

## IDEALS WITH $(d_1, \dots, d_m)$ -LINEAR QUOTIENTS

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ABSTRACT. In this paper, we introduce the class of ideals with  $(d_1, \dots, d_m)$ -linear quotients generalizing the class of ideals with linear quotients. Under suitable conditions, we control the numerical invariants of a minimal free resolution of ideals with  $(d_1, \dots, d_m)$ -linear quotients. In particular, we show that their first module of syzygies is a componentwise linear module.

### 1. INTRODUCTION

Let  $\mathbf{k}$  be a field, and  $R = \mathbf{k}[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables. In this paper, we introduce and study a class of ideals in  $R$  which can be considered as a generalization of the class of ideals with linear quotients (see, [8, 10]).

Let  $I$  be a graded ideal,  $\{f_1, \dots, f_m\}$  be a homogeneous system of generators of  $I$  and  $(d_1, \dots, d_m)$  be an  $m$ -tuple of positive integers supposing  $d_1 = 1$ . We say that  $I$  has  $(d_1, \dots, d_m)$ -linear quotients with respect to the elements  $f_1, \dots, f_m$  if the ideal  $(f_1, \dots, f_{j-1}) : f_j$  has  $d_j$ -linear resolution for all  $j = 2, \dots, m$ . Notice that, this property depends on the order of the generators. If  $d_2 = \dots = d_m = d$ , we simply say that  $I$  has  $d$ -linear quotients with respect to the elements  $f_1, \dots, f_m$  and if  $d = 1$ , we get the usual class of ideals with linear quotients.

Monomial ideals with linear quotients were introduced in [8] and have strong combinatorial implication (see for example, [11]). A very

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MSC(2010): Primary: 13D02; Secondary: 13F20, 16W50

Keywords: Mapping cone,  $(d_1, \dots, d_m)$ -linear quotients, componentwise linear module, regularity.

Received: 17 March 2017, Accepted: 08 January 2018.

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important property of ideals with linear quotients is that they are componentwise linear (see, [10, Corollary 2.4]).

Recall that componentwise linear modules over a polynomial ring has been introduced by Herzog and Hibi, enlarging the class of the graded modules with a  $d$ -linear resolution (see [6]). Interesting results concerning their graded Betti numbers has been proved by Aramova, Conca, Herzog and Hibi (see [1, 2, 3, 6, 7]). Later, Römer (see [12]) studied more homological properties of componentwise linear modules in the general setting of finitely generated modules over Koszul algebras (instead of polynomial rings).

In this paper, we assume that  $I = (f_1, \dots, f_m)$  has  $(d_1, \dots, d_m)$ -linear quotients with respect to  $f_1, \dots, f_m$  and  $\deg(f_1) + d_1 \leq \dots \leq \deg(f_m) + d_m$ . In Theorem 4.2, we study the case of ideals with 2-linear quotients and we prove a property of these ideals which is close to the componentwise linear property. In Theorem 4.7, we study the minimal free resolution of  $R/I$  by iterated mapping cone and precisely we compute the regularity of  $R/I$ . Finally, in Theorem 4.9, we show that  $Syz_1(I)$  is a componentwise linear module.

We organize the paper as it follows: In Section 2, we review some basic definitions, notations and results that we need in subsequent sections. In Section 3, we give a sufficient condition for minimality of a resolution obtained by the mapping cone (see Theorem 3.1). Next, we give some easy and technical lemmas that we need for studying  $Syz_1(I)$ . Section 4 is devoted to the main results about ideals with  $(d_1, \dots, d_m)$ -linear quotients.

Furthermore, the paper includes several examples to illustrate and delimitate the results. Definitely, via these examples, we examine some ideals with  $(d_1, \dots, d_m)$ -linear quotients to see if they have nice properties of ideals with linear quotients or not (see [10, 11]).

## 2. PRELIMINARIES

The *Castelnuovo-Mumford regularity* (or briefly regularity) of a graded finitely generated  $R$ -module  $M$ , is defined as

$$reg(M) = \max\{j - i; \beta_{i,j}(M) \neq 0\}$$

and the projective dimension of  $M$  is defined as

$$pd(M) = \max\{i; \beta_{i,j}(M) \neq 0 \text{ for some } j\},$$

where  $\beta_{i,j}(M)$  is the  $(i, j)$ th graded Betti number of  $M$ .

Let

$$\dots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M$$

be the graded minimal free resolution of  $M$ . Then, the  $p$ -th syzygy module of  $M$ , denoted by  $Syz_p(M)$ , is defined as  $Syz_p(M) = \ker(\delta_{p-1}) = \text{Im}(\delta_p)$ . Recall that for each  $j$ , the differential  $\delta_j$  is given by a matrix  $\mathcal{M}_j$  (which depends on the chosen basis of  $F_j$ s). So  $Syz_p(M)$  is generated by the columns of  $\mathcal{M}_p$ .

Let  $M$  be a graded  $R$ -module. The *initial degree* of  $M$  is defined as

$$\text{indeg}(M) = \min\{d \in \mathbf{Z}; M_d \neq 0\}.$$

For  $d \in \mathbf{Z}$ , we write  $M_{\langle d \rangle}$  for the submodule of  $M$  which is generated by all homogeneous elements of  $M$  with degree  $d$ . Moreover, we write  $M_{\leq d}$  for the module generated by all homogeneous elements in  $M$  whose degrees are less than or equal to  $d$ .

