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ON THE CAPACITY OF EILENBERG-MACLANE AND MOORE SPACES

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Abstract. K. Borsuk in 1979, at the Topological Conference in Moscow, introduced concept of the capacity of a compactum and asked some questions concerning properties of the capacity of compacta. In this paper, we give partial positive answers to three of these questions in some cases. In fact, by describing spaces homotopy dominated by Moore and Eilenberg-MacLane spaces, the capacities of a Moore space $M(A,n)$ and an Eilenberg-MacLane space $K(G,n)$ could be obtained. Also, we compute the capacity of wedge sum of finitely many Moore spaces of different degrees and the capacity of product of finitely many Eilenberg-MacLane spaces of different homotopy types. In particular, we compute the capacity of wedge sum of finitely many spheres of the same or different dimensions.

1. Introduction and Motivation

K. Borsuk in [3] introduced concept of the capacity of a compactum (compact metric space) as follows: the capacity $C(A)$ of a compactum A is the cardinality of the set of all shapes of compacta X for which $\mathcal{S}h(X) \leqslant \mathcal{S}h(A)$. Similarly, we can define the capacity of a topological space A as the cardinality of the set of all shapes of spaces X for which $\mathcal{S}h(X) \leqslant \mathcal{S}h(A)$.

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In the case of polyhedra, the notions shape and shape domination in the above definition could be replaced by the notions homotopy type and homotopy domination, respectively. Indeed, by some known results in shape theory we conclude that for any polyhedron P, there is a 1-1 functorial correspondence between the shapes of compacta shape dominated by P and the homotopy types of CW-complexes (not necessarily finite) homotopy dominated by P (in both pointed and unpointed cases) [13].

It is obvious that the capacity of a topological space is a homotopy invariant, i.e., if topological spaces X and Y have the same homotopy type, then $C(X) = C(Y)$. Accordingly, it will be interesting to know about topological spaces which have finite capacities. Of course, S. Mather in [18] proved that every polyhedron dominates only a countable number of different homotopy types (hence shapes).

In addition, Borsuk in [3] asked: "Is it true that the capacity of every finite polyhedron is finite?". D. Kolodziejczyk in [16] gave a negative answer to this question. However, Kolodziejczyk in [12, 13, 14, 15] investigated some conditions under which a polyhedron has finite capacity. For instance, polyhedra with finite fundamental groups and polyhedra P with abelian fundamental groups $\pi_1(P)$ and finitely generated homology groups $H_i(\tilde{P})$, for $i \geq 2$, have finite capacities.

In this paper, we concentrate on some questions concerning properties of the capacity of compacta which have been stated in [3] and we give partial positive answers to three of these questions in some cases. The first question is:

1. Is $C(X \times Y)$ determined by $C(X)$ and $C(Y)$?

In Section 4, we give a partial positive answer to this question as follows: If X and Y are Eilenberg-MacLane CW-complexes $K(G, n)$ and $K(H, m)$, respectively, such that $n \neq m$ and G and H are Hopfian groups, then $C(X \times Y) = C(X) \times C(Y)$ (see Corollary 4.11).

The second question is:

2. Is $C(X \cup Y)$ determined by $C(X)$, $C(Y)$ and $C(X \cap Y)$?

In [16], by presenting two finite CW-complexes X and Y of dimensions 2 with $C(X), C(Y), C(X \cap Y) < \infty$ but $C(X \cup Y) = \infty$, Kolodziejczyk gave a negative answer to the above question. In Section 3, we show that if $X = M(A, n)$ and $Y = M(B, m)$ are two Moore spaces with abelian Hopfian groups A, B and $n \neq m$, $n, m \geq 2$, then $C(X \vee Y) = C(X) \times C(Y)$ (see Corollary 3.10). Recall that a Moore space is a simply connected CW-complex X with a single non-vanishing homology group for some $n \geq 2$, that is $\tilde{H}_i(X, \mathbb{Z}) = 0$ for $i \neq n$.

The next question is as follows:

3. Is the capacity $C(A)$ determined by the homology properties of A?

In Section 3, we show that the answer to the above question is positive for Moore spaces. In fact, we prove that there is a one-to-one correspondence between the set of all homotopy types of spaces homotopy dominated by $M(A, n)$ and the set of all isomorphism classes of direct summands of A, for $n \geq 2$ (see Proposition 3.5).

Borsuk in [3] asked another question concerning the capacity of finite polyhdera as follows:

Is it true that the capacity of every finite polyhedron is finite?

Kolodziejczyk in [16] presented an example of a finite polyhedron of dimension 2 homotopy dominates infinitely many polyhedra of different homotopy types which is a negative answer to the above question. Moreover, it has been proved that for every non-abelian poly-$\mathbb{Z}$-group G and an integer $n \geq 3$, there exists a polyhedron P with $\pi_1(P) \cong G$ and $\dim P = n$ dominating infinitely many polyhedra of different homotopy types which shows that such examples are not rare (see [12]). In particular, there exist polyhedra with nilpotent fundamental groups and infinite capacities ([12]). However, Kolodziejczyk has given positive answer to the above question under some conditions: in [15] the author proved (using the results of localization theory in the homotopy category of CW-complexes) that every simply connected polyhedron dominates only finitely many different homotopy types. Also, in [14] the author has proved that polyhedra with finite fundamental groups dominate only finitely many different homotopy types. In [13], by extending the methods of [14], it has been proved that for some classes of polyhedra with abelian fundamental groups, the answer to the above question is positive. In addition, Kolodziejczyk proved that every nilpotent polyhedron dominates only finitely many different homotopy types ([14]).

In this paper, we compute the capacities of Moore spaces $M(A, n)$ and Eilenberg-MacLane spaces $K(G, n)$. In fact, we show that the capacities of a Moore space $M(A, n)$ and an Eilenberg-MacLane space $K(G, n)$ are equal to the number of all isomorphism classes of direct summands of A and semidirect factors of G, respectively. Also, we compute the capacity of wedge sum of finitely many Moore spaces of different degrees and the capacity of product of finitely many Eilenberg-MacLane spaces of different homotopy types. In particular, we compute the capacity of wedge sum of finitely many spheres of the same or different dimensions. Note that Borsuk in [3] has mentioned that $C(\mathbb{S}^n) = 2$ and $C(\bigvee_k \mathbb{S}^1) = k + 1$.

W. Holsztynski in [10] proved that the number of homotopy idempotents of a CW-complex is an upper bound for its capacity. Finally, we show that this upper bound is not so good (see Remark 4.13).

2. Preliminaries

In this paper, every topological space is assumed to be connected. We expect that the reader is familiar with the basic notions and facts of shape theory (see [5] and [17]) and retract theory [4]. We need the following results and definitions for the rest of the paper.

Theorem 2.1. [9]. *If a map $f : X \longrightarrow Y$ between connected CW complexes induces isomorphisms $f_* : \pi_n(X) \longrightarrow \pi_n(Y)$ for all n, then f is a homotopy equivalence.*

Definition 2.2. [1]. Let $\lambda : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. By the sufficiency and the realizability conditions, with respect to λ, we mean the following:

(1) **Sufficiency**: if $\lambda(f)$ is an isomorphism, then so is f, where f is a morphism in $\mathcal{C}$. That is, the functor λ reflects isomorphisms.
(2) **Realizability**: two following conditions satisfy:
   - The functor λ is representative, that is, for each object D in $\mathcal{D}$ there is an object C in $\mathcal{C}$ such that $\lambda(C)$ is isomorphic to D. In this case, we say that D is λ-realizable.
   - The functor λ is full, that is, for objects X, Y in $\mathcal{C}$ and for each morphism $f : \lambda(X) \longrightarrow \lambda(Y)$ in $\mathcal{D}$ there is a morphism $f_0 : X \longrightarrow Y$ in $\mathcal{C}$ with $\lambda(f_0) = f$. In this case, we also say that f is λ-realizable.

Definition 2.3. [1]. We call $\lambda : \mathcal{C} \longrightarrow \mathcal{D}$ a detecting functor if λ satisfies both sufficiency and realizability conditions, or equivalently if λ reflects isomorphisms, is representative and full.

A faithful detecting functor is called an equivalence of categories. By a faithful functor, we mean a functor $\lambda : \mathcal{C} \longrightarrow \mathcal{D}$ such that the induced maps $\lambda : Hom(X, Y) \longrightarrow Hom(\lambda X, \lambda Y)$ are injective, for all objects $X, Y \in \mathcal{C}$ (see [1]).

Definition 2.4. [13]. A homomorphism $g : G \longrightarrow H$ of groups is an r-homomorphism if there exists a homomorphism $f : H \longrightarrow G$ such that $g \circ f = id_H$. Then H is an r-image of G.

In particular, let G be a group with a subgroup H. Then H is called a retract of G if there exists a homomorphism $r : G \longrightarrow H$ such that $r \circ i = id_H$, where $i : H \longrightarrow G$ is the inclusion homomorphism.

Lemma 2.5. *Every r-image of an arbitrary group G is a semidirect factor of G and vice versa.*

Proof. It can be concluded from the definition of semidirect factor. □

In general, concepts of semidirect factor and direct summand of a group are different, but one can easily see that those are the same for abelian groups. Using this fact, we have the following corollary.

Corollary 2.6. *Let G be an abelian group. Then the cardinality of the following three sets are equal:*

(1) *The set of all isomorphism classes of r-images of G.*
(2) *The set of all isomorphism classes of retracts of G.*
(3) *The set of all isomorphism classes of direct summands of G.*

Proof. (1) & (3): this is a direct result of Lemma 2.5.

(1) & (2): On the one hand, by definition, any retract of G is an r-image of G. On the other hand, for any r-image H of G, there exist homomorphisms $g : G \longrightarrow H$ and $f : H \longrightarrow G$ such that $g \circ f = id_H$. Then it is easy to show that $f(H)\ \big(\cong H\big)$ is a retract of G. □

Proposition 2.7. *Let G be a finitely generated abelian group with the following form:*

$$\mathbb{Z}^{(k_1)}_{p_1^{\alpha_1}} \oplus \mathbb{Z}^{(k_2)}_{p_2^{\alpha_2}} \oplus \cdots \oplus \mathbb{Z}^{(k_n)}_{p_n^{\alpha_n}},$$

where for $i \neq j$, $p_i^{\alpha_i} \neq p_j^{\alpha_j}$, p_i's are prime numbers, α_i's are non-negative integers, $\mathbb{Z}^{(k_i)}_{p_i^{\alpha_i}}$ is the direct sum of k_i copies of $\mathbb{Z}_{p_i^{\alpha_i}}$, and $\mathbb{Z}_1 = \mathbb{Z}$. Then the number of direct summands of G, up to isomorphism, is equal to

$$(k_1 + 1) \times \cdots \times (k_n + 1).$$

Proof. We have the following three steps to finish the proof.

Step One. For each $1 \leq i \leq n$, the number of direct summands of $\mathbb{Z}^{(k_i)}_{p_i^{\alpha_i}}$, up to isomorphism, is equal to $k_i + 1$.

Clearly, $\mathbb{Z}^{(t)}_{p_i^{\alpha_i}}$ is a direct summand of $\mathbb{Z}^{(k_i)}_{p_i^{\alpha_i}}$ for every $0 \leq t \leq k_i$. Also, $\mathbb{Z}^{(t)}_{p_i^{\alpha_i}} \ncong \mathbb{Z}^{(t')}_{p_i^{\alpha_i}}$ for every $0 \leq t \neq t' \leq k_i$. Now, suppose that C is a direct summand of $\mathbb{Z}^{(k_i)}_{p_i^{\alpha_i}}$. There exists a subgroup D of $\mathbb{Z}^{(k_i)}_{p_i^{\alpha_i}}$ such that $\mathbb{Z}^{(k_i)}_{p_i^{\alpha_i}} \cong C \oplus D$. By [11, Corollary 2.1.7], C is a finitely generated abelian group. Let $C \cong \mathbb{Z}^{(l_1)}_{q_1^{\beta_1}} \oplus \cdots \oplus \mathbb{Z}^{(l_s)}_{q_s^{\beta_s}}$. Since $\mathbb{Z}^{(l_j)}_{q_j^{\beta_j}}$ $(1 \leq j \leq s)$ is a direct summand of C and C is a direct summand of $\mathbb{Z}^{(k_i)}_{p_i^{\alpha_i}}$, $\mathbb{Z}^{(l_j)}_{q_j^{\beta_j}}$ $(1 \leq j \leq s)$

is a direct summand of $\mathbb{Z}^{(k_i)}_{p_i^{\alpha_i}}$. Now, by uniqueness of decomposition of finitely generated abelian groups [11, Theorem 2.2.6, (iii)], for every $1 \leq j \leq s$, we have $q_j = p_i$ and $\beta_j = \alpha_i$. Hence $C \cong \mathbb{Z}^{(s)}_{p_i^{\alpha_i}}$, where $0 \leq s \leq k_i$.

Step Two. The number of direct summands of $\mathbb{Z}^{(k_i)}_{p_i^{\alpha_i}} \oplus \mathbb{Z}^{(k_j)}_{p_j^{\alpha_j}}$ for $i \neq j$, up to isomorphism, is equal to $(k_i + 1)(k_j + 1)$.

It is easy to see that for every $0 \leq t \leq k_i$ and $0 \leq s \leq k_j$, $\mathbb{Z}^{(t)}_{p_i^{\alpha_i}} \oplus \mathbb{Z}^{(s)}_{p_j^{\alpha_j}}$ is a direct summand of $\mathbb{Z}^{(k_i)}_{p_i^{\alpha_i}} \oplus \mathbb{Z}^{(k_j)}_{p_j^{\alpha_j}}$. Now similar to Step One, suppose that C is a direct summand of $\mathbb{Z}^{(k_i)}_{p_i^{\alpha_i}} \oplus \mathbb{Z}^{(k_j)}_{p_j^{\alpha_j}}$ and D is a subgroup of $\mathbb{Z}^{(k_i)}_{p_i^{\alpha_i}} \oplus \mathbb{Z}^{(k_j)}_{p_j^{\alpha_j}}$ such that $\mathbb{Z}^{(k_i)}_{p_i^{\alpha_i}} \oplus \mathbb{Z}^{(k_j)}_{p_j^{\alpha_j}} \cong C \oplus D$. Suppose $C \cong \mathbb{Z}^{(l_1)}_{q_1^{\beta_1}} \oplus \cdots \oplus \mathbb{Z}^{(l_s)}_{q_s^{\beta_s}}$. Since for every $1 \leq m \leq s$, $\mathbb{Z}^{(l_m)}_{q_m^{\beta_m}}$ is a direct summand of $\mathbb{Z}^{(k_i)}_{p_i^{\alpha_i}} \oplus \mathbb{Z}^{(k_j)}_{p_j^{\alpha_j}}$, so similar to the above argument, $C \cong \mathbb{Z}^{(t)}_{p_i^{\alpha_i}} \oplus \mathbb{Z}^{(s)}_{p_j^{\alpha_j}}$ for some $0 \leq t \leq k_i$ and $0 \leq s \leq k_j$.

Step Three: the number of direct summands of $\mathbb{Z}^{(k_1)}_{p_1^{\alpha_1}} \oplus \mathbb{Z}^{(k_2)}_{p_2^{\alpha_2}} \oplus \cdots \oplus \mathbb{Z}^{(k_n)}_{p_n^{\alpha_n}}$, up to isomorphism, is equal to $(k_1 + 1)(k_2 + 1) \cdots (k_n + 1)$.

It is concluded by Step Two and induction on n. □

3. The Capacity of Moore Spaces

In this section, we compute the capacity of Moore spaces. Also, we compute the capacity of wedge sum of finitely many Moore spaces of different degrees. In particular, we compute the capacity of wedge sum of finitely many spheres of the same or different dimensions.

Definition 3.1. [1]. A Moore space of degree n ($n \geq 2$) is a simply connected CW-space X with a single non-vanishing homology group of degree n, that is $\tilde{H}_i(X, \mathbb{Z}) = 0$ for $i \neq n$. We write $X = M(A, n)$, where $A \cong \tilde{H}_n(X, \mathbb{Z})$.

Note that for $n = 1$, the Moore space $M(A, 1)$ can not be defined, because of some problems in existence and uniqueness of the space (for more details, see [9]).

The $(n-1)$-fold suspension [20] of a pseudo projective plane $\mathbb{P}_q = \mathbb{S}^1 \cup_q e_2$ [1], is a Moore space of degree n, that is

$$\Sigma^{n-1}\mathbb{P}_q = M(\mathbb{Z}_q, n).$$

Recall that $\mathbb{P}_q$ is the space obtained by attaching a 2-cell e_2 to $\mathbb{S}^1$ by a map $q : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ of degree q.

It is also obvious that the sphere $\mathbb{S}^n$ is also a Moore space, $\mathbb{S}^n = M(\mathbb{Z}, n)$.

Theorem 3.2. [1]. *The homotopy type of a CW complex Moore space $M(A, n)$ is uniquely determined by A and n $(n > 1)$.*

Let **Ab** be the category of abelian groups. Also for $n \geq 2$, let $\mathbf{M}^n \subset hTop$ be the full subcategory of the category $hTop$ consisting of spaces $M(A, n)$ with $A \in \mathbf{Ab}$. For $n \geq 2$, $\mathbf{FM}^n$ denotes the full subcategory of $\mathbf{M}^n$ consists of all Moore spaces $M(A, n)$, where A is a finitely generated abelian group (see [1]). For each such group we have a direct sum decomposition

$$\mathbb{Z}_{q_1} \oplus \mathbb{Z}_{q_2} \oplus \cdots \oplus \mathbb{Z}_{q_r}, \quad q_i \geq 0$$

of cyclic groups. Associated with this isomorphism there is a homotopy equivalence

$$M(A, n) \simeq \Sigma^{n-1}\Big(\mathbb{P}_{q_1} \vee \mathbb{P}_{q_2} \vee \cdots \vee \mathbb{P}_{q_r}\Big),$$

where $\mathbb{P}_n = \mathbb{S}^1 \cup_n e^2$ is a pseudo-projective plane if $n > 0$, and $\mathbb{P}_0 = \mathbb{S}^1$ (see [1]).

Theorem 3.3. [9]. *For a wedge sum $\bigvee_\alpha X_\alpha$, the inclusions $i_\alpha : X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce an isomorphism $\bigoplus_\alpha i_{\alpha^*} : \bigoplus_\alpha \tilde{H}_n(X_\alpha) \longrightarrow \tilde{H}_n(\bigvee_\alpha X_\alpha)$, provided that the wedge sum is formed at basepoints $x_\alpha \in X_\alpha$, such that the pairs (X_α, x_α)'s are good.*

By a good pair (X, A), we mean a topological space X and a nonempty closed subspace A of X in which A is also a deformation retract of some neighborhood in X. For any CW-complex X and any subcomplex A of X, (X, A) is a good pair (see [9, Proposition A.5]).

Lemma 3.4. *Let A be an abelian group and $n \geq 2$. Then a space X is homotopy dominated by Moore space $M(A, n)$ if and only if X has the homotopy type of $M(B, n)$, where B is a direct summand of A.*

Proof. Suppose that X is homotopy dominated by $M(A, n)$. Then $\tilde{H}_i(X)$ is a direct summand of $\tilde{H}_i(M(A, n))$, for all i. Hence $\tilde{H}_i(X) = 0$ if $i \neq n$ and $\tilde{H}_n(X) = B$, where B is a direct summand of A. Therefore, X has the homotopy type of $M(B, n)$.

Conversely, assume that B is a direct summand of A. So there exists a subgroup C of A such that $A = B \oplus C$. Put $X = M(A, n)$, $Y = M(B, n)$ and $Z = M(C, n)$. Then by Theorem 3.3,

$$\tilde{H}_i(Y \vee Z) \cong \tilde{H}_i(Y) \oplus \tilde{H}_i(Z) \cong \begin{cases} B \oplus C = A, & i = n \\ 0, & i \neq n. \end{cases}$$

This shows that $Y \vee Z$ is an $M(A, n)$. Now by Theorem 3.2, $Y \vee Z$ is homotopy equivalent to X. Since Y is a retract of $Y \vee Z$, it is therefore homotopy dominated by X. Thus the proof is finished. □

The previous lemma and Theorem 3.2 imply the following result.

Proposition 3.5. *There is a one-to-one correspondence between the set of all homotopy types of spaces homotopy dominated by $M(A, n)$ and the set of all isomorphism classes of direct summands of A, for $n \geq 2$.*

The following result is a consequence of Proposition 2.7 and Proposition 3.5.

Proposition 3.6. *Let X be a Moore Space $M(A, m)$ ($m \geq 2$), where A is a finitely generated abelian group of the form*

$$\mathbb{Z}^{(k_1)}_{p_1^{\alpha_1}} \oplus \mathbb{Z}^{(k_2)}_{p_2^{\alpha_2}} \oplus \cdots \oplus \mathbb{Z}^{(k_n)}_{p_n^{\alpha_n}},$$

where for $i \neq j$, $p_i^{\alpha_i} \neq p_j^{\alpha_j}$, p_i's are prime numbers, α_i's are non-negative integers, $\mathbb{Z}^{(k_i)}_{p_i^{\alpha_i}}$ is the direct product of k_i copies of $\mathbb{Z}_{p_i^{\alpha_i}}$ and $\mathbb{Z}_1 = \mathbb{Z}$. Then the capacity of X is equal to

$$(k_1 + 1) \times \cdots \times (k_n + 1).$$

As an example, by Proposition 3.5, the capacity of the Moore space $M(\mathbb{Q}, n)$ is exactly 2. Recall that $\mathbb{Q}$ is not the direct sum of any family of its proper subgroups. Also, by Proposition 3.6, the capacity of $M(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z} \oplus \mathbb{Z}, n)$, $M(\mathbb{Z}_9 \oplus \mathbb{Z}_{64})$, $M(\mathbb{Z}, n)$ and $M(\mathbb{Z}_{p^m}, n)$ are 18, 4, 2 and 2, respectively.

Remark 3.7. The computation of the capacity of wedge sum of finitely many spheres with the same or different dimensions seems interesting. In [3], it has been mentioned that the capacity of $\bigvee_k \mathbb{S}^1$ is equal to $k + 1$. Also, Kolodziejczyk in [13] asked the following question:

Does every polyhedron P with the abelian fundamental group $\pi_1(P)$ dominate only finitely many different homotopy types?

It has been proved that two extensive classes of polyhdera, polyhedra with finite fundamental groups, and polyhedra P with abelian fundamental groups and finitely generated homology groups $H_i(\tilde{P})$ ($i \geq 2$), have finite capacities, where $\tilde{P}$ denotes the universal covering space of P (see [13],[14]). The wedge sum $\mathbb{S}^1 \vee \mathbb{S}^2$ is a simple example of a polyhedron P with abelian fundamental group $\pi_1(P)$ and infinitely generated homology group $H_2(\tilde{P}; \mathbb{Z})$ which the finiteness of its capacity is still unknown. Note that $\mathbb{S}^1 \vee \mathbb{S}^2$ is neither a Moore space nor an Eilenberg-MacLane space.

In the following, we compute the capacity of wedge sum of finitely many Moore spaces with the same or different degrees. In particular, we compute the capacities of $\bigvee_k \mathbb{S}^n$, $\mathbb{S}^m \vee \mathbb{S}^n$ $(m, n \geq 2, m \neq n)$, and the general case $\bigvee_{n\in I}(\vee_{i_n}\mathbb{S}^n)$, where I is a finite subset of $\mathbb{N}\setminus\{1\}$ and the index $i_n \in \mathbb{N}$ denotes the number of copies of $\mathbb{S}^n$ in $\bigvee_{n\in I}(\vee_{i_n}\mathbb{S}^n)$.

Corollary 3.8. *Let $n \geq 2$ be fixed. The capacity of $\bigvee_{\alpha\in I}\mathbb{S}^n$ is finite if and only if I is finite. In particular, $C(\bigvee_{i\in\{1,\cdots,k\}}\mathbb{S}^n) = k+1$, for every $n \geq 1$.*

Proof. Suppose that the capacity of $\bigvee_{\alpha\in I}\mathbb{S}^n$ is finite. Then, since $H_n(\bigvee_{\alpha\in I}\mathbb{S}^n) = \bigoplus_{\alpha\in I}\mathbb{Z}$, by Proposition 3.5, the set I is finite. Conversely, suppose I is finite and $|I| = k$. