:

$$\begin{aligned} T_3(GP^\sigma(n,k)) &= -6(t^+ - t^-) + 54n + 54n \\ &= -6(t^+ - t^-) + 108n, \end{aligned}$$

and in the second case, we have $T_3(GP^\sigma(n,k)) = 54n + 54n = 108n$. Now, we compute $T_4(GP^\sigma(n,k))$. If $3|n$ and $k = \dfrac{n}{3}$, then the number of triangles containing the outer subgraph of $GP^\sigma(n,k)$ is equal to 0 and the number of triangles containing the inner subgraph of $GP^\sigma(n,k)$ is equal to $t^+ - t^-$. On the other hand, $S_4(GP^\sigma(n,k)) = 30n$. Then, we have

$$\begin{aligned} T_4(GP^\sigma(n,\frac{n}{3})) &= 30n - 72(t^+ - t^-) + 216n + 162n + 108n \\ &= 516n - 72(t^+ - t^-). \end{aligned}$$

Suppose that $4|n$ and $k = \dfrac{n}{4}$. Now, $b_ib_{i+\frac{n}{4}}b_{i+\frac{2n}{4}}b_{i+\frac{3n}{4}}$ is a quadrangle in $GP^\sigma(n,k)$. Using Theorem 2.9, we have

$$S_4(GP^\sigma(n,k)) = 30n + 8(q^+ - q^-).$$

Since $t_i = 0$, we get

$$\begin{array}{rcl} T_4(GP^\sigma(n,\dfrac{n}{4})) & = & 30n + 8(q^+ - q^-) + 216n + 162n + 108n \\ & = & 516n + 8(q^+ - q^-). \end{array}$$

If $k = 1$, then $a_ib_ib_{i+1}a_{i+1}a_i$ is a quadrangle in $GP^\sigma(n,k)$. On the other hand, by Theorem 2.9, we get $S_4(GP^\sigma(n,k)) = 30n+8(q^+ - q^-)$. Therefore,

$$\begin{array}{rcl} T_4(GP^\sigma(n,1)) & = & 30n + 8(q^+ - q^-) + 216n + 162n + 108n \\ & = & 516n + 8(q^+ - q^-). \end{array}$$

Otherwise, $q^+ - q^- = 0$. Now, by Theorem 2.9, we get

$$S_4(GP^\sigma(n,k)) = 30n.$$

So,

$$\begin{array}{rcl} T_4(GP^\sigma(n,k)) & = & 30n + 216n + 162n + 108n \\ & = & 516n. \end{array}$$

Now, we compute $T_5(GP^\sigma(n,k))$. Let $5|n$, $k = \dfrac{n}{5}$ or $k = \dfrac{2n}{5}$. So, $b_ib_{i+\frac{n}{5}}b_{i+\frac{2n}{5}}b_{i+\frac{3n}{5}}b_{i+\frac{4n}{5}}b_i$ is a pentagon in $GP^\sigma(n,k)$. If $k = \frac{2n}{5}$, then $b_ib_{i+\frac{n}{5}}b_{i+\frac{2n}{5}}b_{i+\frac{4n}{5}}b_{i+\frac{6n}{5}}b_{i+\frac{8n}{5}}b_i$ is another pentagon of $GP^\sigma(n,k)$. Thus,

$$\begin{array}{rcl} T_5(GP^\sigma(n,\dfrac{n}{5})) & = & T_5(GP^\sigma(n,\dfrac{2n}{5})) = -90n + 270n + 810n + 486n \\ & - & 10(C_5^+ - C_5^-) + 270n + 810n \\ & = & 2556n - 10(C_5^+ - C_5^-). \end{array}$$

If $k = 2$, then $a_ib_ib_{i+2}a_{i+2}a_{i+1}a_i$ is a pentagon in $GP^\sigma(n,k)$ and $t^+ - t^- = 0 = q^+ - q^-$. This implies that,

$$\begin{array}{rcl} T_5(GP^\sigma(n,2)) & = & -90n + 270n + 810n + 486n - 10(C_5^+ - C_5^-) \\ & + & 270n + 810n \\ & = & 2556n - 10(C_5^+ - C_5^-). \end{array}$$

If $4|n$ and $k = \frac{n}{4}$, then $C_5^+ - C_5^- = 0 = t^+ - t^-$. Hence,

$$\begin{aligned} T_5(GP^\sigma(n, \frac{n}{4})) &= -90n + 270n + 810n + 486n + 270n + 810n \\ &+ 120(q^+ - q^-) \\ &= 2556n + 120(q^+ - q^-). \end{aligned}$$

If $3|n$ and $k = \frac{n}{3}$, then $C_5^+ - C_5^- = 0 = q^+ - q^-$. So,

$$\begin{aligned} T_5(GP^\sigma(n, \frac{n}{3})) &= -90n + 270n + 810n + 486n - 600(t^+ - t^-) \\ &+ 270n + 810n \\ &= 2556n - 600(t^+ - t^-). \end{aligned}$$

