ON MAXIMAL IDEALS OF $\mathcal{R}_L$

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Abstract. Let $L$ be a completely regular frame and $\mathcal{R}L$ be the ring of real-valued continuous functions on $L$. We consider the set $\mathcal{R}_L = \{ \varphi \in \mathcal{R}L : \uparrow \varphi(\frac{-1}{n}, \frac{1}{n})$ is a compact frame for any $n \in \mathbb{N} \}$. Suppose that $C_\infty(X)$ is the family of all functions $f \in C(X)$ for which the set $\{ x \in X : |f(x)| \geq \frac{1}{n} \}$ is compact, for every $n \in \mathbb{N}$. Kohls has shown that $C_\infty(X)$ is precisely the intersection of all the free maximal ideals of $C^*(X)$. The aim of this paper is to extend this result to the real continuous functions on a frame and hence we show that $\mathcal{R}_L$ is precisely the intersection of all the free maximal ideals of $\mathcal{R}^*L$. This result is used to characterize the maximal ideals in $\mathcal{R}_L$.

1. Introduction

We denote by $C(X)$ ($C^*(X)$) the ring of all (bounded) real-valued continuous functions on a space $X$ which is a nonempty completely regular Hausdorff space. $C_\infty(X)$, the subring of all functions $C(X)$ which vanish at infinity, was introduced by Kohls in [16] (also, see [2, 1, 3, 18, 20] for more details). He shows that:

Proposition 1.1. [16, Lemma 3.2] The ring $C_\infty(X)$ is the intersection of the free maximal ideals of $C^*(X)$.

Azarpanah and Soundararajan in [4], show that $C_\infty(X)$ is an ideal in $C^*(X)$ but not in $C(X)$, see also [16] and 7D in [14]. In fact, $C_\infty(X)$

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is the subring of $C(X)$ and topological spaces $X$ for which $C_\infty(X)$ is the ideal of $C(X)$ are characterized in [4].

$\mathcal{R}_\infty L$, the family of all functions $f \in \mathcal{RL}$ for which $\uparrow \varphi(\frac{-1}{n}, \frac{1}{n})$ is compact for each $n \in \mathbb{N}$, was introduced by Dube in [6].

In this paper, we are trying to show that $\mathcal{R}_\infty L$ is a subring of $\mathcal{RL}$ and an ideal of $\mathcal{R}^* L$ (see Propositions 3.4 and 3.5) and it is not an ideal of $\mathcal{RL}$ (see Example 3.6). Also, we prove that if for every $a \in L$, $\downarrow a$ is a locally compact frame implies $\mathcal{R}^*(\downarrow a) = \mathcal{R}(\downarrow a)$, then $\mathcal{R}_\infty L$ is an ideal of $\mathcal{RL}$ (see Proposition 3.9). In Section 4, we prove that for every completely regular frame $L$, it is a compact frame if and only if $\mathcal{RL} = \mathcal{R}^* L = \mathcal{R}_\infty L$ (see Proposition 4.4). In Section 5, we show that the ring $\mathcal{R}_\infty L$ is the intersection of all the free maximal ideals in $\mathcal{R}^* L$ (see Proposition 5.7). In the last section, we study maximal ideals in the ring $\mathcal{R}_\infty L$ and we show that if $L$ is a completely regular frame, then every maximal ideal of $\mathcal{R}_\infty L$ is strongly fixed ideal (see Proposition 6.6). In fact, $M$ is a maximal ideal of $\mathcal{R}_\infty L$ if and only if there exists $p \in pt(L)$ such that

1. $M = M_p^* \cap \mathcal{R}_\infty L$, and
2. $p \in \uparrow \varphi(\frac{-1}{n}, \frac{1}{n})$, for some $\varphi \in \mathcal{R}_\infty L$ and $n \in \mathbb{N}$.

2. Preliminaries

Regarding the frame of reals $L(\mathbb{R})$ and the f-ring $\mathcal{RL}$ of continuous real-valued functions on frame $L$, we use the notations of [5]. The bounded part, in the f-ring sense, of $\mathcal{RL}$ is denoted by $\mathcal{R}^* L$ and is characterized by:

$$\varphi \in \mathcal{R}^* L \iff \varphi(p, q) = 1 \text{ for some } p, q \in \mathbb{Q}.$$  

An element $a$ of a frame $L$ is said to be rather below an element $b$, written $a \prec b$, provided that $a^* \lor b = \top$. Also, $a$ is completely below $b$, written $a \ll b$, if there are elements $(c_q)$ indexed by the rational numbers $\mathbb{Q} \cap [0, 1]$ such that $c_0 = a$, $c_1 = b$, and $c_p \prec c_q$ for $p < q$. A frame $L$ is said to be regular if $a = \bigvee \{x \in L : x \ll a \}$ for each $a \in L$, and completely regular if $a = \bigvee \{x \in L : x \ll a \}$ for each $a \in L$.

An element $p$ of $L$ is point (or prime) whenever $p < \top$ and $a \land b \leq p$ implies that $a \leq p$ or $b \leq p$. We denote the set of all points of $L$ by $pt(L)$ or $\Sigma L$.