If  $N$  is a graded submodule of  $M$ , then

$$(M/N)_{\langle a \rangle} \cong (M_{\langle a \rangle} + N)/N.$$

For a module  $M$  minimally generated in degrees  $i_1 < \dots < i_\ell$ , we define  $M_{\{1\}} = M$  and for every  $j = 2, \dots, \ell$ ,

$$M_{\{j\}} := M_{\{j-1\}} / (M_{\{j-1\}})_{\langle \text{indeg}(M_{\{j-1\}}) \rangle} = M_{\{j-1\}} / (M_{\{j-1\}})_{\langle i_{j-1} \rangle}.$$

**Lemma 2.1.** *If  $M$  is a module minimally generated in degrees  $i_1 < \dots < i_\ell$ , then for each  $2 \leq r \leq \ell$ ,*

$$(M_{\{r\}})_{\langle i_r \rangle} \cong (M_{\langle i_1 \rangle} + \dots + M_{\langle i_r \rangle}) / M_{\langle i_1 \rangle} + \dots + M_{\langle i_{r-1} \rangle}.$$

*Proof.* Note that

$$\begin{aligned} (M_{\{r\}})_{\langle i_r \rangle} &= (M_{\{r-1\}} / (M_{\{r-1\}})_{\langle i_{r-1} \rangle})_{\langle i_r \rangle} \\ &= ((M_{\{r-1\}})_{\langle i_r \rangle} + (M_{\{r-1\}})_{\langle i_{r-1} \rangle}) / (M_{\{r-1\}})_{\langle i_{r-1} \rangle} \end{aligned}$$

and if we continue in this way, we get the desired result.  $\square$

Let  $d \in \mathbf{Z}$ . We say that  $M$  has a  $d$ -linear resolution if  $\beta_{i,j}(M) = 0$  for  $j \neq d + i$ , and we say  $M$  is componentwise linear if for all integers  $d$  the module  $M_{\langle d \rangle}$  has a  $d$ -linear resolution.

For more information concerning the componentwise linear modules, see [2, 3, 6, 12]. We select here some good properties of their graded minimal free resolutions.

**Lemma 2.2.** *If  $M$  is a graded  $R$ -module and it has an  $i$ -linear resolution, then  $\mathfrak{m}M$  has an  $i + 1$ -linear resolution, where  $\mathfrak{m} = (x_1, \dots, x_n)$  is the homogeneous maximal ideal of  $R$ .*

**Lemma 2.3.** (see [12, Lemma 3.2.2]) *Let  $M$  be a graded  $R$ -module. Then the following statements are equivalent:*

- (i):  $M$  is componentwise linear;

(ii):  $M/M_{\langle \text{indeg}(M) \rangle}$  is componentwise linear and  $M_{\langle \text{indeg}(M) \rangle}$  has an  $\text{indeg}(M)$ -linear resolution.

The following corollary is an immediate consequence of the above lemma.

**Corollary 2.4.** *Let  $M$  be a graded module minimally generated in degrees  $i_1 < \dots < i_\ell$ . Then  $M$  is a componentwise linear module if and only if for each  $1 \leq j \leq \ell$ ,  $(M_{\{j\}})_{\langle i_j \rangle}$  has an  $i_j$ -linear resolution.*

Following Römer, we define a special subcomplex of the minimal graded free resolution of a module.

**Definition 2.5.** Let  $M$  be a graded  $R$ -module and  $(\mathbf{G}., d.)$  be the minimal graded free resolution of  $M$ . We define the subcomplex  $(\widetilde{\mathbf{G}}., \widetilde{d}.)$  of  $(\mathbf{G}., d.)$  to be

$$\widetilde{G}_i = R(-(i + \text{indeg}(M)))^{\beta_{i, i + \text{indeg}(M)}} \subseteq G_i \text{ and } \widetilde{d} = d|_{\widetilde{\mathbf{G}}}.$$

**Lemma 2.6.** (see [12, Lemma 3.2.4]) *Let  $M$  be a graded  $R$ -module such that  $M_{\langle \text{indeg}(M) \rangle}$  has a linear resolution, and let  $(\mathbf{G}., d.)$  be the minimal graded free resolution of  $M$ . Then:*

- (i):  $\widetilde{\mathbf{G}}.$  is the minimal graded free resolution of  $M_{\langle \text{indeg}(M) \rangle}$ .
- (ii):  $\mathbf{G}./\widetilde{\mathbf{G}}.$  is the minimal graded free resolution of  $M/M_{\langle \text{indeg}(M) \rangle}$ .

**Proposition 2.7.** (see [13, Proposition 2.2]) *Let  $M$  be a componentwise linear  $R$ -module minimally generated in degrees  $i_1 < \dots < i_\ell$ . Then for each  $1 \leq i \leq \text{pd}(M)$ , we have*

$$\beta_{i,j}(M) = 0 \text{ for } j \neq i + i_1, \dots, i_\ell + i.$$

Next, we review some basic properties of ideals with linear quotients.

Let  $I$  be a graded ideal and  $\{f_1, \dots, f_m\}$  be a homogeneous system of generators of  $I$  and  $I_j = (f_1, \dots, f_j)$  for  $j = 1, \dots, m$ . We say that  $I$  has linear quotients with respect to the elements  $f_1, \dots, f_m$ , if the ideal  $I_{j-1} : f_j$  is generated by linear forms for all  $j = 2, \dots, m$ . Notice that this property depends on the order of the generators. Any order of the generators for which we have linear quotients will be called an admissible order. If  $I$  has linear quotients with respect to an admissible order of a homogeneous system of generators, we simply say  $I$  has linear quotients. Ideals with linear quotients have the following properties:

**Proposition 2.8.** (see [10, Corollary 2.4]) *If the graded ideal  $I$  has linear quotients with respect to the elements  $f_1, \dots, f_m$ , then  $I$  is componentwise linear provided that  $\{f_1, \dots, f_m\}$  is a minimal system of generators.*

For a monomial ideal  $I$ , we denote by  $G(I)$  the unique minimal system of monomial generators of  $I$ . In this case, when we say  $I$  has linear quotients, we mean  $I$  has linear quotients with respect to an admissible order of  $G(I)$

**Proposition 2.9.** (see [11, Lemma 2.1]) *If a monomial ideal  $I$  has linear quotients, then there exists a degree increasing admissible order of  $G(I)$ .*

### 3. MAPPING CONE TECHNIQUE

One of the fundamental tools for computing free resolutions is mapping cone technique. Many well-known free resolutions arise as iterated mapping cones. For example, the Taylor resolution of monomial ideals.

The idea of the iterated mapping cone construction is the following: Let  $\{f_1, \dots, f_m\}$  be a homogeneous system of generators for  $I$ , and  $I_j = (f_1, \dots, f_j)$ . Then, for  $j = 2, \dots, m$ , there are exact sequences

$$0 \rightarrow R/(I_{j-1} : f_j) \rightarrow R/I_{j-1} \rightarrow R/I_j \rightarrow 0$$

assuming that a free  $R$ -resolution  $(\mathbf{F}., \delta.)$  of  $R/I_{j-1}$  and a free  $R$ -resolution  $(\mathbf{G}., d.)$  of  $R/(I_{j-1} : f_j)$  are known, we can obtain a resolution  $(\mathbf{M}(\psi), \gamma.)$  of  $R/I_j$  as a *mapping cone* of a complex homomorphism  $\psi : \mathbf{G} \rightarrow \mathbf{F}.$ , which is a lifting of the map  $R/(I_{j-1} : f_j) \rightarrow R/I_{j-1}$ . The mapping cone  $\mathbf{M}(\psi)$  is the complex such that

$$(M(\psi))_i = F_i \oplus G_{i-1},$$

with the differential maps

$$\gamma_i(x, y) = (\psi_{i-1}(y) + \delta_i(x), -d_{i-1}(y)),$$

where  $x \in F_i$  and  $y \in G_{i-1}$ . This complex is exact (see [4, Page 650 and Proposition A3.19.]), so, it is a free resolution for  $R/I_j$ .

It is clear that in this way, we get a free resolution of  $R/I$ . Of course, in general, such a resolution may be non-minimal. For example if  $I = (f_1, f_2, f_3)$  where  $f_1 = x_1^2, f_2 = x_2^3, f_3 = x_1x_2$ , the result of the iterated mapping cone is not a minimal free resolution. But, there are some important classes of ideals for which the minimal free resolution obtained by iterated mapping cone. For example, the Eliahou-Kervaire resolution of stable monomial ideals (as noted by Evans and Charalambous[5]). More in general, if  $I$  has linear quotients with respect to a minimal homogeneous system of generators, then its minimal free resolution can be obtained by iterated mapping cone. This is an immediate consequence of [10, Corollary 2.7].

Here, we give a sufficient condition to check the minimality of a resolution obtained by the mapping cone technique.

**Theorem 3.1.** *Let  $I$  be a graded ideal of  $R$  and  $f$  be a homogeneous form of degree  $d$  which does not belong to  $I$ . Then, we have the following graded short exact sequence:*

$$0 \rightarrow R/(I : f)(-d) \rightarrow R/I \rightarrow R/I + (f) \rightarrow 0.$$

*Assuming that the minimal free resolution of the modules  $R/(I : f)$  and  $R/I$  are already known. Then, the minimal free resolution of  $R/I + (f)$  is obtained by the mapping cone provided that for each  $1 \leq i \leq \text{pd}(R/(I : f))$ ,*

$$\{j; \beta_{i,j}(R/(I : f)) \neq 0\} \cap \{j - d; \beta_{i,j}(R/I) \neq 0\} = \emptyset, \quad (3.1)$$

*and in this case*

**(a):**

$$\beta_{i,j}(R/I + (f)) = \beta_{i,j}(R/I) + \beta_{i-1,j-d}(R/(I : f)),$$

**(b):**

$$\text{reg}(R/(I + (f))) = \max\{\text{reg}(R/I), \text{reg}(R/(I : f)) + d - 1\}$$

**(c):**

$$\text{pd}(R/(I + (f))) = \max\{\text{pd}(R/I), \text{pd}(R/(I : f)) + 1\}.$$

*Proof.* Let  $(\mathbf{F}, \delta)$  be the minimal free resolution of  $R/I$ ,  $(\mathbf{G}, d)$  be the minimal free resolution of  $R/(I : f)$  shifted by  $d$  and  $\psi : \mathbf{G} \rightarrow \mathbf{F}$  be the complex graded homomorphism which is a lifting of the map  $R/(I : f)(-d) \rightarrow R/I$ . It is enough to show that the mapping cone complex is the minimal free resolution of  $R/(I + (f))$ .

Let for each  $r$ ,  $\mathcal{M}_r$  (resp.,  $\mathcal{N}_r$ ) be the matrix of  $\delta_r$  (resp.,  $d_r$ ) with respect to the canonical basis of  $F_r$  and  $F_{r-1}$  (resp.,  $G_r$  and  $G_{r-1}$ ). Also, assume that for each  $r$ ,  $O_r$  be the matrix of  $\psi_r : G_r \rightarrow F_r$ . Then, by the mapping cone construction, the matrix of  $\gamma_r$ , with respect to the canonical basis of  $F_r \oplus G_{r-1}$  and  $F_{r-1} \oplus G_{r-2}$ , is denoted by  $\mathcal{M}'_r$  has the following shape;

$$\mathcal{M}'_r = \left( \begin{array}{c|c} \mathcal{M}_r & O_{r-1} \\ \hline 0 & -\mathcal{N}_{r-1} \end{array} \right).$$

So, the result of the mapping cone is the minimal free resolution if and only if  $\text{Im}(\psi) \subset \mathfrak{m}\mathbf{F}$ .

Let  $e_1, \dots, e_{\beta_i(R/(I:f))}$  be the basis of  $\mathbf{G}$  in the homological degree  $i$ , and  $\eta_1, \dots, \eta_{\beta_i(R/I)}$  be the basis of  $\mathbf{F}$  in the homological degree  $i$ . Then, by the hypothesis  $\psi_i : G_i \rightarrow F_i$  is given by  $\psi_i(e_j) = \sum_{t=1}^{\beta_i(R/I)} a_{it} \eta_t$ , where for each  $1 \leq t \leq \beta_i(R/I)$  if  $a_{it} \neq 0$  then  $\deg(e_j) > \deg(\eta_t)$ . So,  $\deg(a_{it}) > 0$  for each  $i$  and  $t$  when  $a_{it} \neq 0$ . So, the conclusion follows.

The parts (a), (b), (c) are directly followed by the minimality of the obtained resolution.  $\square$

*Remark 3.2.* If  $I = (f_1, \dots, f_m)$  and  $I + (f)$  is *minimally* generated by  $\{f_1, \dots, f_m, f\}$ , then  $\text{Im}(\psi_1) \subseteq \mathfrak{m}F_1$  and we just need to check Equation 3.1 for  $2 \leq j \leq \text{pd}(R/(I : f))$ .

Next, we give an example in which the minimal free resolution is computed by iterated mapping cone by successive using Theorem 3.1. We first recall the definition of lex-segment ideals.

A monomial ideal  $I \subset R$  is called a *lex-segment ideal* if for all monomials  $u \in I$  and all monomials  $v \in R$  with  $\deg(u) = \deg(v)$  and  $v >_{\text{lex}} u$ , one has  $v \in I$ .

**Example 3.3.** Let

$$I = (x_1^2, x_1x_2, \dots, x_1x_n, x_2^m, x_2^{m-1}x_3, \dots, x_2^{m-1}x_i, x_2^{m-1}x_{i+1}^3, x_2^{m-1}x_{i+1}^2x_{i+2}, \dots, x_2^{m-1}x_{i+1}^2x_{n-1}, x_2x_n) \subseteq R$$

where  $m > 1$ . Then the minimal free resolution of  $R/I$  is given by the iterated mapping cone. It is easy to see that in each step, Equation 3.1 holds. Let us just check the final step. Notice that

$$J = (x_1^2, x_1x_2, \dots, x_1x_n, x_2^m, x_2^{m-1}x_3, \dots, x_2^{m-1}x_i, x_2^{m-1}x_{i+1}^3, x_2^{m-1}x_{i+1}^2x_{i+2}, \dots, x_2^{m-1}x_{i+1}^2x_{n-1})$$

is a Lex-segment ideal. So,  $J$  has linear quotients with respect to

$$x_1^2, x_1x_2, \dots, x_1x_n, x_2^m, x_2^{m-1}x_3, \dots, x_2^{m-1}x_i, x_2^{m-1}x_{i+1}^3, x_2^{m-1}x_{i+1}^2x_{i+2}, \dots, x_2^{m-1}x_{i+1}^2x_{n-1}.$$

Therefore,  $J$  is a componentwise linear ideal and by Proposition 2.7,

$$\{j - 2; \beta_{i,j}(R/J) \neq 0\} \subseteq \{i + m - 3, i + m - 1, i - 1\}.$$

$$J : x_2x_n = (x_1, x_2^{m-1}, x_2^{m-2}x_3, \dots, x_2^{m-2}x_i, x_2^{m-2}x_{i+1}^3, x_2^{m-2}x_{i+1}^2x_{i+2}, \dots, x_2^{m-2}x_{i+1}^2x_{n-1})$$

is again a lex-segment ideal and it has linear quotients with respect to

$$x_1, x_2^{m-1}, x_2^{m-2}x_3, \dots, x_2^{m-2}x_i, x_2^{m-2}x_{i+1}^3, x_2^{m-2}x_{i+1}^2x_{i+2}, \dots, x_2^{m-2}x_{i+1}^2x_{n-1}.$$

Thus,  $J : x_2x_n$  is componentwise linear and by Proposition 2.7, we have

$$\{j; \beta_{i,j}(R/(J : x_2x_n)) \neq 0\} \subseteq \{i + m - 2, i + m\}.$$

So, the result follows by Theorem 3.1 and Remark 3.2.

In the following easy and technical lemma,  $I$  is a graded ideal generated by homogeneous forms  $f_1, \dots, f_m$ . For each  $1 \leq j \leq m$ , let  $I_j = (f_1, \dots, f_j)$  and suppose that the ideal  $L_j = (f_1, \dots, f_{j-1}) : f_j$  has initial degree  $d_j$ .

**Lemma 3.4.** *If the minimal free resolution of  $R/I$  is computed by iterated mapping cone and  $j_\ell = \max\{i ; \deg(f_i) + d_i \leq \ell\}$ , then for each  $p \geq 1$ ,*

$$(Syz_p(I))_{<\ell+p-1>} \cong (Syz_p(I_{j_\ell}))_{<\ell+p-1>}.$$

*Proof.* Let  $(\mathbf{F}, \delta)$  be the minimal free resolution of  $R/I_{j_\ell}$ ,  $(\mathbf{G}, d)$  be the minimal free resolution of  $R/(I_{j_\ell} : f_{j_\ell+1})$  shifted by  $\deg(f_{j_\ell+1})$  and  $\psi : \mathbf{G} \rightarrow \mathbf{F}$  be the graded complex homomorphism which is a lifting of the map  $R/(I_{j_\ell} : f_{j_\ell+1})(-\deg(f_{j_\ell+1})) \rightarrow R/I_{j_\ell}$ . Also, assume that  $\mathcal{M}_{p+1}$ ,  $\mathcal{N}_p$  and  $O_p$ , similar to the proof of Theorem 3.1, are the matrices of  $\delta_{p+1}$ ,  $d_p$  and  $\psi_p$ , respectively. Then, the matrix of  $\gamma_{p+1}$  has the following shape:

$$\mathcal{M}'_{p+1} = \left( \begin{array}{c|c} \mathcal{M}_{p+1} & O_p \\ \hline 0 & -\mathcal{N}_p \end{array} \right).$$

Note that  $Syz_p(I_{j_\ell+1})$  is generated by the columns of  $\mathcal{M}'_{p+1}$  and  $Syz_p(I_{j_\ell})$  is generated by the columns of  $\mathcal{M}_{p+1}$ . Also, note that each columns of

$$\left( \begin{array}{c} O_p \\ -\mathcal{N}_p \end{array} \right)$$

as elements of  $Syz_p(I_{j_\ell+1})$  has degree at least  $\deg(f_{j_\ell+1}) + d_{j_\ell+1} + p - 1 \geq \ell + p$ . So, it is clear that

$$(Syz_p(I_{j_\ell+1}))_{\leq \ell+p-1} \cong (Syz_p(I_{j_\ell}))_{\leq \ell+p-1}.$$

Therefore,  $(Syz_p(I_{j_\ell+1}))_{<\ell+p-1>} \cong (Syz_p(I_{j_\ell}))_{<\ell+p-1>}$ . Continuing in this way, we conclude that

$$(Syz_p(I_{j_\ell}))_{<\ell+p-1>} \cong (Syz_p(I))_{<\ell+p-1>}.$$

□

For a graded ideal  $I$ , assume that  $Syz_1(I)$  is minimally generated in the degrees  $i_1 < \dots < i_\ell$  and for each  $1 \leq r \leq \ell$ , let  $N_{r,I} = (Syz_1(I))_{\{r\}}$ .

**Lemma 3.5.** *If the minimal free resolution of  $R/I$  is computed by iterated mapping cone, then for each  $1 \leq r \leq \ell$ , we have:*

$$(N_{r,I})_{<i_r>} \cong (N_{r,I_{i_r}})_{<i_r>}.$$



*Proof.* Note that by Lemma 2.1, for each  $r \geq 2$ , we have

$$(N_{r,I})_{\langle i_r \rangle} \cong ((N_{1,I})_{\langle i_1 \rangle} + \cdots + (N_{1,I})_{\langle i_r \rangle}) / ((N_{1,I})_{\langle i_1 \rangle} + \cdots + (N_{1,I})_{\langle i_{r-1} \rangle}),$$

and  $(N_{r,I_{j_{i_r}}})_{\langle i_r \rangle}$  is isomorphic to

$$((N_{1,I_{j_{i_r}}})_{\langle i_1 \rangle} + \cdots + (N_{1,I_{j_{i_r}}})_{\langle i_r \rangle}) / ((N_{1,I_{j_{i_r}}})_{\langle i_1 \rangle} + \cdots + (N_{1,I_{j_{i_r}}})_{\langle i_{r-1} \rangle}).$$

Now, by Lemma 3.4 it is clear that for each  $s \leq r$ , we have

$$\begin{aligned} (N_{1,I})_{\langle i_s \rangle} &= (Syz_1(I))_{\langle i_s \rangle} \\ &\cong (Syz_1(I_{j_{i_s}}))_{\langle i_s \rangle} \cong (Syz_1(I_{j_{i_r}}))_{\langle i_s \rangle} \\ &= (N_{1,I_{j_{i_r}}})_{\langle i_s \rangle}. \end{aligned}$$

So, the result follows.  $\square$

#### 4. IDEALS WITH $(d_1, \dots, d_m)$ -LINEAR QUOTIENTS

**Definition 4.1.** Let  $I$  be a graded ideal,  $\{f_1, \dots, f_m\}$  be a homogeneous system of generators of  $I$  and  $(d_1, \dots, d_m)$  be an  $m$ -tuple of positive integers with  $d_1 = 1$ . We say that  $I$  has  $(d_1, \dots, d_m)$ -linear quotients with respect to the elements  $f_1, \dots, f_m$  if the ideal  $(f_1, \dots, f_{j-1}) : f_j$  has  $d_j$ -linear resolution for all  $j = 2, \dots, m$ . If  $d_2 = \cdots = d_m = d$ , then we simply say that  $I$  has  $d$ -linear quotients with respect to the elements  $f_1, \dots, f_m$ .

Notice that this property depends on the order of the generators. Any order of the generators for which we have  $(d_1, \dots, d_m)$ -linear quotients will be called an admissible order of generators.

An admissible order of generators, say  $f_1, \dots, f_m$ , is called degree increasing if  $\deg(f_1) + d_1 \leq \cdots \leq \deg(f_m) + d_m$ .

In this section, we study the class of ideals with  $(d_1, \dots, d_m)$ -linear quotients and the particular case of ideals with 2-linear quotients. In the following, we assume that  $\{f_1, \dots, f_m\}$  is a homogeneous system of generators for the graded ideal  $I$  and  $I_j = (f_1, \dots, f_j)$  for all  $j = 1, \dots, m$ .

**Theorem 4.2.** *If  $I$  has 2-linear quotients with respect to the elements  $f_1, \dots, f_m$  and  $\deg(f_1) \leq \cdots \leq \deg(f_m)$ , then for each  $i \geq \deg(f_1)$ , we have*

$$\text{reg}(I_{\langle i \rangle}) = \begin{cases} i + 1 & \text{if } i \in \{\deg(f_i); 1 \leq i \leq m\} \text{ and } m > 1; \\ i & \text{otherwise.} \end{cases}$$

*Proof.* We prove the assertion by induction on  $m$ . For  $m = 1$ , it is obvious that the result is true. Assume that the result is true for  $m \geq 1$ ,  $I$  is a graded ideal which has 2-linear quotients with respect

to  $f_1, \dots, f_{m+1}$  and  $\deg(f_1) \leq \dots \leq \deg(f_{m+1})$ . Let  $J = (f_1, \dots, f_m)$  and  $j = \deg(f_{m+1})$ . Then,  $I = J + (f_{m+1})$ . For each  $i < j$ , since  $I_{\langle i \rangle} = J_{\langle i \rangle}$ , by induction hypothesis the result is true.

Note that  $I_{\langle j \rangle} = J_{\langle j \rangle} + (f_{m+1})$ . By hypothesis,  $J : f_{m+1}$  is an ideal with 2-linear resolution. So, it is generated by elements of degree 2. We will show that

$$J_{\langle j \rangle} : f_{m+1} = J : f_{m+1}.$$

To see it, we prove that each homogeneous generator of degree 2 of  $J : f_{m+1}$  belongs to  $J_{\langle j \rangle} : f_{m+1}$ . Let  $g$  be such a generator. So,  $gf_{m+1} \in J_{\langle \ell \rangle}$  where  $\ell = \deg(g) + \deg(f_{m+1}) > j$ . Since  $J$  is generated by elements of degrees at most  $j$ ,  $J_{\langle \ell \rangle} = \mathfrak{m}^{\ell-j} J_{\langle j \rangle}$ . So,  $gf_{m+1} \in J_{\langle j \rangle}$  and the conclusion follows.

Now, consider the following short exact sequence

$$0 \rightarrow R/(J : f_{m+1})(-j) \rightarrow R/J_{\langle j \rangle} \rightarrow R/I_{\langle j \rangle} \rightarrow 0.$$

By hypothesis,  $\text{reg}(R/(J : f_{m+1})(-j)) = j + 1$  and

$$\text{reg}(R/J_{\langle j \rangle}) = \text{reg}(J_{\langle j \rangle}) - 1 = \begin{cases} j, & \deg(f_m) = j \text{ and } m > 1; \\ j - 1, & \text{otherwise.} \end{cases}$$

By applying the *reg formula* (see [9, Corollary 18.7]) to the above short exact sequence, we have

$$\text{reg}(I_{\langle j \rangle}) = \text{reg}(R/I_{\langle j \rangle}) + 1 = j + 1.$$

So, the assertion follows for  $i = j$ .

If  $i = j + 1$ , consider the following short exact sequence

$$0 \rightarrow I_{\langle j+1 \rangle} \rightarrow I_{\langle j \rangle} \rightarrow I_{\langle j \rangle}/I_{\langle j+1 \rangle} \rightarrow 0.$$

Since  $I_{\langle j+1 \rangle} = \mathfrak{m}I_{\langle j \rangle}$ ,

$$I_{\langle j \rangle}/I_{\langle j+1 \rangle} = \bigoplus \mathbf{k}(-j).$$

So,  $\text{reg}(I_{\langle j \rangle}/I_{\langle j+1 \rangle}) = j$ . Again, by applying the *reg formula* we have  $\text{reg}(I_{\langle j+1 \rangle}) = j + 1$ .

Assume that  $i > j + 1$ . Since  $I$  is generated by elements of degrees at most  $j$ ,  $I_{\langle i \rangle} = \mathfrak{m}^{i-j+1} I_{\langle j+1 \rangle}$  and by Lemma 2.2, we have  $\text{reg}(I_{\langle i \rangle}) = i$ .  $\square$

Next, we present some examples of ideals which satisfies Theorem 4.2.

**Example 4.3.** Let

$$I = (x_1^2 x_2, x_2 x_3^2, x_1 x_3 x_4, x_2^2 x_4^2) \subset \mathbf{k}[x_1, x_2, x_3, x_4].$$

Then  $I$  has 2-linear quotients with respect to  $x_1^2 x_2, x_2 x_3^2, x_1 x_3 x_4, x_2^2 x_4^2$  and satisfies Theorem 4.2.

**Example 4.4.** Let

$$I = (x_1x_2x_5, x_2x_3x_6, x_1x_3x_7, x_1x_4x_6, x_2x_4x_7, x_3x_4x_5) \subset \mathbf{k}[x_1, \dots, x_7].$$

Then  $I$  has 2-linear quotients with respect to

$$x_1x_2x_5, x_2x_3x_6, x_1x_3x_7, x_1x_4x_6, x_2x_4x_7, x_3x_4x_5$$

and satisfies Theorem 4.2.

In the next two examples, we have ideals with 2-linear quotients but the given admissible order of the generators is not degree increasing.

**Example 4.5.** Let

$$I = (x_1x_2x_5x_6, x_1x_2x_3, x_3x_4, x_2x_5x_7) \subset \mathbf{k}[x_1, \dots, x_7].$$

Then  $I$  has 2-linear quotients with respect to

$$x_1x_2x_5x_6, x_1x_2x_3, x_3x_4, x_2x_5x_7.$$

But this ordering of generators is not degree increasing. If we reorder the generators as  $x_3x_4, x_1x_2x_3, x_2x_5x_7, x_1x_2x_5x_6$  then we have a degree increasing admissible order for  $(1, 1, 2, 1)$ -linear quotients property.

**Example 4.6.** Let

$$I = (x_1x_2x_3x_7, x_1x_2x_5x_6, x_4x_5x_6) \subset \mathbf{k}[x_1, \dots, x_7].$$

Then  $I$  has 2-linear quotients with respect to

$$x_1x_2x_3x_7, x_1x_2x_5x_6, x_4x_5x_6.$$

This ordering of generators is not degree increasing and there is no degree increasing admissible order of generators for having some  $(1, d_1, d_2)$ -linear quotients property.

The above example shows that if a monomial ideal  $I$  has  $(d_1, \dots, d_m)$ -linear quotients, then in general we can not conclude that  $G(I)$  has a degree increasing admissible order. This is an important difference with the case of monomial ideals with linear quotients.

**Theorem 4.7.** *If  $I$  has  $(d_1, \dots, d_m)$ -linear quotients with respect to  $f_1, \dots, f_m$  and  $\deg(f_1) + d_1 \leq \dots \leq \deg(f_m) + d_m$ , then the minimal free resolution of  $R/I$  is given by the iterated mapping cone.*

Moreover,

- $\forall i \geq 2$  and  $\forall j \notin \{\deg(f_\ell) + d_\ell + i - 2; 1 \leq \ell \leq m\}$ ,  $\beta_{i,j}(R/I) = 0$ .
- $\text{reg}(R/I) = \deg(f_m) + d_m - 2$ .

*Proof.* Let  $t \geq 1$  and assume that the minimal free resolution of  $R/I_t$  is already known by the iterated mapping cone (for the case  $t = 1$  we just consider the obvious minimal free resolution of  $R/I_1$ ). We can easily see that  $I_t$  is minimally generated by  $f_1, \dots, f_t$  and for each  $i \geq 2$  and  $j \notin \{\deg(f_\ell) + d_\ell + i - 2; 1 \leq \ell \leq t\}$ ,  $\beta_{i,j}(R/I_t) = 0$ . Since, by the assumption,  $\deg(f_1) + d_1 \leq \dots \leq \deg(f_m) + d_m$ , for each  $i \geq 1$ ,

$$\max\{j; \beta_{i,j}(R/I_t) \neq 0\} \leq \deg(f_t) + d_t + i - 2.$$

On the other hand, since  $L_{t+1} = (f_1, \dots, f_t) : f_{t+1}$  has  $d_{t+1}$ -linear resolution, for each  $1 \leq i \leq \text{pd}(R/L_{t+1})$ , we have

$$\min\{j; \beta_{i,j}(R/L_{t+1}) \neq 0\} = d_{t+1} + i - 1.$$

It is clear that  $d_{t+1} + i - 1 > \deg(f_t) + d_t + i - 2 - \deg(f_{t+1})$ . So, Equation (3.1) holds and by Theorem 3.1, the mapping cone arising from the short exact sequence

$$0 \rightarrow R/L_{t+1}(-\deg(f_{t+1})) \rightarrow R/I_t \rightarrow R/I_{t+1} \rightarrow 0,$$

is the minimal free resolution of  $R/I_{t+1}$  and the conclusion follows.  $\square$

**Example 4.8.** Let  $I = (x_1x_2, x_2x_3, x_4x_5, x_1x_3x_4) \subset \mathbf{k}[x_1, x_2, x_3, x_4]$ . Then  $I$  has  $(1, 1, 2, 1)$ -linear quotients and  $I$  satisfies in Theorem 4.7.

In the following, we show that if  $I$  has  $(d_1, \dots, d_m)$ -linear quotients with respect to  $f_1, \dots, f_m$  and  $\deg(f_1) + d_1 \leq \dots \leq \deg(f_m) + d_m$ , then  $\text{Syz}_1(I)$  is a componentwise linear module.

**Theorem 4.9.** *If  $I$  has  $(d_1, \dots, d_m)$ -linear quotients with respect to  $f_1, \dots, f_m$  and  $\deg(f_1) + d_1 \leq \dots \leq \deg(f_m) + d_m$ , then  $\text{Syz}_1(I)$  is a componentwise linear module.*

*Proof.* Suppose that  $\text{Syz}_1(I)$  is minimally generated in degrees  $i_1 < \dots < i_\ell$ . For each  $1 \leq t \leq \ell$ , let

$$j_{i_t} = \max\{i; \deg(f_i) + d_i \leq i_t\}, \quad I_{j_{i_t}} = (f_1, \dots, f_{j_{i_t}})$$

and

$$N_{r,I} = (\text{Syz}_1(I))_{\{r\}}, \quad N_{r,I_{j_{i_t}}} = (\text{Syz}_1(I_{j_{i_t}}))_{\{r\}}.$$

By induction on  $r$ , we show that for each  $1 \leq r \leq \ell$  the module  $N_{r,I}$  (resp.  $N_{r,I_{j_{i_t}}}$  for each  $t \geq r$ ) has the following properties:

- (1)  $\beta_{i,j}(N_{r,I}) = 0 \quad \forall j \neq i_r + i, \dots, i_\ell + i$  (resp.  $\beta_{i,j}(N_{r,I_{j_{i_t}}}) = 0 \quad \forall j \neq i_r + i, \dots, i_t + i$ ).
- (2)  $(N_{r,I})_{\langle i_r \rangle}$  has  $i_r$ -linear resolution (resp.  $(N_{r,I_{j_{i_t}}})_{\langle i_r \rangle}$  has  $i_r$ -linear resolution).

