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IDEALS WITH $(d_1, \ldots, d_m)$-LINEAR QUOTIENTS

L. SHARIFAN*

ABSTRACT. In this paper, we introduce the class of ideals with $(d_1, \ldots, d_m)$-linear quotients generalizing the class of ideals with linear quotients. Under suitable conditions, we control the numerical invariants of a minimal free resolution of ideals with $(d_1, \ldots, d_m)$-linear quotients. In particular, we show that their first module of syzygies is a componentwise linear module.

1. INTRODUCTION

Let $\mathbf{k}$ be a field, and $R = \mathbf{k}[x_1, \ldots, x_n]$ be the polynomial ring in n variables. In this paper, we introduce and study a class of ideals in R which can be considered as a generalization of the class of ideals with linear quotients (see, [8, 10]).

Let I be a graded ideal, $\{f_1, \ldots, f_m\}$ be a homogeneous system of generators of I and $(d_1, \ldots, d_m)$ be an m-tuple of positive integers supposing $d_1 = 1$. We say that I has $(d_1, \ldots, d_m)$-linear quotients with respect to the elements $f_1, \ldots, f_m$ if the ideal $(f_1, \ldots, f_{j-1}) : f_j$ has d_j-linear resolution for all $j = 2, \ldots, m$. Notice that, this property depends on the order of the generators. If $d_2 = \cdots = d_m = d$, we simply say that I has d-linear quotients with respect to the elements $f_1, \ldots, f_m$ and if $d = 1$, we get the usual class of ideals with linear quotients.

Monomial ideals with linear quotients were introduced in [8] and have strong combinatorial implication (see for example, [11]). A very

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$*$Corresponding author.

important property of ideals with linear quotients is that they are componentwise linear (see, [10, Corollary 2.4]).

Recall that componentwise linear modules over a polynomial ring has been introduced by Herzog and Hibi, enlarging the class of the graded modules with a d-linear resolution (see [6]). Interesting results concerning their graded Betti numbers has been proved by Aramova, Conca, Herzog and Hibi (see [1, 2, 3, 6, 7]). Later, Römer (see [12]) studied more homological properties of componentwise linear modules in the general setting of finitely generated modules over Koszul algebras (instead of polynomial rings).

In this paper, we assume that $I = (f_1, \ldots, f_m)$ has $(d_1, \ldots, d_m)$-linear quotients with respect to $f_1, \ldots, f_m$ and $\deg(f_1) + d_1 \leq \cdots \leq \deg(f_m) + d_m$. In Theorem 4.2, we study the case of ideals with 2-linear quotients and we prove a property of these ideals which is close to the componentwise linear property. In Theorem 4.7, we study the minimal free resolution of R/I by iterated mapping cone and precisely we compute the regularity of R/I. Finally, in Theorem 4.9, we show that $Syz_1(I)$ is a componentwise linear module.

We organize the paper as it follows: In Section 2, we review some basic definitions, notations and results that we need in subsequent sections. In Section 3, we give a sufficient condition for minimality of a resolution obtained by the mapping cone (see Theorem 3.1). Next, we give some easy and technical lemmas that we need for studying $Syz_1(I)$. Section 4 is devoted to the main results about ideals with $(d_1, \ldots, d_m)$-linear quotients.

Furthermore, the paper includes several examples to illustrate and delimite the results. Definitely, via these examples, we examine some ideals with $(d_1, \ldots, d_m)$-linear quotients to see if they have nice properties of ideals with linear quotients or not (see [10, 11]).

2. Preliminaries

The *Castelnuovo-Mumford regularity* (or briefly regularity) of a graded finitely generated R-module M, is defined as

$$reg(M) = \max\{j - i;\ \beta_{i,j}(M) \neq 0\}$$

and the projective dimension of M is defined as

$$pd(M) = \max\{i;\ \beta_{i,j}(M) \neq 0 \text{ for some } j\},$$

where $\beta_{i,j}(M)$ is the (i,j)th graded Betti number of M.

Let

$$\cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M$$

be the graded minimal free resolution of M. Then, the $p-$th syzygy module of M, denoted by $Syz_p(M)$, is defined as $Syz_p(M) = \ker(\delta_{p-1}) = \mathrm{Im}(\delta_p)$. Recall that for each j, the differential δ_j is given by a matrix $\mathcal{M}_j$ (which depends on the chosen basis of F_js). So $Syz_p(M)$ is generated by the columns of $\mathcal{M}_p$.

Let M be a graded R-module. The *initial degree* of M is defined as

$$indeg(M) = min\{d \in \mathbf{Z};\ M_d \neq 0\}.$$

For $d \in \mathbf{Z}$, we write $M_{<d>}$ for the submodule of M which is generated by all homogeneous elements of M with degree d. Moreover, we write $M_{\leq d}$ for the module generated by all homogeneous elements in M whose degrees are less than or equal to d.