Finally, if $k = 1$, then we get

$$\begin{aligned} T_5(GP^\sigma(n, 1)) &= -90n + 270n + 810n + 486n + 270n + 810n \\ &+ 30\sum_{i=1}^{2n}(q^+ - q^-) \\ &= 2556n + 30\sum_{i=1}^{2n}(q^+ - q^-). \end{aligned}$$

Otherwise, we have

$$\begin{aligned} T_5(GP^\sigma(n, k)) &= -90n + 270n + 810n + 486n + 270n + 810n \\ &= 2556n. \end{aligned}$$

This completes the proof. □

Theorem 2.11. (*Newton's identity* [2])*: Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of the polynomial*

$$\phi(A, x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

with signed spectral moment S_k. Then

$$a_k = -\frac{1}{k}(S_k + S_{k-1}a_1 + S_{k-2}a_2 + \cdots + S_1a_{k-1}).$$

Now, by Newton's identity we compute coefficients of characteristic polynomial of signed generalized Petersen graphs and coefficients of Laplacian characteristic polynomial of signed generalized Petersen graphs.

Lemma 2.12. *Let $a_i(G^\sigma)$, $1 \leq i \leq 5$, be the coefficients of characteristic polynomial of a signed graph G^σ. Then, we have:*

$$\begin{aligned}
a_1(G^\sigma) &= 0, \\
a_2(G^\sigma) &= -m, \\
a_3(G^\sigma) &= -2(t^+ - t^-), \\
a_4(G^\sigma) &= \frac{1}{2}(m^2 + m - 4(q^+ - q^-) - \sum_{i=1}^{n} d_i^2), \\
a_5(G^\sigma) &= (2m - 6)(t^+ - t^-) - 2(U_4^+ - U_4^-) - 2(C_5^+ - C_5^-).
\end{aligned}$$

Proof. Using Lemma 2.3, Theorem 2.4 and Theorem 2.11, we get the desired result. □

Lemma 2.13. *Let $b_i(G^\sigma)$, for $1 \leq i \leq 4$, be the coefficients of Laplacian characteristic polynomial of a signed graph G^σ. Then, we have:*

$$\begin{aligned}
b_1(G^\sigma) &= -2m, \\
b_2(G^\sigma) &= -\frac{1}{2}(2m - 4m^2 + \sum_{i=1}^{n} d_i^2), \\
b_3(G^\sigma) &= -\frac{1}{3}(4m^3 - 6m^2 + 3(1 - m)\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3 + 6(t^- - t^+)), \\
b_4(G^\sigma) &= -\frac{1}{4}\Big(- 2m - 2m^2 + 8m^3 - \frac{8m^4}{3} + (2 - 10m + 4m^2)\sum_{i=1}^{n} d_i^2 \\
&+ (4 - \frac{8m}{3})\sum_{i=1}^{n} d_i^3 + \sum_{i=1}^{n} d_i^4 + 4\sum_{v_i \sim v_j}^{m} d_i d_j - 8\sum_{i=1}^{n} d_i(t_i^+ - t_i^-) \\
&+ 8(q^+ - q^-) + 16m(t^+ - t^-) - \frac{1}{2}(\sum_{i=1}^{n} d_i^2)^2\Big).
\end{aligned}$$

Proof. Using Theorem 2.5 and Theorem 2.11, we get the desired result. □

Corollary 2.14. *Let $a_i(GP^\sigma(n,k))$ for $1 \leq i \leq 5$, be the coefficients of characteristic polynomial of a signed generalized Petersen graph. Then,* *we have:*

$$\begin{aligned}
a_1(GP^\sigma(n,k)) &= 0,\\
a_2(GP^\sigma(n,k)) &= -3n,\\
a_3(GP^\sigma(n,k)) &= \begin{cases} -2(t^+ - t^-) & \textit{if } 3|n,\ k=\dfrac{n}{3}\\ 0 & \textit{otherwise,}\end{cases}\\
a_4(GP^\sigma(n,k)) &= \begin{cases} \dfrac{1}{2}(9n^2-15n-4(q^+-q^-)) & \textit{if } (4|n,\ k=\frac{n}{4})\\ & \textit{or } (k=1)\\ \dfrac{1}{2}(9n^2-15n) & \textit{otherwise,}\end{cases}\\
a_5(GP^\sigma(n,k)) &= \begin{cases} -2(C_5^+ - C_5^-) & \textit{if } (5|n,\ k=\frac{n}{5} \textit{ or } k=\frac{2n}{5})\\ & \textit{or } (k=2)\\ (6n-12)(t^+-t^-) & \textit{if } 3|n,\ k=\frac{n}{3}\\ 0 & \textit{otherwise.}\end{cases}
\end{aligned}$$