If  $r = 1$ , then  $N_{1,I} = \text{Syz}_1(I)$  (resp.,  $N_{1,I_{j_{i_t}}} = \text{Syz}_1(I_{j_{i_t}})$  for each  $t \geq 1$ ). Since by Theorem 4.7, the minimal free resolution of  $R/I$  (resp.,  $R/I_{j_{i_t}}$ ) is given by the iterated mapping cone, it is clear that  $\beta_{i,j}(N_{1,I}) = 0$  for each  $j \neq i_1 + i, \dots, i_\ell + i$  (resp.,  $\beta_{i,j}(N_{1,I_{j_{i_t}}}) = 0$  for each  $j \neq i_1 + i, \dots, i_t + i$ ). So (1) follows for  $r = 1$ .

By Lemma 3.5,  $(N_{1,I})_{\langle i_1 \rangle} \cong (N_{1,I_{j_{i_1}}})_{\langle i_1 \rangle} \cong (N_{1,I_{j_{i_t}}})_{\langle i_1 \rangle}$  for each  $t \geq 1$ . Moreover, the ideal  $I_{j_{i_1}}$  is generated by  $f_1, \dots, f_{j_{i_1}}$ . By Theorem 4.7, the minimal free resolution of  $R/I_{j_{i_1}}$  is computed by the iterated mapping cone and we have  $i_1 = \deg(f_1) + d_1 = \dots = \deg(f_{j_{i_1}}) + d_{j_{i_1}}$ . So, again by Theorem 4.7,  $\text{Syz}_1(I_{j_{i_1}})$  is generated in degree  $i_1$  and has  $i_1$ -linear resolution. So (2) follows for  $r = 1$ .

Now, assume that (1), (2) is true for  $N_{r-1,I}$  (resp.,  $N_{r-1,I_{j_{i_t}}}$  for each  $t \geq r-1$ ) where  $1 \leq r-1 < \ell$ . We prove that  $N_{r,I}$  (resp.,  $N_{r,I_{j_{i_t}}}$  for each  $t \geq r$ ) satisfies (1), (2).

By definition,

$$N_{r,I} = N_{r-1,I}/(N_{r-1,I})_{\langle i_{r-1} \rangle} \quad (\text{resp. } N_{r,I_{j_{i_t}}} = N_{r-1,I_{j_{i_t}}}/(N_{r-1,I_{j_{i_t}}})_{\langle i_{r-1} \rangle}).$$

By the induction hypothesis,  $(N_{r-1,I})_{\langle i_{r-1} \rangle}$  (resp.,  $(N_{r-1,I_{j_{i_t}}})_{\langle i_{r-1} \rangle}$ ) has  $i_{r-1}$ -linear resolution and  $\beta_{i,j}(N_{r-1,I}) = 0 \quad \forall j \neq i_{r-1} + i, \dots, i_\ell + i$  (resp.,  $\beta_{i,j}(N_{r-1,I_{j_{i_t}}}) = 0 \quad \forall j \neq i_{r-1} + i, \dots, i_t + i$ ).

Since  $(N_{r-1,I})_{\langle i_{r-1} \rangle}$  (resp.,  $(N_{r-1,I_{j_{i_t}}})_{\langle i_{r-1} \rangle}$ ) has  $i_{r-1}$ -linear resolution, by Lemma 2.6, it is clear that  $\beta_{i,j}(N_{r,I}) = 0 \quad \forall j \neq i_r + i, \dots, i_\ell + i$  (resp.,  $\beta_{i,j}(N_{r,I_{j_{i_t}}}) = 0 \quad \forall j \neq i_r + i, \dots, i_t + i$ ). So (1) follows.

Now, by Lemma 3.5,

$$\begin{aligned} (N_{r,I})_{\langle i_r \rangle} &\cong (N_{r,I_{j_{i_r}}})_{\langle i_r \rangle} \\ &\cong ((N_{r-1,I_{j_{i_r}}})/(N_{r-1,I_{j_{i_r}}})_{\langle i_{r-1} \rangle})_{\langle i_r \rangle} \\ &\cong (N_{r,I_{j_{i_t}}})_{\langle i_r \rangle}, \end{aligned}$$

where by the induction hypothesis,  $(N_{r-1,I_{j_{i_r}}})_{\langle i_{r-1} \rangle}$  has  $i_{r-1}$ -linear resolution and  $\beta_{i,j}(N_{r-1,I_{j_{i_r}}}) = 0$ , for each  $j \neq i + i_{r-1}, i + i_r$ . So, by Lemma 2.6,  $(N_{r-1,I_{j_{i_r}}})/(N_{r-1,I_{j_{i_r}}})_{\langle i_{r-1} \rangle}$  is generated in degree  $i_r$  and has  $i_r$ -linear resolution. This means that

$$(N_{r,I})_{\langle i_r \rangle} \cong (N_{r,I_{j_{i_t}}})_{\langle i_r \rangle} \cong N_{r-1,I_{j_{i_r}}}/(N_{r-1,I_{j_{i_r}}})_{\langle i_{r-1} \rangle}$$

has  $i_r$ -linear resolution. So (2) follows for  $r$ .

Now, since (2) holds for each  $1 \leq r \leq \ell$ , by Corollary 2.4,  $\text{Syz}_1(I)$  is a componentwise linear module.  $\square$

## ACKNOWLEDGMENTS

This research was in part supported by a grant from IPM (No. 94130058).

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