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ON MAXIMAL IDEALS OF $\mathcal{R}_\infty L$

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Abstract. Let L be a completely regular frame and $\mathcal{R}L$ be the ring of real-valued continuous functions on L. We consider the set

$$\mathcal{R}_\infty L = \{\varphi \in \mathcal{R}L : \uparrow \varphi(\frac{-1}{n}, \frac{1}{n}) \text{ is a compact frame for any } n \in \mathbb{N}\}.$$

Suppose that $C_\infty(X)$ is the family of all functions $f \in C(X)$ for which the set $\{x \in X : |f(x)| \geq \frac{1}{n}\}$ is compact, for every $n \in \mathbb{N}$. Kohls has shown that $C_\infty(X)$ is precisely the intersection of all the free maximal ideals of $C^*(X)$. The aim of this paper is to extend this result to the real continuous functions on a frame and hence we show that $\mathcal{R}_\infty L$ is precisely the intersection of all the free maximal ideals of $\mathcal{R}^*L$. This result is used to characterize the maximal ideals in $\mathcal{R}_\infty L$.

1. Introduction

We denote by $C(X)$ ($C^*(X)$) the ring of all (bounded) real-valued continuous functions on a space X which is a nonempty completely regular Hausdorff space. $C_\infty(X)$, the subring of all functions $C(X)$ which vanish at infinity, was introduced by Kohls in [16] (also, see [2, 1, 3, 18, 20] for more details). He shows that:

Proposition 1.1. [16, Lemma 3.2] *The ring $C_\infty(X)$ is the intersection of the free maximal ideals of $C^*(X)$.*

Azarpanah and Soundararajan in [4], show that $C_\infty(X)$ is an ideal in $C^*(X)$ but not in $C(X)$, see also [16] and 7D in [14]. In fact, $C_\infty(X)$

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is the subring of $C(X)$ and topological spaces X for which $C_\infty(X)$ is the ideal of $C(X)$ are characterized in [4].

$\mathcal{R}_\infty L$, the family of all functions $f \in \mathcal{R}L$ for which $\uparrow \varphi(\frac{-1}{n}, \frac{1}{n})$ is compact for each $n \in \mathbb{N}$, was introduced by Dube in [6].

In this paper, we are trying to show that $\mathcal{R}_\infty L$ is a subring of $\mathcal{R}L$ and an ideal of $\mathcal{R}^*L$ (see Propositions 3.4 and 3.5) and it is not an ideal of $\mathcal{R}L$ (see Example 3.6). Also, we prove that if for every $a \in L$, $\downarrow a$ is a locally compact frame implies $\mathcal{R}^*(\downarrow a) = \mathcal{R}(\downarrow a)$, then $\mathcal{R}_\infty L$ is an ideal of $\mathcal{R}L$ (see Proposition 3.9). In Section 4, we prove that for every completely regular frame L, it is a compact frame if and only if $\mathcal{R}L = \mathcal{R}^*L = \mathcal{R}_\infty L$ (see Proposition 4.4). In Section 5, we show that the ring $\mathcal{R}_\infty L$ is the intersection of all the free maximal ideals in $\mathcal{R}^*L$ (see Proposition 5.7). In the last section, we study maximal ideals in the ring $\mathcal{R}_\infty L$ and we show that if L is a completely regular frame, then every maximal ideal of $\mathcal{R}_\infty L$ is strongly fixed ideal (see Proposition 6.6). In fact, M is a maximal ideal of $\mathcal{R}_\infty L$ if and only if there exists $p \in pt(L)$ such that

(1) $M = M_p^* \cap \mathcal{R}_\infty L$, and
(2) $p \in \uparrow \varphi(\frac{-1}{n}, \frac{1}{n})$, for some $\varphi \in \mathcal{R}_\infty L$ and $n \in \mathbb{N}$.

2. Preliminaries

Regarding the frame of reals $\mathcal{L}(\mathbb{R})$ and the f-ring $\mathcal{R}L$ of continuous real-valued functions on frame L, we use the notations of [5]. The bounded part, in the f-ring sense, of $\mathcal{R}L$ is denoted by $\mathcal{R}^*L$ and is characterized by:

$$\varphi \in \mathcal{R}^*L \Leftrightarrow \varphi(p,q) = 1 \text{ for some } p,q \in \mathbb{Q}.$$

An element a of a frame L is said to be *rather below* an element b, written $a \prec b$, provided that $a^* \vee b = \top$. Also, a is *completely below* b, written $a \prec\!\!\prec b$, if there are elements (c_q) indexed by the rational numbers $\mathbb{Q} \cap [0,1]$ such that $c_0 = a$, $c_1 = b$, and $c_p \prec c_q$ for $p < q$. A frame L is said to be *regular* if $a = \bigvee\{x \in L : x \prec a\}$ for each $a \in L$, and *completely regular* if $a = \bigvee\{x \in L : x \prec\!\!\prec a\}$ for each $a \in L$.

An element p of L is *point* (or *prime*) whenever $p < \top$ and $a \wedge b \leq p$ implies that $a \leq p$ or $b \leq p$. We denote the set of all points of L by $pt(L)$ or ΣL.

An ideal J of L is completely regular, if for each $x \in J$ there exists $y \in J$ such that $x \prec\!\!\prec y$. The Stone-Čech compactification of L is the frame βL consisting of completely regular ideals of L together with the dense onto frame homomorphism $j_L : \beta L \to L$ given by join. We denote the right adjoint of j_L by r_L, and recall that $r_L(a) = \{x \in L : x \prec\!\!\prec a\}$, for all $a \in L$.