Proof. Since $|E(GP^\sigma(n,k)| = 3n$, then we have $a_2(GP^\sigma(n,k)) = -3n$. Assume that $3|n$ and $k = \dfrac{n}{3}$. Then, $b_i b_{i+\frac{n}{3}} b_{i+\frac{2n}{3}} b_i$ is a triangle of $GP^\sigma(n,k)$. Thus,

$$a_3(GP^\sigma(n,\frac{n}{3})) = -2(t^+ - t^-).$$

Otherwise, there is no triangle in $GP^\sigma(n,k)$ and $a_3(GP^\sigma(n,k)) = 0$. Suppose that $4|n$ and $k = \dfrac{n}{4}$. Then, $b_i b_{i+\frac{n}{4}} b_{i+\frac{2n}{4}} b_{i+\frac{3n}{4}}$ is a quadrangle in $GP^\sigma(n,k)$. So,

$$\begin{aligned}
a_4(GP^\sigma(n,\frac{n}{4})) &= \frac{1}{2}\Big(9n^2+3n-4(q^+-q^-)-18n\Big)\\
&= \frac{1}{2}\Big(9n^2-4(q^+-q^-)-15n\Big).
\end{aligned}$$

Otherwise, $q^+ - q^- = 0$. Thus,

$$\begin{aligned}
a_4(GP^\sigma(n,k)) &= \frac{1}{2}(9n^2+3n-18n)\\
&= \frac{1}{2}(9n^2-15n).
\end{aligned}$$

Now, assume that $k=1$. Then, $a_ib_ib_{i+1}a_{i+1}a_i$ is a quadrangle in $GP^\sigma(n,k)$. Therefore,

$$\begin{aligned}a_4(GP^\sigma(n,1)) &= \frac{1}{2}(9n^2+3n-4(q^+-q^-)-18n)\\ &= \frac{1}{2}(9n^2-4(q^+-q^-)-15n).\end{aligned}$$

Suppose that $5|n$, $k=\dfrac{n}{5}$ or $k=\dfrac{2n}{5}$. If $k=\dfrac{n}{5}$, then $b_ib_{i+\frac{n}{5}}b_{i+\frac{2n}{5}}b_{i+\frac{3n}{5}}b_{i+\frac{4n}{5}}b_i$ is a pentagon in $GP^\sigma(n,k)$ and if $k=\frac{2n}{5}$, then $b_ib_{i+\frac{n}{5}}b_{i+\frac{2n}{5}}b_{i+\frac{4n}{5}}b_{i+\frac{6n}{5}}b_{i+\frac{8n}{5}}b_i$ is an another pentagon of $GP^\sigma(n,k)$. So, $U_4^+-U_4^-=0=t^+-t^-$. This yields that,

$$\begin{aligned}a_5(GP^\sigma(n,\frac{n}{5})) &= a_5(GP^\sigma(n,\frac{2n}{5}))\\ &= -2(C_5^+-C_5^-).\end{aligned}$$

If $k=2$, then $a_ib_ib_{i+2}a_{i+2}a_{i+1}a_i$ is a pentagon in $GP^\sigma(n,k)$ and $U_4^+-U_4^-=0=t^+-t^-$. This implies that $a_5(GP^\sigma(n,2))=-2(C_5^+-C_5^-)$. In other cases, we have $a_5(GP^\sigma(n,k))=0$. If $3|n$, $k=\dfrac{n}{3}$, then $U_4^+-U_4^-=3(t^+-t^-)=n$ and $C_5^+-C_5^-=0$. Hence,

$$\begin{aligned}a_5(GP^\sigma(n,\frac{n}{3})) &= (2(3n)-2(3)-6)(t^+-t^-)\\ &= (6n-12)(t^+-t^-).\end{aligned}$$

Otherwise, $a_5(GP^\sigma(n,k))=0$. This completes the proof. $\square$

Corollary 2.15. *Let $b_i(GP^\sigma(n,k))$, for $1\le i\le 4$, be the coefficients of Laplacian characteristic polynomial of a signed generalized Petersen* *graph. Then, we have:*

$$
\begin{array}{rcl}
b_1(GP^\sigma(n,k)) & = & -6n,\\
b_2(GP^\sigma(n,k)) & = & 18n^2-24n,\\
b_3(GP^\sigma(n,k)) & = & \begin{cases} -36n^3+72n^2-\dfrac{108n}{3} & \textit{if } 3|n,\ k=\frac{n}{3}\\ +2(t^+-t^-) & \\ & \\ -36n^3+72n^2-\dfrac{108n}{3} & \textit{otherwise,}\end{cases}\\
b_4(GP^\sigma(n,k)) & = & \begin{cases} 54n^4-216n^3+288n^2-129n & \textit{if } (4|n,\ k=\frac{n}{4})\\ -2(q^+-q^-) & \textit{or } (k=1)\\ & \\ 54n^4-216n^3+288n^2-129n & \textit{if } 3|n,\ k=\frac{n}{3}\\ -(12n-18)(t^+-t^-) & \\ & \\ 54n^4-216n^3+288n^2-129n & \textit{otherwise.}\end{cases}
\end{array}
$$

Proof. Using Lemma 2.13 and a similar method as used in the proof of Corollary 2.14, one can see that the assertion holds. □

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