An ideal $J$ of $L$ is completely regular, if for each $x \in J$ there exists $y \in J$ such that $x \ll y$. The Stone-Čech compactification of $L$ is the frame $\beta L$ consisting of completely regular ideals of $L$ together with the dense onto frame homomorphism $j_L : \beta L \to L$ given by join. We denote...
the right adjoint of $j_L$ by $r_L$, and recall that $r_L(a) = \{ x \in L : x \prec \prec a \}$, for all $a \in L$.

Let $L$ be a frame, $a \in L$ and $\alpha \in RL$. The sets $\{ r \in Q : \alpha(-,r) \leq a \}$ and $\{ s \in Q : \alpha(s,-) \leq a \}$, are denoted by $L(a,\alpha)$ and $U(a,\alpha)$ respectively. For $a \neq \top$, it is obvious that $r \leq s$, for each $r \in L(a,\alpha)$ and $s \in U(a,\alpha)$. In fact, we have:

**Proposition 2.1.** [8] Let $L$ be a frame and $p$ be a prime element of $L$. There exists a unique map $\tilde{p} : RL \rightarrow \mathbb{R}$ such that $r \leq \tilde{p}(\alpha) \leq s$, for each $\alpha \in RL$, $r \in L(p,\alpha)$ and $s \in U(p,\alpha)$.

**Proposition 2.2.** [8] If $p$ is a prime element of a frame $L$, then $\tilde{p} : RL \rightarrow \mathbb{R}$ is an onto $f$-ring homomorphism.

Let $\alpha \in RL$. We define $\alpha[p] = \tilde{p}(\alpha)$ for all $p \in \Sigma L$, and define

$$Z(\alpha) = \{ p \in \Sigma L : \alpha[p] = 0 \}.$$ 

This set is said to be a zero-set in $L$ (see [11]). For $A \subseteq RL$, we write $Z[A]$ to designate the family of zero-sets $\{ Z(\alpha) : \alpha \in A \}$. The family $Z[RL]$ of all zero-sets in $L$ will also be denoted, for simplicity, by $Z[L]$ (also, see [10, 12, 15] for more details on the zero-sets and their application in $RL$). For undefined terms and notations, the readers are referred to [9, 17].

### 3. Topics in $\mathcal{R}_\infty L$ is an ideal of $RL$ and an ideal of $\mathcal{R}^*L$

The following lemma is proved in [6]. It will be used in this paper.

**Lemma 3.1.** For every $a,b \in L$, if $\uparrow a$ and $\uparrow b$ are compact, then $\uparrow (a \land b)$ is compact.

**Remark 3.2.** For every $a,b \in L$, if $\uparrow a$ is compact and $a \leq b$, then $\uparrow b$ is compact.

**Remark 3.3.** Consider $\varphi \in \mathcal{R}_\infty L$ and $0 < \varepsilon \in \mathbb{Q}$. Then, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} \leq \varepsilon$. Since $\varphi(-\frac{1}{n}, \frac{1}{n}) \leq \varphi(-\varepsilon, \varepsilon)$, we can conclude from the Remark 3.2 that $\uparrow \varphi(-\varepsilon, \varepsilon)$ is compact. Therefore, for every $\varphi \in RL$, $\varphi \in \mathcal{R}_\infty L$ if and only if for every $0 < \varepsilon \in \mathbb{Q}$, $\uparrow \varphi(-\varepsilon, \varepsilon)$ is compact.

For every $p,q,u,v \in \mathbb{Q}$, we put

$$< p, q > := \{ r \in \mathbb{Q} : p < r < q \}$$

and

$$< p, q > < u, v := \{ rs : p < r < q, u < s < v \}.$$ 

In this paper, a subring of a commutative ring with identity does not imply the identity must belong to the subring.
Proposition 3.4. \( R_\infty L \) is a subring of \( RL \).

Proof. Consider \( \varphi, \psi \in R_\infty L \) and \( n \in \mathbb{N} \). Since \( \uparrow \varphi(\frac{-1}{2n}, \frac{1}{2n}) \) and \( \uparrow \psi(\frac{-1}{2n}, \frac{1}{2n}) \) are compact frames, we can conclude from the Lemma 3.1 that \( \uparrow (\varphi(\frac{-1}{2n}, \frac{1}{2n}) \land \psi(\frac{-1}{2n}, \frac{1}{2n})) \) is a compact frame. The fact that
\[
\varphi(\frac{-1}{2n}, \frac{1}{2n}) \land \psi(\frac{-1}{2n}, \frac{1}{2n}) \leq (\varphi + \psi)(\frac{-1}{n}, \frac{1}{n})
\]
ensures us to conclude at once that \( \uparrow (\varphi + \psi)(\frac{-1}{n}, \frac{1}{n}) \) is a compact frame, by Remark 3.2. Therefore, \( \varphi + \psi \in R_\infty L \).

Let \( m \in \mathbb{N} \) such that \( \frac{1}{m} \leq \frac{1}{\sqrt{n}} \). Since \( \uparrow \varphi(\frac{-1}{m}, \frac{1}{m}) \) and \( \uparrow \psi(\frac{-1}{m}, \frac{1}{m}) \) are compact and
\[
\varphi(\frac{-1}{m}, \frac{1}{m}) \land \psi(\frac{-1}{m}, \frac{1}{m}) \leq (\varphi \psi)(\frac{-1}{n}, \frac{1}{n}),
\]
we can conclude from the Lemma 3.1 and the Remark 3.2 that \( \uparrow (\varphi \psi)(\frac{-1}{n}, \frac{1}{n}) \) is compact. Hence, \( \varphi \psi \in R_\infty L \). \( \square \)

Proposition 3.5. \( R_\infty L \) is an ideal of \( R^* L \).