Let L be a frame, $a \in L$ and $\alpha \in \mathcal{R}L$. The sets $\{r \in \mathbb{Q} : \alpha(-, r) \leq a\}$ and $\{s \in \mathbb{Q} : \alpha(s, -) \leq a\}$, are denoted by $L(a, \alpha)$ and $U(a, \alpha)$ respectively. For $a \neq \top$, it is obvious that $r \leq s$, for each $r \in L(a, \alpha)$ and $s \in U(a, \alpha)$. In fact, we have:

Proposition 2.1. [8] *Let L be a frame and p be a prime element of L. There exists a unique map $\widetilde{p} : \mathcal{R}L \longrightarrow \mathbb{R}$ such that $r \leq \widetilde{p}(\alpha) \leq s$, for each $\alpha \in \mathcal{R}L$, $r \in L(p, \alpha)$ and $s \in U(p, \alpha)$.*

Proposition 2.2. [8] If p is a prime element of a frame L, then $\widetilde{p} : \mathcal{R}L \longrightarrow \mathbb{R}$ is an onto f-ring homomorphism.

Let $\alpha \in \mathcal{R}L$. We define $\alpha[p] = \tilde{p}(\alpha)$ for all $p \in \Sigma L$, and define

$$Z(\alpha) = \{p \in \Sigma L : \alpha[p] = 0\}.$$

This set is said to be a zero-set in L (see [11]). For $A \subseteq \mathcal{R}L$, we write $Z[A]$ to designate the family of zero-sets $\{Z(\alpha) : \alpha \in A\}$. The family $Z[\mathcal{R}L]$ of all zero-sets in L will also be denoted, for simplicity, by $Z[L]$ (also, see [10, 12, 15] for more details on the zero-sets and their application in $\mathcal{R}L$). For undefined terms and notations, the readers are referred to [9, 17].

3. Topics in $\mathcal{R}_\infty L$ is an ideal of $\mathcal{R}L$ and an ideal of $\mathcal{R}^*L$

The following lemma is proved in [6]. It will be used in this paper.

Lemma 3.1. *For every $a, b \in L$, if $\uparrow a$ and $\uparrow b$ are compact, then $\uparrow (a \wedge b)$ is compact.*

Remark 3.2. For every $a, b \in L$, if $\uparrow a$ is compact and $a \leq b$, then $\uparrow b$ is compact.

Remark 3.3. Consider $\varphi \in \mathcal{R}_\infty L$ and $0 < \varepsilon \in \mathbb{Q}$. Then, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} \leq \varepsilon$. Since $\varphi(\frac{-1}{n}, \frac{1}{n}) \leq \varphi(-\varepsilon, \varepsilon)$, we can conclude from the Remark 3.2 that $\uparrow \varphi(-\varepsilon, \varepsilon)$ is compact. Therefore, for every $\varphi \in \mathcal{R}L$, $\varphi \in \mathcal{R}_\infty L$ if and only if for every $0 < \varepsilon \in \mathbb{Q}$, $\uparrow \varphi(-\varepsilon, \varepsilon)$ is compact.

For every $p, q, u, v \in \mathbb{Q}$, we put

$$< p, q > := \{r \in \mathbb{Q} : p < r < q\}$$

and

$$< p, q >< u, v > := \{rs : p < r < q,\ u < s < v\}.$$

In this paper, a subring of a commutative ring with identity does not imply the identity must belong to the subring.

Proposition 3.4. *$\mathcal{R}_\infty L$ is a subring of $\mathcal{R}L$.*

Proof. Consider $\varphi, \psi \in \mathcal{R}_\infty L$ and $n \in \mathbb{N}$. Since $\uparrow \varphi(\frac{-1}{2n}, \frac{1}{2n})$ and $\uparrow \psi(\frac{-1}{2n}, \frac{1}{2n})$ are compact frames, we can conclude from the Lemma 3.1 that $\uparrow (\varphi(\frac{-1}{2n}, \frac{1}{2n}) \wedge \psi(\frac{-1}{2n}, \frac{1}{2n}))$ is a compact frame. The fact that

$$\varphi(\frac{-1}{2n}, \frac{1}{2n}) \wedge \psi(\frac{-1}{2n}, \frac{1}{2n}) \leq (\varphi + \psi)(\frac{-1}{n}, \frac{1}{n})$$

enables us to conclude at once that $\uparrow (\varphi + \psi)(\frac{-1}{n}, \frac{1}{n})$ is a compact frame, by Remark 3.2. Therefore, $\varphi + \psi \in \mathcal{R}_\infty L$.