If N is a graded submodule of M, then

$$(M/N)_{<a>} \cong (M_{<a>} + N)/N.$$

For a module M minimally generated in degrees $i_1 < \cdots < i_\ell$, we define $M_{\{1\}} = M$ and for every $j = 2, \ldots, \ell$,

$$M_{\{j\}} := M_{\{j-1\}}/(M_{\{j-1\}})_{<indeg(M_{\{j-1\}})>} = M_{\{j-1\}}/(M_{\{j-1\}})_{<i_{j-1}>}.$$

Lemma 2.1. *If M is a module minimally generated in degrees $i_1 < \cdots < i_\ell$,, then for each $2 \leq r \leq \ell$,*

$$(M_{\{r\}})_{<i_r>} \cong (M_{<i_1>} + \cdots + M_{<i_r>})/M_{<i_1>} + \cdots + M_{<i_{r-1}>}).$$

Proof. Note that

$$\begin{aligned}(M_{\{r\}})_{<i_r>} &= (M_{\{r-1\}}/(M_{\{r-1\}})_{<i_{r-1}>})_{<i_r>} \\ &= ((M_{\{r-1\}})_{<i_r>} + (M_{\{r-1\}})_{<i_{r-1}>})/(M_{\{r-1\}})_{<i_{r-1}>}\end{aligned}$$

and if we continue in this way, we get the desired result. □

Let $d \in \mathbf{Z}$. We say that M has a d-linear resolution if $\beta_{i,j}(M) = 0$ for $j \neq d + i$, and we say M is componentwise linear if for all integers d the module $M_{<d>}$ has a d-linear resolution.

For more information concerning the componentwise linear modules, see [2, 3, 6, 12]. We select here some good properties of their graded minimal free resolutions.

Lemma 2.2. *If M is a graded R-module and it has an i-linear resolution, then $\mathfrak{m}M$ has an $i+1-$linear resolution, where $\mathfrak{m} = (x_1, \ldots, x_n)$ is the homogeneous maximal ideal of R.*

Lemma 2.3. (see [12, Lemma 3.2.2]) *Let M be a graded $R-$module. Then the following statements are equivalent:*

(i): *M is componentwise linear;*

(ii): *$M/M_{<indeg(M)>}$ is componentwise linear and $M_{<indeg(M)>}$ has an $indeg(M)$- linear resolution.*

The following corollary is an immediate consequence of the above lemma.

Corollary 2.4. *Let M be a graded module minimally generated in degrees $i_1 < \cdots < i_\ell$. Then M is a componentwise linear module if and only if for each $1 \leq j \leq \ell$, $(M_{\{j\}})_{<i_j>}$ has an i_j-linear resolution.*

Following Römer, we define a special subcomplex of the minimal graded free resolution of a module.

Definition 2.5. Let M be a graded $R-$module and $(\mathbf{G}., d.)$ be the minimal graded free resolution of M. We define the subcomplex $(\widetilde{\mathbf{G}}., \widetilde{d}.)$ of $(\mathbf{G}., d.)$ to be

$$\widetilde{G_i} = R(-(i + indeg(M)))^{\beta_{i,i+indeg(M)}} \subseteq G_i \text{ and } \widetilde{d}. = d.|_{\widetilde{\mathbf{G}}.}.$$

Lemma 2.6. (see [12, Lemma 3.2.4]) *Let M be a graded $R-$module such that $M_{<indeg(M)>}$ has a linear resolution, and let $(\mathbf{G}., d.)$ be the minimal graded free resolution of M. Then:*

(i): *$\widetilde{\mathbf{G}}.$ is the minimal graded free resolution of $M_{<indeg(M)>}$.*

(ii): *$\mathbf{G}./\widetilde{\mathbf{G}}.$ is the minimal graded free resolution of $M/M_{<indeg(M)>}$.*

Proposition 2.7. (see [13, Proposition 2.2]) *Let M be a componentwise linear R-module minimally generated in degrees $i_1 < \cdots < i_\ell$. Then for each $1 \leq i \leq pd(M)$, we have*

$$\beta_{i,j}(M) = 0 \text{ for } j \neq i + i_1, \ldots, i_\ell + i.$$

Next, we review spme basic properties of ideals with linear quotients.

Let I be a graded ideal and $\{f_1, \ldots, f_m\}$ be a homogeneous system of generators of I and $I_j = (f_1, \ldots, f_j)$ for $j = 1, \ldots, m$. We say that I has linear quotients with respect to the elements $f_1, \ldots, f_m$, if the ideal $I_{j-1} : f_j$ is generated by linear forms for all $j = 2, \ldots, m$. Notice that this property depends on the order of the generators. Any order of the generators for which we have linear quotients will be called an admissible order. If I has linear quotients with respect to an admissible order of a homogeneous system of generators, we simply say I has linear quotients. Ideals with linear quotients have the following properties:

Proposition 2.8. (see [10, Corollary 2.4]) *If the graded ideal I has linear quotients with respect to the elements $f_1, \ldots, f_m$, then I is componentwise linear provided that $\{f_1, \ldots, f_m\}$ is a minimal system of generators.*

For a monomial ideal I, we denote by $G(I)$ the unique minimal system of monomial generators of I. In this case, when we say I has linear quotients, we mean I has linear quotients with respect to an admissible order of $G(I)$

Proposition 2.9. (see [11, Lemma 2.1]) *If a monomial ideal I has linear quotients, then there exists a degree increasing admissible order of $G(I)$.*

3. Mapping cone technique

One of the fundamental tools for computing free resolutions is mapping cone technique. Many well-known free resolutions arise as iterated mapping cones. For example, the Taylor resolution of monomial ideals.