Proof. Consider \( \varphi \in R_\infty L \) and \( n \in \mathbb{N} \). Since for all \( m \in \mathbb{N} \),
\[
\varphi(-m, m) \in \uparrow \varphi(\frac{-1}{n}, \frac{1}{n})
\]
and
\[
\top = \bigvee_{m \in \mathbb{N}} \varphi(-m, m),
\]
we conclude that there are \( m_1, m_1, \ldots, m_k \in \mathbb{N} \) such that
\[
\top = \bigvee_{1 \leq i \leq k} \varphi(-m_i, m_i).
\]
If \( m = Max\{m_1, m_2, \ldots, m_k\} \) then \( \varphi(-m, m) = \top \), that is \( \varphi \in R^* L \). Therefore, \( R_\infty L \subseteq R^* L \).

Now, suppose that \( \varphi \in R_\infty L \) and \( \psi \in R^* L \). It suffices to show that \( \varphi \psi \in R_\infty L \). There exists \( m \in \mathbb{N} \) such that \( \psi(-m, m) = \top \), by hypothesis. Consider \( n \in \mathbb{N} \). Since
\[
< -\frac{1}{mn}, \frac{1}{mn} > < -m, m > \subseteq < -\frac{1}{n}, \frac{1}{n} >,
\]
we have
\[
\varphi(\frac{-1}{mn}, \frac{1}{mn}) = \varphi(\frac{-1}{mn}, \frac{1}{mn}) \land \psi(-m, m) \leq (\varphi \psi)(\frac{-1}{n}, \frac{1}{n}).
\]
ON MAXIMAL IDEALS OF $R^\infty L$

Since $\uparrow \varphi(-\frac{1}{mn}, \frac{1}{mn})$ is a compact frame, we can conclude from the Remark 3.2 that $\uparrow (\varphi\psi)(-\frac{1}{n}, \frac{1}{n})$ is a compact frame, hence $\varphi\psi \in R^\infty L$.

The following example shows that $R^\infty L$ is not an ideal of $RL$ in general.

**Example 3.6.** We consider the function $\alpha : L\mathbb{R} \to P(\mathbb{N})$ defined by

$$\alpha(p, q) = \{ n \in \mathbb{N} : p < \frac{1}{n} < q \},$$

for every $p, q \in \mathbb{Q}$. We claim that $\alpha$ is a frame map. To prove this, we check the relations (R1)-(R4) to identities in $P(\mathbb{N})$ (see [5]).

(R1). For every $p, q, r, s \in \mathbb{Q}$, we have

$$\alpha(p, q) \land \alpha(r, s) = \{ n \in \mathbb{N} : p < \frac{1}{n} < q \} \cap \{ n \in \mathbb{N} : r < \frac{1}{n} < s \} = \{ n \in \mathbb{N} : p \land r < \frac{1}{n} < q \land s \} = \alpha((p, q) \land (r, s)).$$

(R2). For every $p, q, r, s \in \mathbb{Q}$ with $p \leq r < q \leq s$, we have

$$\alpha(p, q) \lor \alpha(r, s) = \{ n \in \mathbb{N} : p < \frac{1}{n} < q \} \cup \{ n \in \mathbb{N} : r < \frac{1}{n} < s \} = \{ n \in \mathbb{N} : p \lor r < \frac{1}{n} < q \lor s \} = \{ n \in \mathbb{N} : p < \frac{1}{n} < s \} = \alpha(p, s).$$

(R3). For every $p, q \in \mathbb{Q}$, we have

$$\bigvee_{p < r < s < q} \alpha(r, s) = \bigcup_{p < r < s < q} \{ n \in \mathbb{N} : r < \frac{1}{n} < s \} = \{ n \in \mathbb{N} : p < \frac{1}{n} < q \} = \alpha(p, q).$$

(R4). It is clear that

$$\mathbb{N} = \top_{P(\mathbb{N})} = \alpha(0, 2) \leq \bigcup_{p, q \in \mathbb{Q}} \alpha(p, q) \leq \mathbb{N},$$

then $\bigvee_{p, q \in \mathbb{Q}} \alpha(p, q) = \top_{P(\mathbb{N})}$. Therefore, $\alpha \in R(P(\mathbb{N}))$. 
Since, for any \( n \in \mathbb{N} \),
\[
\alpha\left(-\frac{1}{n}, \frac{1}{n}\right) = \{ m \in \mathbb{N} : n < m \} = \{ n+1, n+2, n+3, \ldots \},
\]
we infer that \( \uparrow \alpha\left(-\frac{1}{n}, \frac{1}{n}\right) \) is a finite frame and hence it is a compact frame. Hence, \( \alpha \in \mathcal{R}_\infty(\mathcal{P}(\mathbb{N})) \). Since
\[
\uparrow 1\left(-\frac{1}{n}, \frac{1}{n}\right) = \uparrow \perp = \mathcal{P}(\mathbb{N})
\]
is not a compact frame, we conclude that \( 1 \notin \mathcal{R}_\infty(\mathcal{P}(\mathbb{N})) \). Since
\[
\coz(\alpha) = \alpha(-, 0) \lor \alpha(0, -) = \mathbb{N} = \top_{\mathcal{P}(\mathbb{N})},
\]
we conclude that
(1) \( \alpha \) is unit and \( \alpha \in \mathcal{R}_\infty(\mathcal{P}(\mathbb{N})) \).
(2) \( \mathcal{R}_\infty(\mathcal{P}(\mathbb{N})) \subseteq \mathcal{R}(\mathcal{P}(\mathbb{N})) \).
(3) \( \mathcal{R}_\infty(\mathcal{P}(\mathbb{N})) \) is not an ideal of \( \mathcal{R}(\mathcal{P}(\mathbb{N})) \).