Let $m \in \mathbb{N}$ such that $\frac{1}{m} \leq \frac{1}{\sqrt{n}}$. Since $\uparrow \varphi(\frac{-1}{m}, \frac{1}{m})$ and $\uparrow \psi(\frac{-1}{m}, \frac{1}{m})$ are compact and

$$\varphi(\frac{-1}{m}, \frac{1}{m}) \wedge \psi(\frac{-1}{m}, \frac{1}{m}) \leq (\varphi\psi)(\frac{-1}{n}, \frac{1}{n}),$$

we can conclude from the Lemma 3.1 and the Remark 3.2 that $\uparrow (\varphi\psi)(\frac{-1}{n}, \frac{1}{n})$ is compact. Hence, $\varphi\psi \in \mathcal{R}_\infty L$. □

Proposition 3.5. *$\mathcal{R}_\infty L$ is an ideal of $\mathcal{R}^* L$.*

Proof. Consider $\varphi \in \mathcal{R}_\infty L$ and $n \in \mathbb{N}$. Since for all $m \in \mathbb{N}$,

$$\varphi\ (-m, m) \in \uparrow\ \varphi(\frac{-1}{n}, \frac{1}{n})$$

and

$$\top = \bigvee_{m \in \mathbb{N}} \varphi(-m, m),$$

we conclude that there are $m_1, m_1, \ldots, m_k \in \mathbb{N}$ such that

$$\top = \bigvee_{1 \leqslant i \leqslant k} \varphi(-m_i, m_i).$$

If $m = Max\{m_1, m_2, \ldots, m_k\}$ then $\varphi(-m, m) = \top$, that is $\varphi \in \mathcal{R}^* L$. Therefore, $\mathcal{R}_\infty L \subseteq \mathcal{R}^* L$.

Now, suppose that $\varphi \in \mathcal{R}_\infty L$ and $\psi \in \mathcal{R}^* L$. It suffices to show that $\varphi\psi \in \mathcal{R}_\infty L$. There exists $m \in \mathbb{N}$ such that $\psi(-m, m) = \top$, by hypothesis. Consider $n \in \mathbb{N}$. Since

$$< -\frac{1}{mn}, \frac{1}{mn} >< -m, m > \subseteq < -\frac{1}{n}, \frac{1}{n} >,$$

we have

$$\varphi(-\frac{1}{mn}, \frac{1}{mn}) = \varphi(-\frac{1}{mn}, \frac{1}{mn}) \wedge \psi(-m, m) \leq (\varphi\psi)(-\frac{1}{n}, \frac{1}{n}).$$

Since $\uparrow \varphi(-\frac{1}{mn}, \frac{1}{mn})$ is a compact frame, we can conclude from the Remark 3.2 that $\uparrow (\varphi\psi)(-\frac{1}{n}, \frac{1}{n})$ is a compact frame, hence $\varphi\psi \in \mathcal{R}_\infty L$. □

The following example shows that $\mathcal{R}_\infty L$ is not an ideal of $\mathcal{R}L$ in general.

Example 3.6. We consider the function $\alpha : \mathcal{L}\mathbb{R} \to \mathcal{P}(\mathbb{N})$ defined by

$$\alpha(p, q) = \{\, n \in \mathbb{N} : p < \frac{1}{n} < q\},$$

for every $p, q \in \mathbb{Q}$. We claim that α is a frame map. To prove this, we check the relations (R1)-(R4) to identities in $\mathcal{P}(\mathbb{N})$ (see [5]).

(R1). For every $p, q, r, s \in \mathbb{Q}$, we have

$$\begin{aligned} \alpha(p,q) \wedge \alpha(r,s) &= \{\, n \in \mathbb{N} : p < \frac{1}{n} < q\} \cap \{\, n \in \mathbb{N} : r < \frac{1}{n} < s\} \\ &= \{\, n \in \mathbb{N} : p \vee r < \frac{1}{n} < q \wedge s\} \\ &= \alpha(p \vee r, q \wedge s) \\ &= \alpha((p,q) \wedge (r,s)). \end{aligned}$$

(R2). For every $p, q, r, s \in \mathbb{Q}$ with $p \le r < q \le s$, we have

$$\begin{aligned} \alpha(p,q) \vee \alpha(r,s) &= \{n \in \mathbb{N} : p < \frac{1}{n} < q\} \cup \{n \in \mathbb{N} : r < \frac{1}{n} < s\} \\ &= \{n \in \mathbb{N} : p \wedge r < \frac{1}{n} < q \vee s\} \\ &= \{n \in \mathbb{N} : p < \frac{1}{n} < s\} \\ &= \alpha(p, s). \end{aligned}$$

(R3). For every $p, q \in \mathbb{Q}$, we have

$$\begin{aligned} \bigvee\nolimits_{p<r<s<q} \alpha(r,s) &= \bigcup\nolimits_{p<r<s<q} \{n \in \mathbb{N} : r < \frac{1}{n} < s\} \\ &= \{\, n \in \mathbb{N} : p < \frac{1}{n} < q\} \\ &= \alpha(p, q). \end{aligned}$$

(R4). It is clear that

$$\mathbb{N} = \top_{\mathcal{P}(\mathbb{N})} = \alpha(0, 2) \le \bigcup_{p,q \in \mathbb{Q}} \alpha(p, q) \le \mathbb{N},$$

then $\bigvee_{p,q\in\mathbb{Q}} \alpha(p,q) = \top_{\mathcal{P}(\mathbb{N})}$. Therefore, $\alpha \in \mathcal{R}(\mathcal{P}(\mathbb{N}))$.