The idea of the iterated mapping cone construction is the following: Let $\{f_1, \ldots, f_m\}$ be a homogeneous system of generators for I, and $I_j = (f_1, \ldots, f_j)$. Then, for $j = 2, \ldots, m$, there are exact sequences

$$0 \to R/(I_{j-1} : f_j) \to R/I_{j-1} \to R/I_j \to 0$$

assuming that a free $R-$resolution $(\mathbf{F}., \delta.)$ of R/I_{j-1} and a free R-resolution $(\mathbf{G}., d.)$ of $R/(I_{j-1} : f_j)$ are known, we can obtain a resolution $(\mathbf{M}(\psi), \gamma.)$ of R/I_j as a *mapping cone* of a complex homomorphism $\psi : \mathbf{G}. \to \mathbf{F}.$, which is a lifting of the map $R/(I_{j-1} : f_j) \to R/I_{j-1}$. The mapping cone $\mathbf{M}(\psi)$ is the complex such that

$$(M(\psi))_i = F_i \oplus G_{i-1},$$

with the differential maps

$$\gamma_i(x, y) = (\psi_{i-1}(y) + \delta_i(x), -d_{i-1}(y)),$$

where $x \in F_i$ and $y \in G_{i-1}$. This complex is exact (see [4, Page 650 and Proposition A3.19.]), so, it is a free resolution for R/I_j.

It is clear that in this way, we get a free resolution of R/I. Of course, in general, such a resolution may be non-minimal. For example if $I = (f_1, f_2, f_3)$ where $f_1 = x_1^2, f_2 = x_2^3, f_3 = x_1x_2$, the result of the iterated mapping cone is not a minimal free resolution. But, there are some important classes of ideals for which the minimal free resolution obtained by iterated mapping cone. For example, the Eliahou-Kervaire resolution of stable monomial ideals (as noted by Evans and Charalambous[5]). More in general, if I has linear quotients with respect to a minimal homogeneous system of generators, then its minimal free resolution can be obtained by iterated mapping cone. This is an immediate consequence of [10, Corollary 2.7].

Here, we give a sufficient condition to check the minimality of a resolution obtained by the mapping cone technique.

Theorem 3.1. *Let I be a graded ideal of R and f be a homogeneous form of degree d which does not belong to I. Then, we have the following graded short exact sequence:*

$$0 \to R/(I : f)(-d) \to R/I \to R/I + (f) \to 0.$$

Assuming that the minimal free resolution of the modules $R/(I : f)$ and R/I are already known. Then, the minimal free resolution of $R/I + (f)$ is obtained by the mapping cone provided that for each $1 \leq i \leq pd(R/(I : f))$,

$$\{j;\ \beta_{i,j}(R/(I : f)) \neq 0\} \cap \{j - d\ ;\ \beta_{i,j}(R/I) \neq 0\} = \emptyset, \tag{3.1}$$

and in this case

(a):

$$\beta_{i,j}(R/I + (f)) = \beta_{i,j}(R/I) + \beta_{i-1,j-d}(R/(I : f)),$$

(b):

$$reg(R/(I + (f)) = \max\{reg(R/I), reg(R/(I : f)) + d - 1\}$$

(c):

$$pd(R/(I + (f)) = \max\{pd(R/I), pd(R/(I : f)) + 1\}.$$

Proof. Let $(\mathbf{F}., \delta.)$ be the minimal free resolution of R/I, $(\mathbf{G}., d.)$ be the minimal free resolution of $R/(I : f)$ shifted by d and $\psi : \mathbf{G}. \to \mathbf{F}.$ be the complex graded homomorphism which is a lifting of the map $R/(I : f)(-d) \to R/I$. It is enough to show that the mapping cone complex is the minimal free resolution of $R/(I + (f))$.

Let for each r, $\mathcal{M}_r$ (resp., $\mathcal{N}_r$) be the matrix of δ_r (resp., d_r) with respect to the canonical basis of F_r and F_{r-1} (resp., G_r and G_{r-1}). Also, assume that for each r, O_r be the matrix of $\psi_r : G_r \to F_r$. Then, by the mapping cone construction, the matrix of γ_r, with respect to the canonical basis of $F_r \oplus G_{r-1}$ and $F_{r-1} \oplus G_{r-2}$, is denoted by $\mathcal{M}'_r$ has the following shape;

$$\mathcal{M}'_r = \left(\begin{array}{c|c} \mathcal{M}_r & O_{r-1} \\ \hline 0 & -\mathcal{N}_{r-1} \end{array}\right).$$

So, the result of the mapping cone is the minimal free resolution if and only if $Im(\psi) \subset \mathfrak{m}\mathbf{F}.$.

