Journal of Algebraic Systems
Vol. 6, No. 1, (2018), pp 71-80

ON THE CHARACTERISTIC DEGREE OF FINITE GROUPS

Z. SEPEHRIZADEH AND M. R. RISMANCHIAN*

Abstract. In this article, we introduce and study the concept of characteristic degree of a subgroup in a finite group. We define the characteristic degree of a subgroup H in a finite group G as the ratio of the number of all pairs $(h, \alpha) \in H \times \mathrm{Aut}(G)$ such that $h^\alpha \in H$, by the order of $H \times \mathrm{Aut}(G)$, where $\mathrm{Aut}(G)$ is the automorphisms group of G. This quantity measures the probability that H can be characteristic in G. We determine the upper and lower bounds for this probability. We also obtain a special lower bound, when H is a cyclic p-subgroup of G.

1. Introduction

In 1968, Erdös and Turán investigated some statistical aspects in group theory. Then, in 1975, Gustafson continued the subject of commutativity degree. The commutativity degree of afinite group G is defined as follows:

$$d(G) = \frac{1}{|G|^2} \left|\{(x, y) \in G \times G \mid xy = yx\}\right|,$$

which is the probability that two randomly chosen elements of G commute, see also [2, 3]. In 1975, Sherman [9] introduced the probability of an automorphism of G fixes an arbitrary element of G, which is a generalization of commutativity degree.

In [5] Moghaddam et al. extended this concept to n-nilpotency degree. Also in [4], they studied the probability of an automorphism of

MSC(2010): 20D15, 20F28, 20P05.
Keywords: Autocommutativity degree, Characteristic degree, p-group.
Received: 23 October 2017, Accepted: 04 March 2018.
*Corresponding author .

G fixing an element of a subgroup H of G. The authors in [8] studied another generalization in this area. Starting with *autocommutativity degree*, the concept of *relative autocommutativity degree* is defined as following:

$$d_{\mathrm{aut}}(H,G) = \frac{|\{(h,\alpha) \in H \times \mathrm{Aut}(G) \mid [h,\alpha] = 1\}|}{|H||\mathrm{Aut}(G)|},$$

where $[h,\alpha] = h^{-1}h^{\alpha}$ is the autocommutator element of G. If one puts $H = G$, then $d_{\mathrm{aut}}(G) = d_{\mathrm{aut}}(G,G)$ is the autocommutivity degree of G, see [6]. We intend to extend these concepts from autocommutativity degree to the characteristic degree.
Now, we introduce *characteristic degree* of a subgroup H in a finite group G, as follows:

$$d_c(H,G) = \frac{|\{(h,\alpha) \in H \times \mathrm{Aut}(G) \mid h^{\alpha} \in H\}|}{|H||\mathrm{Aut}(G)|}.$$

In fact, this formula estimates how much the subgroup H is close of being characteristic in G. If we replace $\mathrm{Aut}(G)$ by G, and α by g, then the formula of the normality degree will be obtained, which was introduced by Saeedi et al. [7].

It is clear that $d_c(H,G) = 1$ if and only if H is a characteristic subgroup of G. The autocommutivity degree $d_{\mathrm{aut}}(G) = d_{\mathrm{aut}}(G,G)$, the relative autocommutivity degree $d_{\mathrm{aut}}(H,G)$ and the characteristic degree may be compared by the following inequalities:

$$d_{\mathrm{aut}}(G) \le d_{\mathrm{aut}}(H,G) \le d_c(H,G).$$

The first inequality can be derived from Theorem 2.3 of [6] and the last inequality is trivial. The rest of article is organized as follows. In Section 2, we will provide some useful results that will be used in the proof of our main theorems. In Section 3, we derive some properties and results involving general lower and upper bounds for the characteristic degrees. We also obtain a lower bound for $d_c(H,G)$, when H is a cyclic p-subgroup of G.

2. Preliminary and technical lemmas

In this section, we introduce some new definitions and provide some technical lemmas which are needed in proving our main theorems.

Definition 2.1. Let H be a subgroup of a finite group G. For a fixed element $h \in H$, we define

$$\mathrm{Chr}_G(h) := \{\alpha \in \mathrm{Aut}(G) \mid h^{\alpha} \in H\ \},$$

also for a fixed element $\alpha \in \mathrm{Aut}(G)$, we define

$$\mathrm{Chr}_H(\alpha) := \{h \in H \mid h^\alpha \in H \}.$$

It is obvious that, $\mathrm{Chr}_H(\alpha)$ is a subgroup of H, for all $\alpha \in \mathrm{Aut}(G)$.

Definition 2.2. Let G be a group and H be a subgroup of G. Then the characterizer of H in G is defined as

$$\mathrm{Chr}_G(H) := \{\alpha \in \mathrm{Aut}(G) \mid H^\alpha \subseteq H \}.$$

It is clear that

$$\mathrm{Chr}_G(H) = \bigcap_{h\in H} \mathrm{Chr}_G(h).$$

The following preliminary lemma is an immediate result.

Lemma 2.3. *Let G be a finite group and H be a subgroup of G. Then $\mathrm{Chr}_G(H)$ is a subgroup of $\mathrm{Aut}(G)$.*

Now, we need the following technical lemma, which is similar to the one in [7, Lemma 2.2].

Lemma 2.4. *Let H be a subgroup of a finite group G. Then*

$$d_c(H,G) = \frac{1}{|H||\mathrm{Aut}(G)|}\sum_{h\in H}|\mathrm{Chr}_G(h)| = \frac{1}{|H||\mathrm{Aut}(G)|}\sum_{\alpha\in \mathrm{Aut}(G)}|Chr_H(\alpha)|.$$

Proof. By definition, it follows that

$$\begin{aligned} d_c(H,G) &= \frac{|\{(h,\alpha)\in H\times \mathrm{Aut}(G) \mid h^\alpha \in H\}|}{|H||\mathrm{Aut}(G)|} \\ &= \frac{\sum_{h\in H}|\{\alpha\in \mathrm{Aut}(G) \mid h^\alpha\in H\}|}{|H||\mathrm{Aut}(G)|} \\ &= \frac{1}{|H||\mathrm{Aut}(G)|}\sum_{h\in H}|\mathrm{Chr}_G(h)|. \end{aligned}$$

The proof of the second equality is similar to the first one. □

Lemma 2.5. (*J. H Christopher and L. R. Darren* [1]) *Let G_1 and G_2 be two groups such that $(|G_1|,|G_2|) = 1$. Then,*

$$\mathrm{Aut}(G_1\times G_2) \simeq \mathrm{Aut}(G_1)\times \mathrm{Aut}(G_2).$$

The following remark is used in the proof of next section, see [4] for more details.

Remark 2.6. Let $G = G_1 \times G_2$, $H = H_1 \times H_2$, and $h_i \in H_i$, $\; i = 1, 2$. One can easily see that any automorphism α of $\mathrm{Aut}(G_i)$, $(i = 1, 2)$ may be extended to an automorphism α^e of $\mathrm{Aut}(G)$, in such a way that for example $(g_1 g_2)^{\alpha^e} = g_1^\alpha g_2$, for all $g_1, g_2 \in G$. We denote all such extended automorphisms in $\mathrm{Aut}(G)$ by $\mathrm{Aut}(G_i^e)$, which are one-to-one corresponding with the ones in $\mathrm{Aut}(G_i)$. So, it becomes clear that $|\mathrm{Aut}(G_i^e)| = |\mathrm{Aut}(G_i)|$, for $i = 1, 2$, $\mathrm{Aut}(G_1^e) \cap \mathrm{Aut}(G_2^e) = \langle \mathrm{id}_G \rangle$ and $\mathrm{Aut}(G_1^e)\mathrm{Aut}(G_2^e)\mathrm{Chr}_G(h_1 h_2) \subseteq \mathrm{Aut}(G)$. Put

$$t = \frac{|\mathrm{Aut}(G_1^e)\mathrm{Aut}(G_2^e)||\mathrm{Chr}_G(h_1 h_2)|}{|\mathrm{Aut}(G_1^e)\mathrm{Aut}(G_2^e) \cap \mathrm{Chr}_G(h_1 h_2)|},$$

hence

$$\begin{aligned} t &= \frac{|\mathrm{Aut}(G_1^e)||\mathrm{Aut}(G_2^e)||\mathrm{Chr}_G(h_1 h_2)|}{|\mathrm{Aut}(G_1^e) \cap \mathrm{Chr}_G(h_1 h_2)||\mathrm{Aut}(G_2^e) \cap \mathrm{Chr}_G(h_1 h_2)|} \\ &= \frac{|\mathrm{Aut}(G_1^e)||\mathrm{Aut}(G_2^e)||\mathrm{Chr}_G(h_1 h_2)|}{|\mathrm{Chr}_{G_1}(h_1)||\mathrm{Chr}_{G_2}(h_2)|} \\ &\leq |\mathrm{Aut}(G)|, \end{aligned}$$

which implies that $\dfrac{|\mathrm{Chr}_G(h_1 h_2)|}{|\mathrm{Aut}(G)|} \leq \dfrac{|\mathrm{Chr}_{G_1}(h_1)||\mathrm{Chr}_{G_2}(h_2)|}{|\mathrm{Aut}(G_1^e)||\mathrm{Aut}(G_2^e)|}$.

3. Main results

Assume H is a subgroup of a finite group G. Computing $d_c(H, G)$, we must concentrate properties of H or G, as the characteristic subgroup of H in G depends on both structures of H and G. Note that, in most of probabilistic articles a similar version of our following result have been proved, see for instance [4, 7].

Theorem 3.1. *Let H_1 and H_2 be subgroups of groups G_1 and G_2, respectively. Then*

$$d_c(H_1 \times H_2, G_1 \times G_2) \leq d_c(H_1, G_1) \times d_c(H_2, G_2).$$

In particular, the equality holds if $(|G_1|, |G_2|) = 1$.

Proof. Put $G = G_1 \times G_2$, $H = H_1 \times H_2$ and $h_i \in H_i$, $i = 1, 2$. Then, from Lemma 2.4 and the Remark 2.6, it follows that

$$\begin{aligned}
d_c(H,G) &= \frac{1}{|H|}\sum_{h\in H}\frac{|\mathrm{Chr}_G(h)|}{|\mathrm{Aut}(G)|}\\
&= \frac{1}{|H_1||H_2|}\sum_{(h_1,h_2)\in H}\frac{|\mathrm{Chr}_G(h_1,h_2)|}{|\mathrm{Aut}(G)|}\\
&\leq \frac{1}{|H_1|}\frac{1}{|H_2|}\sum_{h_1\in H_1}\sum_{h_2\in H_2}\frac{|\mathrm{Chr}_{G_1}(h_1)|}{|\mathrm{Aut}(G_1)|}\frac{|\mathrm{Chr}_{G_2}(h_2)|}{|\mathrm{Aut}(G_2)|}\\
&= \frac{1}{|H_1|}\sum_{h_1\in H_1}\frac{|\mathrm{Chr}_{G_1}(h_1)|}{|\mathrm{Aut}(G_1)|}\frac{1}{|H_2|}\sum_{h_2\in H_2}\frac{|\mathrm{Chr}_{G_2}(h_2)|}{|\mathrm{Aut}(G_2)|}\\
&= d_c(H_1,G_1)d_c(H_2,G_2).
\end{aligned}$$

Now, using the assumption $(|G_1|, |G_2|) = 1$ and Lemma 2.5, the equality is obtained. □

The following theorems provide the general upper and lower bounds for the characteristic degree of H in G, see also [7, Theorem 2.5].

Theorem 3.2. *Let H be subgroup of a finite group G and p be the smallest prime dividing $|H|$. Then*

$$d_c(H,G) \leq \frac{1}{p} + \left(1 - \frac{1}{p}\right)\frac{1}{[\mathrm{Aut}(G) : \mathrm{Chr}_G(H)]}.$$

Proof. By Lemma 2.4, we have

$$\begin{aligned}
d_c(H,G) &= \frac{1}{|H||\mathrm{Aut}(G)|}\sum_{\alpha\in \mathrm{Aut}(G)} |\mathrm{Chr}_H(\alpha)|\\
&= \frac{1}{|H||\mathrm{Aut}(G)|}\left(\sum_{\alpha\in \mathrm{Chr}_G(H)} |\mathrm{Chr}_H(\alpha)| + \sum_{\alpha\notin \mathrm{Chr}_G(H)} |\mathrm{Chr}_H(\alpha)|\right)\\
&\leq \frac{1}{|H||\mathrm{Aut}(G)|}\Big(|\mathrm{Chr}_G(H)||H| + \sum_{\alpha\notin \mathrm{Chr}_G(H)} \frac{|H|}{p}\Big)\\
&= \frac{1}{|\mathrm{Aut}(G)|}\left(|\mathrm{Chr}_G(H)| + \frac{1}{p}(|\mathrm{Aut}(G)| - |\mathrm{Chr}_G(H)|)\right)\\
&= \frac{|\mathrm{Chr}_G(H)|}{|\mathrm{Aut}(G)|}\left(1+\frac{1}{p}\left(\frac{|\mathrm{Aut}(G)|}{|\mathrm{Chr}_G(H)|}-1\right)\right)\\
&= \frac{|\mathrm{Chr}_G(H)|}{|\mathrm{Aut}(G)|}\left(\left(1-\frac{1}{p}\right)+\frac{1}{p}\frac{|\mathrm{Aut}(G)|}{|\mathrm{Chr}_G(H)|}\right)\\
&= \frac{1}{p}+\left(1-\frac{1}{p}\right)\frac{1}{[\mathrm{Aut}(G):\mathrm{Chr}_G(H)]}.
\end{aligned}$$

□

Theorem 3.3. *Let H be subgroup of a finite group G. Then*

$$d_c(H,G) \geq \frac{1}{|H|} + \frac{1}{[\mathrm{Aut}(G):\mathrm{Chr}_G(H)]}\left(1-\frac{1}{|H|}\right).$$

Proof. Suppose that $A = \{(1,\alpha) \mid \alpha \in \mathrm{Aut}(G)\}$ and

$$B = \{(h,\alpha) \mid h \in H, \alpha \in \mathrm{Chr}_G(H), h \neq 1\}.$$

Then, it is clear that

$$\begin{aligned}
d_c(H,G) &= \frac{|\{(h,\alpha)\in H\times \mathrm{Aut}(G) \mid h^\alpha \in H\}|}{|H||\mathrm{Aut}(G)|}\\
&\geq \frac{|A|+|B|}{|H||\mathrm{Aut}(G)|}\\
&= \frac{|\mathrm{Aut}(G)| + |\mathrm{Chr}_G(H)|(|H|-1)}{|H||\mathrm{Aut}(G)|}\\
&= \frac{1}{|H|} + \frac{1}{[\mathrm{Aut}(G):\mathrm{Chr}_G(H)]} - \frac{|\mathrm{Chr}_G(H)|}{|H||\mathrm{Aut}(G)|}\\
&= \frac{1}{|H|} + \frac{1}{[\mathrm{Aut}(G):\mathrm{Chr}_G(H)]}\left(1-\frac{1}{|H|}\right),
\end{aligned}$$

and the proof is complete. □

The following corollary classifies the characteristic degree in two cases.

Corollary 3.4. *Let H be a non-characteristic subgroup of a finite group G. Then the following statements hold:*

(i) *If $\mathrm{Chr}_G(H)$ is a normal subgroup of $\mathrm{Aut}(G)$, then $d_c(H,G) \leq \frac{2+p}{2p}$.*

(ii) *If $\mathrm{Chr}_G(H)$ is a non-normal subgroup of $\mathrm{Aut}(G)$, then $d_c(H,G) \leq \frac{2+p}{3p} \leq \frac{2}{3}$, where p is the smallest prime divisor of $|H|$.*

Proof. (i) Since H is a non-characteristic subgroup of G, then we get that $\mathrm{Chr}_G(H)$ is purely contained in $\mathrm{Aut}(G)$, and hence $[\mathrm{Aut}(G) : \mathrm{Chr}_G(H)] \geq 2$. By Theorem 3.2, we get

$$\begin{aligned} d_c(H,G) &\leq \frac{1}{p} + \left(1 - \frac{1}{p}\right) \frac{1}{[\mathrm{Aut}(G) : \mathrm{Chr}_G(H)]} \\ &\leq \frac{1}{p} + \left(1 - \frac{1}{p}\right) \frac{1}{2} \\ &= \frac{2+p}{2p}. \end{aligned}$$

(ii) Since $\mathrm{Chr}_G(H)$ is a non-normal subgroup of $\mathrm{Aut}(G)$, then we have $[\mathrm{Aut}(G) : \mathrm{Chr}_G(H)] \neq 2$, which implies that $[\mathrm{Aut}(G) : \mathrm{Chr}_G(H)] \geq 3$. Therefore, by Theorem 3.2, we have

$$d_c(H,G) \leq \frac{1}{p} + \left(1 - \frac{1}{p}\right) \frac{1}{3} = \frac{2+p}{3p}.$$

□

Example 3.5. Consider any non-characteristic subgroup H of S_3,

$$H \simeq \langle (1\ 2) \rangle \simeq \langle (1\ 3) \rangle \simeq \langle (2\ 3) \rangle .$$

It is easy to see that $|\mathrm{Aut}(S_3)| = 6$. If we put

$$A = \{(1, \alpha) \in H \times \mathrm{Aut}(S_3) \mid \alpha(1) = 1\}$$

and

$$B = \{(h, \alpha) \in H \times \mathrm{Aut}(S_3) \mid \alpha(h) \in H\},$$

then we have

$$\begin{aligned} d_c(H, S_3) &= \frac{|A| + |B|}{|H||\mathrm{Aut}(S_3)|} \\ &= \frac{6+2}{12} = \frac{2}{3}. \end{aligned}$$

One can see that the above example attained the bound in Corollary 3.4 (ii), and so it is the best possible upper bound.

From now on, we assume that H is cyclic. A simple verification shows that among all cyclic groups, cyclic p-groups have the following property. In fact, if we take the additional property that H is a p-group of order p^m and let $H_i = H^{p^i}$, for $i = 0, \ldots, m$, then

$$H = H_0 \supset H_1 \supset \cdots \supset H_m = H^{p^m} = 1$$

is the chain of all subgroups of H. Now, if $C_i = \mathrm{Chr}_G(H_i)$, then we can see that

$$\mathrm{Chr}_G(H) = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_m = \mathrm{Aut}(G),$$

and consequently $d_c(H, G)$ can be computed if the order of C_i is given. Suppose that H is a non-characteristic subgroup of G and $0 = i_0 < i_1 < \cdots < i_k \leq m$, such that

$$C_{i_0} = \cdots = C_{i_1 - 1} \subset C_{i_1} = \cdots = C_{i_2 - 1} \subset \cdots \subset C_{i_k} = \cdots = C_m,$$

where $C = C_{i_0}$ and $C_m = \mathrm{Aut}(G)$. Utilizing the above notations, we have the following result which gives a lower bound and generalizes [7, Theorem 4.2].

Theorem 3.6. *Let H be a cyclic p-subgroup of order p^m of a finite group G. Then*

$$d_c(H, G) \geq \frac{1}{[\mathrm{Aut}(G) : \mathrm{Chr}_G(H)]} \sum_{j=0}^{k-1} \left(\frac{1}{p^{i_j}} - \frac{1}{p^{i_{j+1}}} \right) q^j + \frac{1}{p^{i_k}}, \tag{3.1}$$

where q is the smallest prime divisor of $|\mathrm{Aut}(G)|$.

Proof. By definition, we have

$$\begin{aligned} d_c(H,G) &= \frac{|\{(h,\alpha) \in H \times \mathrm{Aut}(G) \mid h^\alpha \in H\}|}{|H||\mathrm{Aut}(G)|} \\ &= \frac{1}{|H||\mathrm{Aut}(G)|}\left(|H_{i_0}||C_{i_0}| + \sum_{j=1}^{k} |H_{i_j}||C_{i_j} \setminus C_{i_{j-1}}|\right) \\ &= \frac{1}{|H||\mathrm{Aut}(G)|}\left(p^m|C_{i_0}| + \sum_{j=1}^{k} p^{m-i_j}(|C_{i_j}| - |C_{i_{j-1}}|)\right) \\ &= \frac{1}{|H||\mathrm{Aut}(G)|}\left(\sum_{j=0}^{k-1}(p^{m-i_j} - p^{m-i_{j+1}})|C_{i_j}| + p^{m-i_k}|C_{i_k}|)\right) \\ &= \frac{1}{[\mathrm{Aut}(G) : C]}\left(\sum_{j=0}^{k-1}\left(\frac{1}{p^{i_j}} - \frac{1}{p^{i_{j+1}}}\right)[C_{i_j} : C]\right) + \frac{1}{p^{i_K}}. \end{aligned}$$

Let q be the smallest prime divisor of $|G|$. Then ,$[C_{i_j} : C_{i_{j-1}}] \geq q$, which implies that $[C_{i_j} : C] \geq q^j$, for $j = 1, \ldots, k$. So, we have

$$d_c(H,G) \geq \frac{1}{[\mathrm{Aut}(G) : C]}\left(\sum_{j=0}^{k-1}\left(\frac{1}{p^{i_j}} - \frac{1}{p^{i_{j+1}}}\right)\right)q^j + \frac{1}{p^{i_K}}.$$

□

Clearly, the equality in relation (3.1) holds if $k = 1$, then $[C_{i_0} : C] = 1$. In this case, we have

$$d_c(H,G) = \frac{1}{[\mathrm{Aut}(G) : \mathrm{Chr}_G(H)]}\left(1 - \frac{1}{p^{i_1}}\right) + \frac{1}{p^{i_1}},$$

where $C = C_0 = \cdots = C_{i_1-1} \subset C_{i_1} = \cdots = C_m = \mathrm{Aut}(G)$. Also, the lower bound in (3.1) takes the minimum and maximum values if $k = 1$ and $i_1 = m, 1$, respectively, which satisfies Theorems 3.2 and 3.3.

Acknowledgments

The authors would like to thank the referee for his/her useful comments that helped to improve the presentation of the article.

References

1. J. H. Christopher and L. R. Darren, Automorphism of finite abelian groups, *Amer. Math. Monthly* **114** (2007), 917-923.

2. P. Erdös and P. Turán, On some problems of statistical group theory, *Acta. Math. Acad. Sci. Hung.* **19** (1968), 413-435.
3. W. H. Gustafson, What is the probability that two group elements commute?, *Amer. Math. Monthly* **80** (1973), 1031-1034.
4. M. R. R. Moghaddam, F. Saeedi and E. Khamseh, The probability of an automorphism fixing a subgroup element of a finite group, *Asian-Eur. J. Math.* **4**(2) (2011), 301-308.
5. M. R. R. Moghaddam, A. R. Salemkar and K. Chitti, n-Isoclinism classes and n-nilpotency degree of finite groups, *Algebra Colloq.* **12** (2) (2005), 255-261.
6. M. R. Rismanchian and Z. Sepehrizadeh, Autoisoclinism classes and autocommutativity degrees of finite groups, *Hacet. J. Math. Stat.* **44** (4) (2015), 893-899.
7. F. Saeedi, M. Farrokhi D.G and S. H. Jafari, Subgroup normality degrees of finite groups I, *Arch. Math.* **96** (2011), 215-224.
8. Z. Sepehrizadeh and M. R. Rismanchian, Probability that an autocommutator element of a finite group equals to a fixed element, *Filomat* **31** (20) (2017), 6241-6246.
9. G. J. Sherman, What is the probability an automorphism fixes a group element?, *Amer. Math. Monthly* **82** (1975), 261-264.

Zohreh Sepehrizadeh
Department of Pure Mathematics, Shahrekord University , P.O.Box 115, Shahrekord, Iran.
Email: zohreh.sepehri@gmail.com

Mohammad Reza Rismanchian
Department of Pure Mathematics, Shahrekord University , P.O.Box 115, Shahrekord, Iran.
Email: rismanchian133@gmail.com, rismanchian@sku.ac.ir