Since, for any $n \in \mathbb{N}$,

$$\alpha(\frac{-1}{n}, \frac{1}{n}) = \{m \in \mathbb{N} : n < m\} = \{n+1, n+2, n+3, ...\},$$

we infer that $\uparrow \alpha(\frac{-1}{n}, \frac{1}{n})$ is a finite frame and hence it is a compact frame. Hence, $\alpha \in \mathcal{R}_\infty(\mathcal{P}(\mathbb{N}))$. Since

$$\uparrow \mathbf{1}(\frac{-1}{n}, \frac{1}{n}) = \uparrow \bot = \mathcal{P}(\mathbb{N})$$

is not a compact frame, we conclude that $\mathbf{1} \notin \mathcal{R}_\infty(\mathcal{P}(\mathbb{N}))$. Since

$$coz(\alpha) = \alpha(-, 0) \vee \alpha(0, -) = \mathbb{N} = \top_{\mathcal{P}(\mathbb{N})},$$

we conclude that

(1) α is unit and $\alpha \in \mathcal{R}_\infty(\mathcal{P}(\mathbb{N}))$.
(2) $\mathcal{R}_\infty(\mathcal{P}(\mathbb{N})) \subsetneqq \mathcal{R}(\mathcal{P}(\mathbb{N}))$.
(3) $\mathcal{R}_\infty(\mathcal{P}(\mathbb{N}))$ is not an ideal of $\mathcal{R}(\mathcal{P}(\mathbb{N}))$.

Let L be a frame. We say that a is *way below* b (or *relatively compact with respect to* b) and write $a \ll b$ if for any $S \subseteq L$ with $b \leq \bigvee S$, there exists a finite set $F \subseteq S$ such that $a \leq \bigvee F$.

A frame L is called *continuous* (or *locally compact*) whenever, for each $a \in L$, $a = \bigvee_{x \ll a} x$.

Lemma 3.7. *For every completely regular frame L and $\varphi \in \mathcal{R}_\infty L$, $\downarrow coz(\varphi)$ is a locally compact frame.*

Proof. Consider $a \in \downarrow coz(\varphi)$. Let $x \prec a \wedge \varphi\big((-, -\frac{1}{n})$
$\vee(\frac{1}{n}, -)\big)$
and $S \subseteq L$ with $a \wedge \varphi\big((-, -\frac{1}{n}) \vee (\frac{1}{n}, -)\big) \leq \bigvee S$. Then

$$\begin{aligned}
\varphi(-\frac{1}{n}, \frac{1}{n}) &\leq \big(\varphi\big((-, -\tfrac{1}{n}) \vee (\tfrac{1}{n}, -)\big)\big)^* \\
&\leq a^* \vee \big(\varphi\big((-, -\tfrac{1}{n}) \vee (\tfrac{1}{n}, -)\big)\big)^* \\
&= \big(a \wedge \varphi\big((-, -\tfrac{1}{n}) \vee (\tfrac{1}{n}, -)\big)\big)^* \\
&\leq x^*.
\end{aligned}$$

Using $\varphi \in \mathcal{R}_\infty L$, we conclude from Remark 3.2 that $\uparrow x^*$ is a compact frame.

Since

$$\top = x^* \vee \big(a \wedge \varphi\big((-, -\frac{1}{n}) \vee (\frac{1}{n}, -)\big)\big) \leq x^* \vee \bigvee S,$$

we infer that there are $s_1, \dots s_k \in S$ such that $\top = \bigvee_{i=1}^k (x^* \vee s_i)$, which implies that $x \leq \bigvee_{i=1}^k s_i$. Hence, if $x \prec a \wedge \varphi\big((-, -\frac{1}{n}) \vee (\frac{1}{n}, -)\big)$, then $x \ll a \wedge \varphi\big((-,-\frac{1}{n}) \vee (\frac{1}{n},-)\big)$, for every $x \in L$. Therefore, the complete regularity of L insures that

$$\begin{aligned} a &= a \wedge coz(\varphi) \\ &= \bigvee_{n\in\mathbb{N}} \big(a \wedge \varphi\big((-,-\tfrac{1}{n}) \vee (\tfrac{1}{n},-)\big)\big) \\ &= \bigvee_{n\in\mathbb{N}} \bigvee\{\, x \in L : x \prec a \wedge \varphi\big((-,-\tfrac{1}{n}) \vee (\tfrac{1}{n},-)\big)\,\} \\ &\leq \bigvee_{n\in\mathbb{N}} \bigvee\{\, x \in L : x \ll a \wedge \varphi\big((-,-\tfrac{1}{n}) \vee (\tfrac{1}{n},-)\big)\,\} \\ &\leq \bigvee_{\substack{x\in L,\\ x\ll a}} x \\ &\leq a, \end{aligned}$$

and this completes the proof. □

Lemma 3.8. *Let $\alpha \in \mathcal{R}L$ and $\rho_3 : \mathcal{L}(\mathbb{R}) \to \mathcal{L}(\mathbb{R})$ by $\rho_3(p,q) = (p^3,q^3)$. Then the following statements hold:*

1. *$\rho_3 \in \mathcal{R}(\mathcal{L}(\mathbb{R}))$.*
2. *$\rho_3^3 = id_{\mathcal{L}(\mathbb{R})}$.*
3. *$(\alpha \circ \rho_3)^3 = \alpha$.*
4. *$coz(\alpha \circ \rho_3) = coz(\alpha)$.*
5. *If $\alpha \in \mathcal{R}_\infty L$, then $\alpha \circ \rho_3 \in \mathcal{R}_\infty L$.*

Proof. By [13], to complete the proof it suffices to show that if $\alpha \in \mathcal{R}_\infty L$, then $\alpha \circ \rho_3 \in \mathcal{R}_\infty L$. Consider $\alpha \in \mathcal{R}_\infty L$. Since for every $n \in \mathbb{N}$, $\uparrow \alpha \circ \rho_3(-\frac{1}{n},\frac{1}{n}) = \uparrow \alpha(-\frac{1}{n^3},\frac{1}{n^3})$ is a compact frame, we conclude that $\alpha \circ \rho_3 \in \mathcal{R}_\infty L$. □

Proposition 3.9. *Let L be a completely regular frame and for every $a \in L$, if $\downarrow a$ is a locally compact frame, then $\mathcal{R}^*(\downarrow a) = \mathcal{R}(\downarrow a)$. Then $\mathcal{R}_\infty L$ is an ideal of $\mathcal{R}L$.*

Proof. Consider $\alpha \in \mathcal{R}L$ and $\beta \in \mathcal{R}_\infty L$. We put $\beta^{\frac{1}{3}} = \beta \circ \rho_3$. By Lemma 3.8, we have $\alpha\beta^{\frac{1}{3}} \in \mathcal{R}L$, which implies that $\overline{\alpha} : \mathcal{L}\mathbb{R} \to \downarrow coz(\beta)$ given by $\overline{\alpha}(u) = \alpha\beta^{\frac{1}{3}}(u) \wedge coz(\beta)$ is an element of $\mathcal{R}(\downarrow coz(\beta))$. Since, by Lemma 3.7, $\downarrow coz(\beta)$ is a locally compact frame, we conclude that there exists $n \in \mathbb{N}$ such that

$$\alpha\beta^{\frac{1}{3}}\big((-,-n) \vee (n,-)\big) \wedge coz(\beta) = \overline{\alpha}\big((-,-n) \vee (n,-)\big) = \bot.$$

By

$$\alpha\beta^{\frac{1}{3}}\big((-,-n) \vee (n,-)\big) \leq coz(\alpha\beta^{\frac{1}{3}}) \leq coz(\beta),$$

we infer that

$$\alpha\beta^{\frac{1}{3}}\big((-,-n) \vee (n,-)\big) = \bot,$$

which follows that $\alpha\beta^{\frac{1}{3}} \in \mathcal{R}^*L$. Since, by Lemma 3.8, $\beta^{\frac{1}{3}} \in \mathcal{R}_\infty L$, we conclude from Proposition 3.5 and Lemma 3.8 that $\alpha\beta = \alpha\beta^{\frac{1}{3}}(\beta^{\frac{1}{3}})^2 \in \mathcal{R}_\infty L$ and this completes the proof. □

4. When is $\mathcal{R}_\infty L$ equal to $\mathcal{R}L$?

In this section, we characterize frames L for which $\mathcal{R}_\infty L = \mathcal{R}L$. Let I be an ideal in $\mathcal{R}L$ or $\mathcal{R}^*L$. If $\bigvee\{coz(\varphi) : \varphi \in I\} < \top$, we call I a fixed ideal; if $\bigvee\{coz(\varphi) : \varphi \in I\} = \top$, then I is a free ideal.

Lemma 4.1. *If I is a free ideal in $\mathcal{R}L$ and $a \in Coz(L)$ is a compact element of $Coz(L)$, then there exists $\varphi \in I$ such that $a = coz(\varphi)$.*

Proof. Evidently

$$a = a \wedge \top = \bigvee\{a \wedge coz(\varphi) : \varphi \in I\},$$

it follows that there are $\varphi_1, \ldots, \varphi_n \in I$ such that

$$a = a \wedge \bigvee_{i=1}^{n} coz(\varphi_i) = a \wedge coz(\varphi_1^2 + \cdots + \varphi_n^2).$$

Since $Coz(I)$ is an ideal of $Coz(L)$ and

$$a \leq coz(\varphi_1^2 + \cdots + \varphi_n^2) \in Coz(I)$$

we include that there exists $\varphi \in I$ such that $a = coz(\varphi)$. □

Corollary 4.2. *The set*

$$\{a \in Coz(L) : a \text{ is a compact element of } Coz(L)\}$$

is a subset of

$$\bigcap\{Coz(I) : I \text{ is a free ideal in } \mathcal{R}L \}.$$

Proof. By Lemma 4.1, it is clear. □

The following proposition is proved by Dube in [6, Lemma 4.7], but here, in the proof of this proposition, a different approach is used.

Proposition 4.3. *For every completely regular frame L, the following statements are equivalent:*

(1) *L is a compact frame;*
(2) *Every proper ideal I in $\mathcal{R}L$ is fixed;*
(3) *Every maximal ideal I in $\mathcal{R}L$ is fixed.*

Proof. (1) $\Rightarrow$ (2). Let I be a proper free ideal in $\mathcal{R}L$, then by Lemma 4.1, there exists $\varphi \in I$ such that $\top = coz(\varphi)$. It then follows that I contents a unit element. Hence, $I = \mathcal{R}L$ and this is a contradiction.

(2) $\Rightarrow$ (3). It is clear.

(3) $\Rightarrow$ (1). Let $\{a_\lambda\}_{\lambda\in\Lambda} \subseteq L$ such that $\top = \bigvee_{\lambda\in\Lambda} a_\lambda$. It is clear that

$$I = \{\varphi \in \mathcal{R}L : \exists \Lambda' \subseteq \Lambda\big(|\Lambda'| < \infty,\, coz(\varphi) \leq \bigvee_{\lambda\in\Lambda'} a_\lambda\big)\}$$

is an ideal of $\mathcal{R}L$. If $I \neq \mathcal{R}L$, then there exists a maximal ideal M such that $I \subseteq M$. Since L is completely regular frame, we infer that

$$\top = \bigvee_{\lambda\in\Lambda} a_\lambda = \bigvee Coz(I) \leq \bigvee Coz(M),$$

i.e., $\top = \bigvee Coz(M)$, which is a contradiction. Now, we can assume that $I = \mathcal{R}L$. Then there exists $\Lambda' \subseteq \Lambda$ such that $|\Lambda'| < \infty$ and

$$\top = coz(\mathbf{1}) = \bigvee_{\lambda\in\Lambda'} a_\lambda,$$

this completes the proof of the proposition. □

Proposition 4.4. *For every completely regular frame L, then L is a compact frame if and only if $\mathcal{R}L = \mathcal{R}^*L = \mathcal{R}_\infty L$.*

Proof. Necessity.

Consider $\varphi \in \mathcal{R}L$, $n \in \mathbb{N}$ and $a = \varphi(-\frac{1}{n}, \frac{1}{n})$. Since $L = \uparrow \bot$ is a compact frame and $\bot \leq a$, we can conclude from the Remark 3.2 that $\uparrow a$ is a compact frame, i.e., $\varphi \in \mathcal{R}_\infty L$.

Sufficiency. Since $\mathbf{1} \in \mathcal{R}_\infty L$, we infer that

$$L = \uparrow \bot = \uparrow \mathbf{1}(-1, 1)$$

is a compact frame. □

5. Intersection of free maximal ideals

In [16, Lemma 3.2], the intersection of the free maximal ideals in $C^*(X)$ was characterized as the set of all functions that vanish at infinity (that is all functions $f \in C(X)$ such that $\{x : |f(x)| \geq \frac{1}{n}\}$ is compact for all $n \in \mathbb{N}$). In this section, we show that this is also true for $\mathcal{R}^*(L)$.

Proposition 5.1. *If I is a proper free ideal in $\mathcal{R}L$, then*

$$\varphi(-\frac{1}{n}, \frac{1}{n}) \notin Coz(I),$$

for every $\varphi \in \mathcal{R}_\infty L$ and $n \in \mathbb{N}$.

Proof. Consider $\varphi \in \mathcal{R}L$ and $n \in \mathbb{N}$. Then

$$\top = \bigvee I = \bigvee\{coz(\alpha) \vee \varphi(-\frac{1}{n}, \frac{1}{n}) : \alpha \in I\}$$

and since $\uparrow \varphi(-\frac{1}{n}, \frac{1}{n})$ is compact, we conclude that there are $\alpha_1, \dots, \alpha_k \in I$ such that

$$\top = \big(\bigvee_{i=1}^{k} coz(\alpha_i)\big) \vee \varphi(-\frac{1}{n}, \frac{1}{n}) = coz\big(\sum_{i=1}^{k} \alpha_i^2\big) \vee \varphi(-\frac{1}{n}, \frac{1}{n})$$

and $\sum_{i=1}^{k}\alpha_i^2 \in I$. If $\varphi(-\frac{1}{n},\frac{1}{n}) \in Coz(I)$, then $\top \in Coz(I)$, i.e., $I = \mathcal{R}L$, which is a contradiction. Hence, $\varphi(-\frac{1}{n},\frac{1}{n}) \notin Coz(I)$. □

It is well known that $\mathfrak{t}_L : \mathcal{R}(\beta L) \to \mathcal{R}^*L$ given by $\mathfrak{t}_L(\alpha) = j_L\alpha$ is the ring isomorphism. Also, we will denote $\varphi^\beta = \mathfrak{t}_L^{-1}(\varphi)$, for every $\varphi \in \mathcal{R}^*L$ (see [7]).

For each $\top_{\beta L} \neq I \in \beta L$, the ideal M^I of $\mathcal{R}L$ defined by

$$M^I = \{\varphi \in \mathcal{R}L : r_L(coz(\varphi)) \subseteq I\}$$

and $M^{*I} = M^I \cap \mathcal{R}^*L$. Also,

$$M^{*I} = \{\varphi \in \mathcal{R}^*L : coz(\varphi^\beta) \subseteq I\}.$$

We need the following propositions which are proved in [7].

Proposition 5.2. [7, Proposition 3.8] *Maximal ideals of $\mathcal{R}^*L$ are precisely the ideals M^{*I}, for $I \in pt(\beta L)$. They are distinct for distinct I.*

Proposition 5.3. [7, Proposition 3.9] *For every $I \in pt(\beta L)$, M^{*I} is fixed maximal ideal in $\mathcal{R}^*L$ if and only if $\bigvee I < \top$.*

The following lemma plays an important role in this note.

Lemma 5.4. [10, Lemma 4.2] *For every $p \in pt(L)$ and $\varphi \in \mathcal{R}L$, $\varphi[p] = 0$ if and only if $coz(\varphi) \leq p$.*

Remark 5.5. For every frame L, we put

$$L^\star = \{I \in pt(\beta L) : \bigvee I = \top\}.$$

Also, for every $A \subseteq pt(L)$ and $\varphi \in \mathcal{R}L$, $\varphi[A] = \{\varphi[p] : p \in A\}$.

Proposition 5.6. *For every $\varphi \in \mathcal{R}^*L$, the following statements are equivalent:*

(1) $\varphi \in \bigcap_{I\in L^\star} M^{*I}$;
(2) $\varphi^\beta[L^\star] = \{0\}$;
(3) *For every* $0 < \varepsilon \in \mathbb{Q}$ *and* $I \in L^\star$, $|\varphi^\beta[I]| < \varepsilon$;
(4) *For every* $n \in \mathbb{N}$,

$$\{I \in pt(\beta L)|\, |\varphi^\beta[I]| \geq \frac{1}{n}\} = \{I \in pt(\beta L) - L^\star|\, |\varphi^\beta[I]| \geq \frac{1}{n}\}.$$

Proof. (1) ⇔ (2). By Lemma 5.4, we have

$$\begin{aligned}\varphi \in \bigcap_{I\in L^\star} M^{*I} &\Leftrightarrow \forall I \in L^\star(coz(\varphi^\beta) \subseteq I)\\ &\Leftrightarrow \forall I \in L^\star(\varphi^\beta[I] = 0)\\ &\Leftrightarrow \varphi^\beta[L^\star] = \{0\}.\end{aligned}$$

The rest is straightforward. □

Theorem 5.7. *The ring $\mathcal{R}_\infty L$ is the intersection of all the free maximal ideals in $\mathcal{R}^*L$.*

Proof. Let $\varphi \in \mathcal{R}_\infty L$ and $I \in L^\star$ such that $\varphi \notin M^{*I}$. Then

$$\bigvee_{n\in\mathbb{N}} \varphi^\beta((-,-\frac{1}{n}) \vee (\frac{1}{n},-)) = coz(\varphi^\beta) \nsubseteq I.$$

So, there exists $n_0 \in \mathbb{N}$ such that

$$\varphi^\beta((-,-\frac{1}{n_0}) \vee (\frac{1}{n_0},-)) \nsubseteq I,$$

which implies that

$$\varphi^\beta((-,-\frac{1}{n_0}) \vee (\frac{1}{n_0},-)) \vee I = \top_{\beta L}$$

and there exists $a \in I$ and

$$x \in \varphi^\beta((-,-\frac{1}{n_0}) \vee (\frac{1}{n_0},-))$$

such that $x \vee a = \top$. Since

$$x \le \bigvee \varphi^\beta((-,-\frac{1}{n_0}) \vee (\frac{1}{n_0},-)) = \varphi((-,-\frac{1}{n_0}) \vee (\frac{1}{n_0},-)),$$

we conclude that

$$\varphi((-,-\frac{1}{n_0}) \vee (\frac{1}{n_0},-)) \vee a = \top,$$

which implies

$$\varphi(-\frac{1}{n_0},\frac{1}{n_0}) \le (\varphi((-,-\frac{1}{n_0}) \vee (\frac{1}{n_0},-)))^* \le a.$$

It is clear that

$$A = \{\, x \vee a : x \in I \,\} \subseteq \uparrow\varphi(-\frac{1}{n_0},\frac{1}{n_0})$$

and $\bigvee A = \top$. Since $\uparrow\varphi(-\frac{1}{n_0},\frac{1}{n_0})$ is compact frame, we conclude that there exist $x_1,\ldots,x_m \in I$ such that

$$\top = \bigvee_{i=1}^{m}(x_i \vee a) \in I,$$

which is a contradiction.

Conversely, let $\varphi \in \bigcap_{I\in L^\star} M^{*I}$, $n \in \mathbb{N}$ and

$$\{\, a_\lambda \,\}_{\lambda\in\Lambda} \subseteq \uparrow\varphi(-\frac{1}{n},\frac{1}{n})$$

such that $\bigvee_{\lambda\in\Lambda} a_\lambda = \top$. Suppose that for every $\Lambda' \subseteq \Lambda$, if Λ' is finite set, then $\bigvee_{\lambda\in\Lambda'} a_\lambda \neq \top$. Hence, there exists $I \in L^\star$ such that $\{ a_\lambda \}_{\lambda\in\Lambda} \subseteq I$. By the statement (4) of Proposition 5.6, we have $\varphi^\beta[I] = 0$, so that $coz(\varphi^\beta) \subseteq I$, by Lemma 5.4. Since

$$\varphi(-\frac{1}{n}, \frac{1}{n}) \leq a_\lambda \in I,$$

we conclude that

$$\bigvee \varphi^\beta(-\frac{1}{n}, \frac{1}{n}) = \varphi(-\frac{1}{n}, \frac{1}{n}) \in I,$$

which follows that

$$\varphi^\beta(-\frac{1}{n}, \frac{1}{n}) \subseteq I.$$

Therefore,

$$L = \varphi^\beta(-\frac{1}{n}, \frac{1}{n}) \vee coz(\varphi^\beta) \subseteq I,$$

i.e., $L = I \in L^\star$, which is a contradiction. □

6. Maximal ideals of $\mathcal{R}_\infty L$

We turn our attention now to the fixed maximal ideals in the rings $\mathcal{R}_\infty L$.

Lemma 6.1. *Let $\varphi \in \mathcal{R}L$, $p \in pt(L)$ and $n \in \mathbb{N}$, then $\varphi(\frac{-1}{n}, \frac{1}{n}) \leq p$ if and only if $|\varphi[p]| \geq \frac{1}{n}$.*

Proof. Necessity.

Let $\varphi(\frac{-1}{n}, \frac{1}{n}) \leq p$ and $|\varphi[p]| < \frac{1}{n}$. If $t = \varphi[p]$, then, by Proposition 2.1,

$$\bigvee\{\varphi(-, r) \vee \varphi(s, -) : r, s \in \mathbb{Q}, r < t < s\} \leq p,$$

it follows that

$$\top = \varphi(\frac{-1}{n}, \frac{1}{n}) \vee \bigvee\{\varphi(-, r)\} \vee \varphi(s, -) : r, s \in \mathbb{Q}, r < t < s\} \leq p,$$

which is a contradiction.

Sufficiency. Let $|\varphi[p]| \geq \frac{1}{n}$. Then, by Proposition 2.1,

$$\varphi(\frac{-1}{n}, \frac{1}{n}) \leq \bigvee\{\varphi(-, r) \vee \varphi(s, -)|r, s \in \mathbb{Q}, r < \varphi[p] < s\} \leq p.$$

This completes the proof of the lemma. □

Proposition 6.2. *For every $A \subseteq pt(L)$, then $\varphi[A] = 0$ for every $\varphi \in \mathcal{R}_\infty L$, if and only if for every $\varphi \in \mathcal{R}_\infty L$ and $n \in \mathbb{N}$, if $p \in A$, then $p \notin \uparrow \varphi(\frac{-1}{n}, \frac{1}{n})$.*

Proof. Necessity. Let $\varphi \in \mathcal{R}_\infty L$, $p \in A$ and $n \in \mathbb{N}$. Suppose that $p \in \uparrow \varphi(\frac{-1}{n}, \frac{1}{n})$. Then, by Lemma 6.1, $|\varphi[p]| \geq \frac{1}{n}$. Hence, $\varphi[p] \neq 0$, which is a contradiction.

Sufficiency. Let $\varphi \in \mathcal{R}_\infty L$ and $p \in A$. By Lemma 6.1, $|\varphi[p]| < \frac{1}{n}$, for every $n \in \mathbb{N}$. Hence $\varphi[p] = 0$. □

For each $a \in L$ with $a < \top$, define the subset M_a of $\mathcal{R}L$ by

$$M_a = \{\varphi \in \mathcal{R}L : coz(\varphi) \leq a\}$$

and $M_a^* = M_a \cap \mathcal{R}^* L$. Clearly, M_a is an ideal, and, in fact, $M_a = M^{r_L(a)}$.

Corollary 6.3. *If $p \in pt(L)$ then, $\mathcal{R}_\infty L \subseteq M_p^*$ if and only if for every $\varphi \in \mathcal{R}_\infty L$ and $n \in \mathbb{N}$, $p \notin \uparrow \varphi(\frac{-1}{n}, \frac{1}{n})$.*

Proof. By Proposition 6.2, it is clear. □

For a proof of the following proposition, see [19, Corollary 3.6].

Proposition 6.4. *Let A be a commutative algebra over the rational numbers with unity. Let I be an ideal of A. Then an ideal D of I is a maximal ideal of I if and only if $D = M \cap I$ for some maximal ideal M in A, with $I \nsubseteq M$.*

An ideal I in a subalgebra A of $\mathcal{R}L$ is called strongly fixed ideal if $\bigcap_{\varphi \in I} Z(\varphi) \neq \emptyset$, otherwise, I is said to be strongly free ideal.

For a proof of the following proposition, see [7, Proposition 3.3] or [10, Proposition 4.8, Corollary 4.9].

Proposition 6.5. *The fixed maximal ideals of $\mathcal{R}L$ ($\mathcal{R}^*L$) are precisely the ideals M_p (M_p^*) for $p \in Pt(L)$. They are distinct for distinct points.*

Proposition 6.6. *If L is a completely regular frame, then every maximal ideal of $\mathcal{R}_\infty L$ is strongly fixed ideal. In fact, M is a maximal ideal of $\mathcal{R}_\infty L$ if and only if there exists $p \in pt(L)$ such that*

(1) *$M = M_p^* \cap \mathcal{R}_\infty L$, and*

(2) *$p \in \uparrow \varphi(\frac{-1}{n}, \frac{1}{n})$, for some $\varphi \in \mathcal{R}_\infty L$ and $n \in \mathbb{N}$.*

Proof. Let M be a maximal ideal of $\mathcal{R}_\infty L$, then by Propositions 5.2 and 6.4, there exists $I \in pt(\beta L)$ such that $M = M^{*I} \cap \mathcal{R}_\infty L$, with $\mathcal{R}_\infty L \nsubseteq M^{*I}$. By Theorem 5.7, M^{*I} is a fixed maximal ideal of $\mathcal{R}^*L$.

Then, there exists $p \in pt(L)$ such that $M^{*I} = M^*_p$, by Proposition 6.5. Therefore, we have

(1) $M = M^*_p \cap \mathcal{R}_\infty L$, and

(2) $p \in \uparrow \varphi(\frac{-1}{n}, \frac{1}{n})$, for some $\varphi \in \mathcal{R}_\infty L$ and $n \in \mathbb{N}$, by Corollary 6.3.

Conversely, by Corollary 6.3 and Propositions 6.4 and 6.5, it is clear that M is a maximal ideal of $\mathcal{R}_\infty L$. □

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