Let $e_1, \ldots, e_{\beta_i(R/(I:f))}$ be the basis of $\mathbf{G}.$ in the homological degree i, and $\eta_1, \ldots, \eta_{\beta_i(R/I)}$ be the basis of $\mathbf{F}.$ in the homological degree i. Then, by the hypothesis $\psi_i : G_i \to F_i$ is given by $\psi_i(e_j) = \sum_{t=1}^{\beta_i(R/I)} a_{it}\eta_t$, where for each $1 \leq t \leq \beta_i(R/I)$ if $a_{it} \neq 0$ then $\deg(e_j) > \deg(\eta_t)$. So, $\deg(a_{it}) > 0$ for each i and t when $a_{it} \neq 0$. So, the conclusion follows.

The parts (a), (b), (c) are directly followed by the minimality of the obtained resolution. $\square$

Remark 3.2. If $I=(f_1,\dots,f_m)$ and $I+(f)$ is *minimally* generated by $\{f_1,\dots,f_m,f\}$, then $Im(\psi_1)\subseteq \mathfrak{m}F_1$ and we just need to check Equation 3.1 for $2\le j\le pd(R/(I:f))$.

Next, we give an example in which the minimal free resolution is computed by iterated mapping cone by successive using Theorem 3.1. We first recall the definition of lex-segment ideals.

A monomial ideal $I\subset R$ is called a *lex-segment ideal* if for all monomials $u\in I$ and all monomials $v\in R$ with $\deg(u)=\deg(v)$ and $v>_{lex}u$, one has $v\in I$.

Example 3.3. Let

$$\begin{aligned} I= \quad & (x_1^2,x_1x_2,\dots,x_1x_n,x_2^m,x_2^{m-1}x_3,\dots,x_2^{m-1}x_i,\\ & x_2^{m-1}x_{i+1}^3,x_2^{m-1}x_{i+1}^2x_{i+2},\dots,x_2^{m-1}x_{i+1}^2x_{n-1},x_2x_n)\subseteq R\end{aligned}$$

where $m>1$. Then the minimal free resolution of R/I is given by the iterated mapping cone. It is easy to see that in each step, Equation 3.1 holds. Let us just check the final step. Notice that

$$\begin{aligned} J= \quad & (x_1^2,x_1x_2,\dots,x_1x_n,x_2^m,x_2^{m-1}x_3,\dots,x_2^{m-1}x_i,x_2^{m-1}x_{i+1}^3,\\ & x_2^{m-1}x_{i+1}^2x_{i+2},\dots,x_2^{m-1}x_{i+1}^2x_{n-1})\end{aligned}$$

is a Lex-segment ideal. So, J has linear quotients with respect to

$$\begin{aligned} & x_1^2,x_1x_2,\dots,x_1x_n,x_2^m,x_2^{m-1}x_3,\dots,x_2^{m-1}x_i,x_2^{m-1}x_{i+1}^3,\\ & x_2^{m-1}x_{i+1}^2x_{i+2},\dots,x_2^{m-1}x_{i+1}^2x_{n-1}.\end{aligned}$$

Therefore, J is a componentwise linear ideal and by Proposition 2.7,

$$\{j-2;\ \beta_{i,j}(R/J)\neq 0\}\subseteq\{i+m-3,i+m-1,i-1\}.$$

$$\begin{aligned} J:x_2x_n= \quad & (x_1,x_2^{m-1},x_2^{m-2}x_3,\dots,x_2^{m-2}x_i,x_2^{m-2}x_{i+1}^3,\\ & x_2^{m-2}x_{i+1}^2x_{i+2},\dots,x_2^{m-2}x_{i+1}^2x_{n-1})\end{aligned}$$

is again a lex-segment ideal and it has linear quotients with respect to

$$x_1,x_2^{m-1},x_2^{m-2}x_3,\dots,x_2^{m-2}x_i,x_2^{m-2}x_{i+1}^3,x_2^{m-2}x_{i+1}^2x_{i+2},\dots,x_2^{m-2}x_{i+1}^2x_{n-1}.$$

Thus, $J:x_2x_n$ is componentwise linear and by Proposition 2.7, we have

$$\{j;\ \beta_{i,j}(R/(J:x_2x_n))\neq 0\}\subseteq\{i+m-2,i+m\}.$$

So, the result follows by Theorem 3.1 and Remark 3.2.

In the following easy and technical lemma, I is a graded ideal generated by homogeneous forms $f_1, \dots, f_m$. For each $1 \le j \le m$, let $I_j = (f_1, \dots, f_j)$ and suppose that the ideal $L_j = (f_1, \dots, f_{j-1}) : f_j$ has initial degree d_j.