Let \( L \) be a frame. We say that \( a \) is way below \( b \) (or relatively compact with respect to \( b \)) and write \( a \ll b \) if for any \( S \subseteq L \) with \( b \leq \bigvee S \), there exists a finite set \( F \subseteq S \) such that \( a \leq \bigvee F \).

A frame \( L \) is called continuous (or locally compact) whenever, for each \( a \in L \), \( a = \bigvee_{x \ll a} x \).

**Lemma 3.7.** For every completely regular frame \( L \) and \( \varphi \in \mathcal{R}_\infty L \), \( \downarrow \coz(\varphi) \) is a locally compact frame.

**Proof.** Consider \( a \in \downarrow \coz(\varphi) \). Let \( x \prec a \land \varphi((-,-\frac{1}{n}) \lor (\frac{1}{n},-)) \leq \bigvee S \). Then
\[
\varphi\left(-\frac{1}{n}, \frac{1}{n}\right) \leq \left(\varphi((-,-\frac{1}{n}) \lor (\frac{1}{n},-))\right)^* \\
\leq a^* \lor \left(\varphi((-,-\frac{1}{n}) \lor (\frac{1}{n},-))\right)^* \\
= \left( a \land \varphi((-,-\frac{1}{n}) \lor (\frac{1}{n},-))\right)^* \\
\leq x^*.
\]
Using \( \varphi \in \mathcal{R}_\infty L \), we conclude from Remark 3.2 that \( \uparrow x^* \) is a compact frame.

Since
\[
\top = x^* \lor \left( a \land \varphi((-,-\frac{1}{n}) \lor (\frac{1}{n},-))\right) \leq x^* \lor \bigvee S,
\]
we infer that there are \( s_1, \ldots, s_k \in S \) such that \( \top = \bigvee_{i=1}^k (x^* \lor s_i) \), which implies that \( x \leq \bigvee_{i=1}^k s_i \). Hence, if \( x \prec a \land \varphi((-,-\frac{1}{n}) \lor (\frac{1}{n},-)) \),
then } x \ll a \land \varphi \left( \left( -, -\frac{1}{n} \right) \lor \left( \frac{1}{n}, - \right) \right), \text{ for every } x \in L. \text{ Therefore, the complete regularity of } L \text{ insures that }
\begin{align*}
a &= a \land \cos (\varphi) \\
&= \bigvee_{n \in \mathbb{N}} \left( a \land \varphi \left( \left( -, -\frac{1}{n} \right) \lor \left( \frac{1}{n}, - \right) \right) \right) \\
&= \bigvee_{n \in \mathbb{N}} \left\{ x \in L : x \ll a \land \varphi \left( \left( -, -\frac{1}{n} \right) \lor \left( \frac{1}{n}, - \right) \right) \right\} \\
&\leq \bigvee_{n \in \mathbb{N}} \left\{ x \in L : x \ll a \land \varphi \left( \left( -, -\frac{1}{n} \right) \lor \left( \frac{1}{n}, - \right) \right) \right\} \\
&\leq \bigvee_{x \in L, x \ll a} a,
\end{align*}
and this completes the proof.

\textbf{Lemma 3.8.} Let } \alpha \in \mathcal{RL} \text{ and } \rho_3 : \mathcal{L}(\mathbb{R}) \to \mathcal{L}(\mathbb{R}) \text{ by } \rho_3(p, q) = (p^3, q^3). \text{ Then the following statements hold:}
\begin{enumerate}
\item \( \rho_3 \in \mathcal{R}(\mathcal{L}(\mathbb{R})). \)
\item \( \rho_3^3 = id_{\mathcal{L}(\mathbb{R})}. \)
\item \( \alpha \circ \rho_3^3 = \alpha. \)
\item \( \cos(\alpha \circ \rho_3) = \cos(\alpha). \)
\item \( \text{If } \alpha \in \mathcal{R}_\infty L, \text{ then } \alpha \circ \rho_3 \in \mathcal{R}_\infty L. \)
\end{enumerate}

\textit{Proof.} By [13], to complete the proof it suffices to show that if } \alpha \in \mathcal{R}_\infty L, \text{ then } \alpha \circ \rho_3 \in \mathcal{R}_\infty L. \text{ Consider } \alpha \in \mathcal{R}_\infty L. \text{ Since for every } n \in \mathbb{N}, \uparrow \alpha \circ \rho_3(-\frac{1}{n}, \frac{1}{n}) = \uparrow \alpha(-\frac{1}{n}, \frac{1}{n}) \text{ is a compact frame, we conclude that } \alpha \circ \rho_3 \in \mathcal{R}_\infty L. \qed

\textbf{Proposition 3.9.} Let } L \text{ be a completely regular frame and for every } a \in L, \text{ if } \downarrow a \text{ is a locally compact frame, then } \mathcal{R}^*(\downarrow a) = \mathcal{R}(\downarrow a). \text{ Then } \mathcal{R}_\infty L \text{ is an ideal of } \mathcal{RL}.