Lemma 3.4. *If the minimal free resolution of R/I is computed by iterated mapping cone and $j_\ell = \max\{i\,; \deg(f_i) + d_i \le \ell\}$, then for each $p \ge 1$,*

$$(Syz_p(I))_{<\ell+p-1>} \cong (Syz_p(I_{j_\ell}))_{<\ell+p-1>}.$$

Proof. Let $(\mathbf{F}., \delta.)$ be the minimal free resolution of R/I_{j_ℓ}, $(\mathbf{G}., d.)$ be the minimal free resolution of $R/(I_{j_\ell} : f_{j_\ell+1})$ shifted by $\deg(f_{j_\ell+1})$ and $\psi : \mathbf{G}. \to \mathbf{F}.$ be the graded complex homomorphism which is a lifting of the map $R/(I_{j_\ell} : f_{j_\ell+1})(-\deg(f_{j_\ell+1})) \to R/I_{j_\ell}$. Also, assume that $\mathcal{M}_{p+1}, \mathcal{N}_p$ and O_p, similar to the proof of Theorem 3.1, are the matrices of δ_{p+1}, d_p and ψ_p, respectively. Then, the matrix of γ_{p+1} has the following shape:

$$\mathcal{M}'_{p+1} = \left(\begin{array}{c|c} \mathcal{M}_{p+1} & O_p \\ \hline 0 & -\mathcal{N}_p \end{array} \right).$$

Note that $Syz_p(I_{j_\ell+1})$ is generated by the columns of $\mathcal{M}'_{p+1}$ and $Syz_p(I_{j_\ell})$ is generated by the columns of $\mathcal{M}_{p+1}$. Also, note that each columns of

$$\left(\begin{array}{c} O_p \\ \hline -\mathcal{N}_p \end{array} \right)$$

as elements of $Syz_p(I_{j_\ell+1})$ has degree at least $\deg(f_{j_\ell+1}) + d_{j_\ell+1} + p - 1 \ge \ell + p$. So, it is clear that

$$(Syz_p(I_{j_\ell+1}))_{\le \ell+p-1} \cong (Syz_p(I_{j_\ell}))_{\le \ell+p-1}.$$

Therefore, $(Syz_p(I_{j_\ell+1}))_{<\ell+p-1>} \cong (Syz_p(I_{j_\ell}))_{<\ell+p-1>}$. Continuing in this way, we conclude that

$$(Syz_p(I_{j_\ell}))_{<\ell+p-1>} \cong (Syz_p(I))_{<\ell+p-1>}.$$

□

For a graded ideal I, assume that $Syz_1(I)$ is minimally generated in the degrees $i_1 < \cdots < i_\ell$ and for each $1 \le r \le \ell$, let $N_{r,I} = (Syz_1(I))_{\{r\}}$.

Lemma 3.5. *If the minimal free resolution of R/I is computed by iterated mapping cone, then for each $1 \le r \le \ell$, we have:*

$$(N_{r,I})_{<i_r>} \cong (N_{r,I_{j_{i_r}}})_{<i_r>}.$$

Proof. Note that by Lemma 2.1, for each $r \geq 2$, we have

$$(N_{r,I})_{<i_r>} \cong$$
$$((N_{1,I})_{<i_1>} + \cdots + (N_{1,I})_{<i_r>})/((N_{1,I})_{<i_1>} + \cdots + (N_{1,I})_{<i_{r-1}>}),$$

and $(N_{r,I_{j_{i_r}}})_{<i_r>}$ is isomorphic to

$$((N_{1,I_{j_{i_r}}})_{<i_1>} + \cdots + (N_{1,I_{j_{i_r}}})_{<i_r>})/((N_{1,I_{j_{i_r}}})_{<i_1>} + \cdots + (N_{1,I_{j_{i_r}}})_{<i_{r-1}>}).$$

Now, by Lemma 3.4 it is clear that for each $s \leq r$, we have

$$\begin{aligned}(N_{1,I})_{<i_s>} &= (Syz_1(I))_{<i_s>}\\ &\cong (Syz_1(I_{j_{i_s}}))_{<i_s>} \cong (Syz_1(I_{j_{i_r}}))_{<i_s>}\\ &= (N_{1,I_{j_{i_r}}})_{<i_s>}.\end{aligned}$$

So, the result follows. □

4. Ideals with $(d_1,\dots,d_m)$-linear quotients

Definition 4.1. Let I be a graded ideal, $\{f_1,\dots,f_m\}$ be a homogeneous system of generators of I and $(d_1,\dots,d_m)$ be an m-tuple of positive integers with $d_1 = 1$. We say that I has $(d_1,\dots,d_m)$-linear quotients with respect to the elements $f_1,\dots,f_m$ if the ideal $(f_1,\dots,f_{j-1}) : f_j$ has d_j-linear resolution for all $j = 2,\dots,m$. If $d_2 = \cdots = d_m = d$, then we simply say that I has d-linear quotients with respect to the elements $f_1,\dots,f_m$.

Notice that this property depends on the order of the generators. Any order of the generators for which we have $(d_1,\dots,d_m)$-linear quotients will be called an admissible order of generators.

An admissible order of generators, say $f_1,\dots,f_m$, is called degree increasing if $\deg(f_1) + d_1 \leq \cdots \leq \deg(f_m) + d_m$.

In this section, we study the class of ideals with $(d_1,\dots,d_m)$-linear quotients and the particular case of ideals with 2-linear quotients. In the following, we assume that $\{f_1,\dots,f_m\}$ is a homogeneous system of generators for the graded ideal I and $I_j = (f_1,\dots,f_j)$ for all $j = 1,\dots,m$.

Theorem 4.2. *If I has 2-linear quotients with respect to the elements $f_1,\dots,f_m$ and $\deg(f_1) \leq \cdots \leq \deg(f_m)$, then for each $i \geq \deg(f_1)$, we have*

$$reg(I_{<i>}) = \begin{cases} i+1 & \text{if } i \in \{\deg(f_i);\ 1 \leq i \leq m\} \text{ and } m > 1;\\ i & \text{otherwise.}\end{cases}$$

Proof. We prove the assertion by induction on m. For $m = 1$, it is obvious that the result is true. Assume that the result is true for $m \geq 1$, I is a graded ideal which has 2-linear quotients with respect to $f_1, \dots, f_{m+1}$ and $\deg(f_1) \le \cdots \le \deg(f_{m+1})$. Let $J = (f_1, \dots, f_m)$ and $j = \deg(f_{m+1})$. Then, $I = J + (f_{m+1})$. For each $i < j$, since $I_{<i>} = J_{<i>}$, by induction hypothesis the result is true.

Note that $I_{<j>} = J_{<j>} + (f_{m+1})$. By hypothesis, $J : f_{m+1}$ is an ideal with 2-linear resolution. So, it is generated by elements of degree 2. We will show that

$$J_{<j>} : f_{m+1} = J : f_{m+1}.$$

To see it, we prove that each homogeneous generator of degree 2 of $J : f_{m+1}$ belongs to $J_{<j>} : f_{m+1}$. Let g be such a generator. So, $gf_{m+1} \in J_{<\ell>}$ where $\ell = \deg(g) + \deg(f_{m+1}) > j$. Since J is generated by elements of degrees at most j, $J_{<\ell>} = \mathfrak{m}^{\ell-j} J_{<j>}$. So, $gf_{m+1} \in J_{<j>}$ and the conclusion follows.

Now, consider the following short exact sequence

$$0 \to R/(J : f_{m+1})(-j) \to R/J_{<j>} \to R/I_{<j>} \to 0.$$

By hypothesis, $reg(R/(J : f_{m+1})(-j)) = j + 1$ and

$$reg(R/J_{<j>}) = reg(J_{<j>}) - 1 = \begin{cases} j, & \deg(f_m) = j \text{ and } m > 1; \\ j-1, & \text{otherwise.} \end{cases}$$

By applying the *reg formula* (see [9, Corollary 18.7]) to the above short exact sequence, we have

$$reg(I_{<j>}) = reg(R/I_{<j>}) + 1 = j + 1.$$

So, the assertion follows for $i = j$.

If $i = j + 1$, consider the following short exact sequence

$$0 \to I_{<j+1>} \to I_{<j>} \to I_{<j>}/I_{<j+1>} \to 0.$$

Since $I_{<j+1>} = \mathfrak{m} I_{<j>}$,

$$I_{<j>}/I_{<j+1>} = \bigoplus \mathbf{k}(-j).$$

So, $reg(I_{<j>}/I_{<j+1>}) = j$. Again, by applying the reg formula we have $reg(I_{<j+1>}) = j + 1$.

Assume that $i > j+1$. Since I is generated by elements of degrees at most j, $I_{<i>} = \mathfrak{m}^{i-j+1} I_{<j+1>}$ and by Lemma 2.2, we have $reg(I_{<i>}) = i$. □

Next, we present some examples of ideals which satisfies Theorem 4.2.

Example 4.3. Let

$$I = (x_1^2x_2, x_2x_3^2, x_1x_3x_4, x_2^2x_4^2) \subset \mathbf{k}[x_1, x_2, x_3, x_4].$$

Then I has 2-linear quotients with respect to $x_1^2x_2, x_2x_3^2, x_1x_3x_4, x_2^2x_4^2$ and satisfies Theorem 4.2.

Example 4.4. Let

$$I = (x_1x_2x_5, x_2x_3x_6, x_1x_3x_7, x_1x_4x_6, x_2x_4x_7, x_3x_4x_5) \subset \mathbf{k}[x_1, \ldots, x_7].$$

Then I has 2-linear quotients with respect to

$$x_1x_2x_5, x_2x_3x_6, x_1x_3x_7, x_1x_4x_6, x_2x_4x_7, x_3x_4x_5$$

and satisfies Theorem 4.2.

In the next two examples, we have ideals with 2-linear quotients but the given admissible order of the generators is not degree increasing.

Example 4.5. Let

$$I = (x_1x_2x_5x_6, x_1x_2x_3, x_3x_4, x_2x_5x_7) \subset \mathbf{k}[x_1, \ldots, x_7].$$

Then I has 2-linear quotients with respect to

$$x_1x_2x_5x_6, x_1x_2x_3, x_3x_4, x_2x_5x_7.$$

But this ordering of generators is not degree increasing. If we reorder the generators as $x_3x_4, x_1x_2x_3, x_2x_5x_7, x_1x_2x_5x_6$ then we have a degree increasing admissible order for $(1, 1, 2, 1)$-linear quotients property.

Example 4.6. Let

$$I = (x_1x_2x_3x_7, x_1x_2x_5x_6, x_4x_5x_6) \subset \mathbf{k}[x_1, \ldots, x_7].$$

Then I has 2-linear quotients with respect to

$$x_1x_2x_3x_7, x_1x_2x_5x_6, x_4x_5x_6.$$

This ordering of generators is not degree increasing and there is no degree increasing admissible order of generators for having some $(1, d_1, d_2)$-linear quotients property.

The above example shows that if a monomial ideal I has $(d_1, \ldots, d_m)$-linear quotients, then in general we can not conclude that $G(I)$ has a degree increasing admissible order. This is an important difference with the case of monomial ideals with linear quotients.

Theorem 4.7. *If I has $(d_1, \ldots, d_m)$-linear quotients with respect to $f_1, \ldots, f_m$ and $\deg(f_1) + d_1 \leq \cdots \leq \deg(f_m) + d_m$, then the minimal free resolution of R/I is given by the iterated mapping cone.*

Moreover,

- $\forall i \geq 2$ *and* $\forall j \notin \{\deg(f_\ell)+d_\ell+i-2;\ 1 \leq \ell \leq m\}$, $\beta_{i,j}(R/I) = 0$.
- $reg(R/I) = \deg(f_m) + d_m - 2$.

Proof. Let $t \geq 1$ and assume that the minimal free resolution of R/I_t is already known by the iterated mapping cone (for the case $t = 1$ we just consider the obvious minimal free resolution of R/I_1). We can easily see that I_t is minimally generated by $f_1, \dots, f_t$ and for each $i \geq 2$ and $j \notin \{\deg(f_\ell) + d_\ell + i - 2;\ 1 \leq \ell \leq t\}$, $\beta_{i,j}(R/I_t) = 0$. Since, by the assumption, $\deg(f_1) + d_1 \leq \cdots \leq \deg(f_m) + d_m$, for each $i \geq 1$,

$$\max\{j;\ \beta_{i,j}(R/I_t) \neq 0\} \leq \deg(f_t) + d_t + i - 2.$$

On the other hand, since $L_{t+1} = (f_1, \dots, f_t) : f_{t+1}$ has d_{t+1}-linear resolution, for each $1 \leq i \leq pd(R/L_{t+1})$, we have

$$\min\{j;\ \beta_{i,j}(R/L_{t+1}) \neq 0\} = d_{t+1} + i - 1.$$

It is clear that $d_{t+1} + i - 1 > \deg(f_t) + d_t + i - 2 - \deg(f_{t+1})$. So, Equation (3.1) holds and by Theorem 3.1, the mapping cone arising from the short exact sequence

$$0 \to R/L_{t+1}(-\deg(f_{t+1})) \to R/I_t \to R/I_{t+1} \to 0,$$

is the minimal free resolution of R/I_{t+1} and the conclusion follows. $\square$

Example 4.8. Let $I = (x_1x_2, x_2x_3, x_4x_5, x_1x_3x_4) \subset \mathbf{k}[x_1, x_2, x_3, x_4]$. Then I has $(1, 1, 2, 1)$-linear quotients and I satisfies in Theorem 4.7.

In the following, we show that if I has $(d_1, \dots, d_m)$-linear quotients with respect to $f_1, \dots, f_m$ and $\deg(f_1) + d_1 \leq \cdots \leq \deg(f_m) + d_m$, then $Syz_1(I)$ is a componentwise linear module.

Theorem 4.9. *If I has $(d_1, \dots, d_m)$-linear quotients with respect to $f_1, \dots, f_m$ and $\deg(f_1) + d_1 \leq \cdots \leq \deg(f_m) + d_m$, then $Syz_1(I)$ is a componentwise linear module.*

Proof. Suppose that $Syz_1(I)$ is minimally generated in degrees $i_1 < \cdots < i_\ell$. For each $1 \leq t \leq \ell$, let

$$j_{i_t} = \max\{i;\ \deg(f_i) + d_i \leq i_t\}, \quad I_{j_{i_t}} = (f_1, \dots, f_{j_{i_t}})$$

and

$$N_{r,I} = (Syz_1(I))_{\{r\}}, \quad N_{r,I_{j_{i_t}}} = (Syz_1(I_{j_{i_t}})_{\{r\}}.$$

By induction on r, we show that for each $1 \leq r \leq \ell$ the module $N_{r,I}$ (resp. $N_{r,I_{j_{i_t}}}$ for each $t \geq r$) has the following properties:

(1) $\beta_{i,j}(N_{r,I}) = 0 \quad \forall j \neq i_r + i, \cdots, i_\ell + i$ (resp. $\beta_{i,j}(N_r, I_{j_{i_t}}) = 0 \quad \forall j \neq i_r + i, \cdots, i_t + i$).