\textit{Proof.} Consider } \alpha \in \mathcal{RL} \text{ and } \beta \in \mathcal{R}_\infty L. \text{ We put } \beta^{\frac{1}{2}} = \beta \circ \rho_3. \text{ By Lemma 3.8, we have } \alpha \beta^{\frac{1}{2}} \in \mathcal{RL} \text{, which implies that } \overline{\alpha} : \mathcal{L}(\mathbb{R}) \to \downarrow \cos(\beta) \text{ given by } \overline{\alpha}(u) = \alpha \beta^{\frac{1}{2}}(u) \land \cos(\beta) \text{ is an element of } \mathcal{R}(\downarrow \cos(\beta)). \text{ Since, by Lemma 3.7, } \downarrow \cos(\beta) \text{ is a locally compact frame, we conclude that there exists } n \in \mathbb{N} \text{ such that}
\begin{align*}
\alpha \beta^{\frac{1}{2}} \left( (-, -n) \lor (n, -) \right) \land \cos(\beta) &= \overline{\alpha} \left( (-, -n) \lor (n, -) \right) = \sqcap.
\end{align*}
By
\begin{align*}
\alpha \beta^{\frac{1}{2}} \left( (-, -n) \lor (n, -) \right) \leq \cos(\alpha \beta^{\frac{1}{2}}) \leq \cos(\beta),
\end{align*}
we infer that
\begin{align*}
\alpha \beta^{\frac{1}{2}} \left( (-, -n) \lor (n, -) \right) = \sqcap,
\end{align*}
which follows that } \alpha \beta^{\frac{1}{2}} \in \mathcal{R}^* L. \text{ Since, by Lemma 3.8, } \beta^{\frac{1}{2}} \in \mathcal{R}_\infty L, \text{ we conclude from Proposition 3.5 and Lemma 3.8 that } \alpha \beta = \alpha \beta^{\frac{1}{2}} \left( \beta^{\frac{1}{2}} \right)^{2} \in \mathcal{R}_\infty L \text{ and this completes the proof.} \qed
4. When is $R \infty L$ equal to $RL$?

In this section, we characterize frames $L$ for which $R \infty L = RL$. Let $I$ be an ideal in $RL$ or $R^*L$. If $\bigvee \{coz(\phi) : \phi \in I\} < \top$, we call $I$ a fixed ideal; if $\bigvee \{coz(\phi) : \phi \in I\} = \top$, then $I$ is a free ideal.

**Lemma 4.1.** If $I$ is a free ideal in $RL$ and $a \in Coz(L)$ is a compact element of $Coz(L)$, then there exists $\phi \in I$ such that $a = coz(\phi)$.

**Proof.** Evidently

$$a = a \wedge \top = \bigvee \{a \wedge coz(\phi) : \phi \in I\},$$

it follows that there are $\varphi_1, \ldots, \varphi_n \in I$ such that

$$a = a \wedge \bigvee_{i=1}^{n} coz(\varphi_i) = a \wedge coz(\varphi_1^2 + \cdots + \varphi_n^2).$$

Since $Coz(I)$ is an ideal of $Coz(L)$ and

$$a \leq coz(\varphi_1^2 + \cdots + \varphi_n^2) \in Coz(I)$$

we include that there exists $\varphi \in I$ such that $a = coz(\varphi)$. \qed

**Corollary 4.2.** The set

$$\{a \in Coz(L) : a \text{ is a compact element of } Coz(L)\}$$

is a subset of

$$\bigcap \{Coz(I) : I \text{ is a free ideal in } RL \}.$$  

**Proof.** By Lemma 4.1, it is clear. \qed

The following proposition is proved by Dube in [6, Lemma 4.7], but here, in the proof of this proposition, a different approach is used.

**Proposition 4.3.** For every completely regular frame $L$, the following statements are equivalent:

1. $L$ is a compact frame;
2. Every proper ideal $I$ in $RL$ is fixed;
3. Every maximal ideal $I$ in $RL$ is fixed.

**Proof.** (1) $\Rightarrow$ (2). Let $I$ be a proper free ideal in $RL$, then by Lemma 4.1, there exists $\varphi \in I$ such that $\top = coz(\varphi)$. It then follows that $I$ contains a unit element. Hence, $I = RL$ and this is a contradiction.

(2) $\Rightarrow$ (3). It is clear.

(3) $\Rightarrow$ (1). Let $\{a_\lambda\}_{\lambda \in \Lambda} \subseteq L$ such that $\top = \bigvee_{\lambda \in \Lambda} a_\lambda$. It is clear that

$$I = \{\varphi \in RL : \exists \Lambda' \subseteq \Lambda (|\Lambda'| < \infty, coz(\varphi) \leq \bigvee_{\lambda \in \Lambda'} a_\lambda)\}$$
is an ideal of $\mathcal{R} L$. If $I \neq \mathcal{R} L$, then there exists a maximal ideal $M$ such that $I \subseteq M$. Since $L$ is completely regular frame, we infer that

$$
\top = \bigvee_{\lambda \in \Lambda} a_\lambda = \bigvee \text{Coz}(I) \leq \bigvee \text{Coz}(M),
$$

i.e., $\top = \bigvee \text{Coz}(M)$, which is a contradiction. Now, we can assume that $I = \mathcal{R} L$. Then there exists $\Lambda' \subseteq \Lambda$ such that $|\Lambda'| < \infty$ and

$$
\top = \text{Coz}(1) = \bigvee_{\lambda \in \Lambda'} a_\lambda,
$$

this completes the proof of the proposition.

**Proposition 4.4.** For every completely regular frame $L$, then $L$ is a compact frame if and only if $\mathcal{R} L = \mathcal{R}^* L = \mathcal{R}_\infty L$.

**Proof.** Necessity.

Consider $\varphi \in \mathcal{R} L$, $n \in \mathbb{N}$ and $a = \varphi(-\frac{1}{n}, \frac{1}{n})$. Since $L = \uparrow \bot$ is a compact frame and $\bot \leq a$, we can conclude from the Remark 3.2 that $\uparrow a$ is a compact frame, i.e., $\varphi \in \mathcal{R}_\infty L$.

Sufficiency. Since $1 \in \mathcal{R}_\infty L$, we infer that $L = \uparrow \bot = \uparrow 1(-1, 1)$ is a compact frame.

**5. Intersection of free maximal ideals**

In [16, Lemma 3.2], the intersection of the free maximal ideals in $C^*(X)$ was characterized as the set of all functions that vanish at infinity (that is all functions $f \in C(X)$ such that $\{x : |f(x)| \geq \frac{1}{n}\}$ is compact for all $n \in \mathbb{N}$). In this section, we show that this is also true for $\mathcal{R}^*(L)$.

**Proposition 5.1.** If $I$ is a proper free ideal in $\mathcal{R} L$, then

$$
\varphi(-\frac{1}{n}, \frac{1}{n}) \notin \text{Coz}(I),
$$

for every $\varphi \in \mathcal{R}_\infty L$ and $n \in \mathbb{N}$.

**Proof.** Consider $\varphi \in \mathcal{R} L$ and $n \in \mathbb{N}$. Then

$$
\top = \bigvee I = \bigvee \{\text{Coz}(\alpha) \vee \varphi(-\frac{1}{n}, \frac{1}{n}) : \alpha \in I\}
$$

and since $\uparrow \varphi(-\frac{1}{n}, \frac{1}{n})$ is compact, we conclude that there are $\alpha_1, \ldots, \alpha_k \in I$ such that

$$
\top = \left( \bigvee_{i=1}^k \text{Coz}(\alpha_i) \right) \vee \varphi(-\frac{1}{n}, \frac{1}{n}) = \text{Coz} \left( \sum_{i=1}^k \alpha_i^2 \right) \vee \varphi(-\frac{1}{n}, \frac{1}{n})$$
and $\sum_{i=1}^{k} \alpha_i^2 \in I$. If $\varphi(-\frac{1}{n}, \frac{1}{n}) \in Coz(I)$, then $\top \in Coz(I)$, i.e., $I = RL$, which is a contradiction. Hence, $\varphi(-\frac{1}{n}, \frac{1}{n}) \notin Coz(I)$. 

It is well known that $t_L : RL(\beta L) \rightarrow R^*L$ given by $t_L(\alpha) = j_L\alpha$ is the ring isomorphism. Also, we will denote $\varphi^\beta = t_L^{-1}(\varphi)$, for every $\varphi \in R^*L$ (see [7]).

For each $\top \beta L \neq I \in \beta L$, the ideal $M^I$ of $RL$ defined by

$$M^I = \{ \varphi \in RL : r_L(coz(\varphi)) \subseteq I \}$$

and $M^*I = M^I \cap R^*L$. Also,

$$M^*I = \{ \varphi \in R^*L : coz(\varphi^\beta) \subseteq I \}.$$ We need the following propositions which are proved in [7].

**Proposition 5.2.** [7, Proposition 3.8] Maximal ideals of $R^*L$ are precisely the ideals $M^I$, for $I \in pt(\beta L)$. They are distinct for distinct $I$.

**Proposition 5.3.** [7, Proposition 3.9] For every $I \in pt(\beta L)$, $M^I$ is fixed maximal ideal in $R^*L$ if and only if $\forall I < \top$.

The following lemma plays an important role in this note.

**Lemma 5.4.** [10, Lemma 4.2] For every $p \in pt(L)$ and $\varphi \in RL$, $\varphi[p] = 0$ if and only if $coz(\varphi) \leq p$.

**Remark 5.5.** For every frame $L$, we put

$$L^* = \{ I \in pt(\beta L) : \bigvee I = \top \}.$$ Also, for every $A \subseteq pt(L)$ and $\varphi \in RL$, $\varphi[A] = \{ \varphi[p] : p \in A \}$.

**Proposition 5.6.** For every $\varphi \in R^*L$, the following statements are equivalent:

1. $\varphi \in \bigcap_{I \in L^*} M^I$;
2. $\varphi^\beta[L^*] = \{0\}$;
3. For every $0 < \varepsilon \in \mathbb{Q}$ and $I \in L^*$, $|\varphi^\beta[I]| < \varepsilon$;
4. For every $n \in \mathbb{N}$,

$$\{ I \in pt(\beta L) : |\varphi^\beta[I]| \geq \frac{1}{n} \} = \{ I \in pt(\beta L) - L^* : |\varphi^\beta[I]| \geq \frac{1}{n} \}.$$ 

**Proof.** (1) $\iff$ (2). By Lemma 5.4, we have

$$\varphi \in \bigcap_{I \in L^*} M^I \iff \forall I \in L^*(coz(\varphi^\beta) \subseteq I)$$
$$\iff \forall I \in L^*(\varphi^\beta[I] = 0)$$
$$\iff \varphi^\beta[L^*] = \{0\}.$$ The rest is straightforward. 

$\square$
Theorem 5.7. The ring $\mathcal{R}_\infty L$ is the intersection of all the free maximal ideals in $\mathcal{R}^* L$.

Proof. Let $\varphi \in \mathcal{R}_\infty L$ and $I \in L^*$ such that $\varphi \notin M^* I$. Then

$$\bigvee_{n \in \mathbb{N}} \varphi^\beta((-, -\frac{1}{n}) \vee (\frac{1}{n}, -)) = \text{coz}(\varphi^\beta) \subseteq I.$$ 

So, there exists $n_0 \in \mathbb{N}$ such that

$$\varphi^\beta((-, -\frac{1}{n_0}) \vee (\frac{1}{n_0}, -)) \subseteq I,$$

which implies that

$$\varphi^\beta((-, -\frac{1}{n_0}) \vee (\frac{1}{n_0}, -)) \vee I = \top_{\beta L},$$

and there exists $a \in I$ and

$$x \in \varphi^\beta((-, -\frac{1}{n_0}) \vee (\frac{1}{n_0}, -))$$

such that $x \vee a = \top$. Since

$$x \leq \bigvee \varphi^\beta((-, -\frac{1}{n_0}) \vee (\frac{1}{n_0}, -)) = \varphi((-, -\frac{1}{n_0}) \vee (\frac{1}{n_0}, -)),$$

we conclude that

$$\varphi((-, -\frac{1}{n_0}) \vee (\frac{1}{n_0}, -)) \vee a = \top,$$

which implies

$$\varphi(-\frac{1}{n_0}, \frac{1}{n_0}) \leq (\varphi((-, -\frac{1}{n_0}) \vee (\frac{1}{n_0}, -)))^* \leq a.$$

It is clear that

$$A = \{ x \vee a : x \in I \} \subseteq \uparrow \varphi(-\frac{1}{n_0}, \frac{1}{n_0})$$

and $\bigvee A = \top$. Since $\uparrow \varphi(-\frac{1}{n_0}, \frac{1}{n_0})$ is compact frame, we conclude that there exist $x_1, \ldots, x_m \in I$ such that

$$\top = \bigvee_{i=1}^m (x_i \vee a) \in I,$$

which is a contradiction.

Conversely, let $\varphi \in \bigcap_{I \in L^*} M^* I$, $n \in \mathbb{N}$ and

$$\{ a_\lambda \}_{\lambda \in \Lambda} \subseteq \uparrow \varphi(-\frac{1}{n}, \frac{1}{n})$$
such that $\bigvee_{\lambda \in \Lambda} a_\lambda = \top$. Suppose that for every $\Lambda' \subseteq \Lambda$, if $\Lambda'$ is finite set, then $\bigvee_{\lambda \in \Lambda'} a_\lambda \neq \top$. Hence, there exists $I \in L^*$ such that $\{ a_\lambda \}_{\lambda \in \Lambda} \subseteq I$.

By the statement (4) of Proposition 5.6, we have $\varphi^\beta[I] = 0$, so that $coz(\varphi^\beta) \subseteq I$, by Lemma 5.4. Since

$$\varphi(-\frac{1}{n}, \frac{1}{n}) \leq a_\lambda \in I,$$

we conclude that

$$\bigvee \varphi^\beta(-\frac{1}{n}, \frac{1}{n}) = \varphi(-\frac{1}{n}, \frac{1}{n}) \in I,$$

which follows that

$$\varphi^\beta(-\frac{1}{n}, \frac{1}{n}) \subseteq I.$$

Therefore,

$$L = \varphi^\beta(-\frac{1}{n}, \frac{1}{n}) \vee coz(\varphi^\beta) \subseteq I,$$

i.e., $L = I \in L^*$, which is a contradiction.

6. Maximal ideals of $R_{\infty}L$

We turn our attention now to the fixed maximal ideals in the rings $R_{\infty}L$.

Lemma 6.1. Let $\varphi \in RL$, $p \in pt(L)$ and $n \in \mathbb{N}$, then $\varphi(-\frac{1}{n}, \frac{1}{n}) \leq p$ if and only if $|\varphi[p]| \geq \frac{1}{n}$.


Let $\varphi(-\frac{1}{n}, \frac{1}{n}) \leq p$ and $|\varphi[p]| < \frac{1}{n}$. If $t = \varphi[p]$, then, by Proposition 2.1,

$$\bigvee \{ \varphi(-, r) \vee \varphi(s, -) : r, s \in \mathbb{Q}, r < t < s \} \leq p,$$

it follows that

$$\top = \varphi(-\frac{1}{n}, \frac{1}{n}) \vee \bigvee \{ \varphi(-, r) \vee \varphi(s, -) : r, s \in \mathbb{Q}, r < t < s \} \leq p,$$

which is a contradiction.

Sufficiency. Let $|\varphi[p]| \geq \frac{1}{n}$. Then, by Proposition 2.1,

$$\varphi(-\frac{1}{n}, \frac{1}{n}) \leq \bigvee \{ \varphi(-, r) \vee \varphi(s, -) : r, s \in \mathbb{Q}, r < \varphi[p] < s \} \leq p.$$

This completes the proof of the lemma. \qed
**Proposition 6.2.** For every \( A \subseteq pt(L) \), then \( \varphi[A] = 0 \) for every \( \varphi \in R_{\infty}L \), if and only if for every \( \varphi \in R_{\infty}L \) and \( n \in \mathbb{N} \), if \( p \in A \), then \( p \not\in \uparrow \varphi\left(\frac{-1}{n}, \frac{1}{n}\right) \).

*Proof. Necessity.* Let \( \varphi \in R_{\infty}L \), \( p \in A \) and \( n \in \mathbb{N} \). Suppose that \( p \in \uparrow \varphi\left(\frac{-1}{n}, \frac{1}{n}\right) \). Then, by Lemma 6.1, \( |\varphi[p]| \geq \frac{1}{n} \). Hence, \( \varphi[p] \neq 0 \), which is a contradiction.

* Sufficiency. Let \( \varphi \in R_{\infty}L \) and \( p \in A \). By Lemma 6.1, \( |\varphi[p]| < \frac{1}{n} \), for every \( n \in \mathbb{N} \). Hence \( \varphi[p] = 0 \).

For each \( a \in L \) with \( a < \top \), define the subset \( M_a \) of \( R_L \) by

\[
M_a = \{ \varphi \in R_L : \text{coz}(\varphi) \leq a \}
\]

and \( M_a^* = M_a \cap R^*L \). Clearly, \( M_n \) is an ideal, and, in fact, \( M_a = M^{rL(a)} \).

**Corollary 6.3.** If \( p \in pt(L) \) then, \( R_{\infty}L \subseteq M_p^* \) if and only if for every \( \varphi \in R_{\infty}L \) and \( n \in \mathbb{N} \), \( p \not\in \uparrow \varphi\left(\frac{-1}{n}, \frac{1}{n}\right) \).

*Proof. By Proposition 6.2, it is clear.*

For a proof of the following proposition, see [19, Corollary 3.6].

**Proposition 6.4.** Let \( A \) be a commutative algebra over the rational numbers with unity. Let \( I \) be an ideal of \( A \). Then an ideal \( D \) of \( I \) is a maximal ideal of \( I \) if and only if \( D = M \cap I \) for some maximal ideal \( M \) in \( A \), with \( I \not\subseteq M \).

An ideal \( I \) in a subalgebra \( A \) of \( R_L \) is called strongly fixed ideal if \( \bigcap_{\varphi \in I} Z(\varphi) \neq \emptyset \), otherwise, \( I \) is said to be strongly free ideal.

For a proof of the following proposition, see [7, Proposition 3.3] or [10, Proposition 4.8, Corollary 4.9].

**Proposition 6.5.** The fixed maximal ideals of \( R_L (R^*L) \) are precisely the ideals \( M_p \) (\( M_p^* \)) for \( p \in Pt(L) \). They are distinct for distinct points.

**Proposition 6.6.** If \( L \) is a completely regular frame, then every maximal ideal of \( R_{\infty}L \) is strongly fixed ideal. In fact, \( M \) is a maximal ideal of \( R_{\infty}L \) if and only if there exists \( p \in pt(L) \) such that

1. \( M = M_p^* \cap R_{\infty}L \), and
2. \( p \in \uparrow \varphi\left(\frac{-1}{n}, \frac{1}{n}\right) \), for some \( \varphi \in R_{\infty}L \) and \( n \in \mathbb{N} \).

*Proof. Let \( M \) be a maximal ideal of \( R_{\infty}L \), then by Propositions 5.2 and 6.4, there exists \( I \in pt(\beta L) \) such that \( M = M^*I \cap R_{\infty}L \), with \( R_{\infty}L \not\subseteq M^*I \). By Theorem 5.7, \( M^*I \) is a fixed maximal ideal of \( R^*L \).
Then, there exists $p \in pt(L)$ such that $M^*I = M^*_p$, by Proposition 6.5. Therefore, we have

1. $M = M^*_p \cap R_\infty L$, and
2. $p \in \varphi\left(\frac{1}{n}, \frac{1}{n}\right)$, for some $\varphi \in R_\infty L$ and $n \in \mathbb{N}$, by Corollary 6.3.

Conversely, by Corollary 6.3 and Propositions 6.4 and 6.5, it is clear that $M$ is a maximal ideal of $R_\infty L$. □

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On maximal ideals of $R_\infty L$

Ali Akbar Estaji and Ahmad Mahmoudi Darghadam

\[ R_\infty L = \{ \varphi \in RL : \varphi(x) \uparrow \left( \frac{1}{n}, \frac{1}{n} \right), n \in \mathbb{N} \}. \]

$x \in X : |f(x)| \geq \frac{1}{n}$

\[ R_\infty L = \{ f \in C(X) : C_\infty(X) \text{ is} \}

\[ \text{크림일ike:} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크림일ike,} \quad \text{크립