(2) $(N_{r,I})_{<i_r>}$ has i_r-linear resolution (resp. $(N_{r,I_{j_{i_t}}})_{<i_r>}$ has i_r-linear resolution).

If $r=1$, then $N_{1,I}=Syz_1(I)$ (resp., $N_{1,I_{j_{i_t}}}=Syz_1(I_{j_{i_t}})$ for each $t\geq 1$). Since by Theorem 4.7, the minimal free resolution of R/I (resp., $R/I_{j_{i_t}}$) is given by the iterated mapping cone, it is clear that $\beta_{i,j}(N_{1,I})=0$ for each $j\neq i_1+i,\cdots,i_\ell+i$ (resp., $\beta_{i,j}(N_{1,I_{j_{i_t}}})=0$ for each $j\neq i_1+i,\cdots,i_t+i$). So (1) follows for $r=1$.

By Lemma 3.5, $(N_{1,I})_{<i_1>}\cong(N_{1,I_{j_{i_1}}})_{<i_1>}\cong(N_{1,I_{j_{i_t}}})_{<i_1>}$ for each $t\geq 1$. Moreover, the ideal $I_{j_{i_1}}$ is generated by $f_1,\dots,f_{j_{i_1}}$. By Theorem 4.7, the minimal free resolution of $R/I_{j_{i_1}}$ is computed by the iterated mapping cone and we have $i_1=\deg(f_1)+d_1=\cdots=\deg(f_{j_{i_1}})+d_{j_{i_1}}$. So, again by Theorem 4.7, $Syz_1(I_{j_{i_1}})$ is generated in degree i_1 and has i_1-linear resolution. So (2) follows for $r=1$.

Now, assume that (1), (2) is true for $N_{r-1,I}$ (resp., $N_{r-1,I_{j_{i_t}}}$ for each $t\geq r-1$) where $1\leq r-1<\ell$. We prove that $N_{r,I}$ (resp., $N_{r,I_{j_{i_t}}}$ for each $t\geq r$) satisfies (1), (2).

By definition,

$$N_{r,I}=N_{r-1,I}/(N_{r-1,I})_{<i_{r-1}>}\ (\text{resp. } N_{r,I_{j_{i_t}}}=N_{r-1,I_{j_{i_t}}}/(N_{r-1,I_{j_{i_t}}})_{<i_{r-1}>}).$$

By the induction hypothesis, $(N_{r-1,I})_{<i_{r-1}>}$ (resp., $(N_{r-1,I_{j_{i_t}}})_{<i_{r-1}>}$) has i_{r-1}-linear resolution and $\beta_{i,j}(N_{r-1,I})=0\ \ \forall j\neq i_{r-1}+i,\cdots,i_\ell+i$ (resp., $\beta_{i,j}(N_{r-1,I_{j_{i_t}}})=0\ \ \forall j\neq i_{r-1}+i,\cdots,i_t+i$).

Since $(N_{r-1,I})_{<i_{r-1}>}$ (resp., $(N_{r-1,I_{j_{i_t}}})_{<i_{r-1}>}$) has i_{r-1}-linear resolution, by Lemma 2.6, it is clear that $\beta_{i,j}(N_{r,I})=0\ \ \forall j\neq i_r+i,\dots,i_\ell+i$ (resp., $\beta_{i,j}(N_{r,I_{j_{i_t}}})=0\ \ \forall j\neq i_r+i,\dots,i_t+i$). So (1) follows.

Now, by Lemma 3.5,

$$\begin{aligned}(N_{r,I})_{<i_r>}&\cong(N_{r,I_{j_{i_r}}})_{<i_r>}\\&\cong((N_{r-1,I_{j_{i_r}}})/(N_{r-1,I_{j_{i_r}}})_{<i_{r-1}>})_{<i_r>}\\&\cong(N_{r,I_{j_{i_t}}})_{<i_r>},\end{aligned}$$

where by the induction hypothesis, $(N_{r-1,I_{j_{i_r}}})_{<i_{r-1}>}$ has i_{r-1}-linear resolution and $\beta_{i,j}(N_{r-1,I_{i_r}})=0$, for each $j\neq i+i_{r-1},i+i_r$. So, by Lemma 2.6, $(N_{r-1,I_{j_{i_r}}})/(N_{r-1,I_{j_{i_r}}})_{<i_{r-1}>}$ is generated in degree i_r and has i_r-linear resolution. This means that

$$(N_{r,I})_{<i_r>}\cong(N_{r,I_{j_{i_t}}})_{<i_r>}\cong N_{r-1,I_{j_{i_r}}}/(N_{r-1,I_{j_{i_r}}})_{<i_{r-1}>}$$

has i_r-linear resolutio. So (2) follows for r.

Now, since (2) holds for each $1\leq r\leq\ell$, by Corollary 2.4, $Syz_1(I)$ is a componentwise linear module.

□

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Leila Sharifan

Department of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran

and

School of Mathematics, Institute for research in Fundamental Sciences (IPM), P. O. Box: 19395-5746, Tehran, Iran.

Email: leila-sharifan@aut.ac.ir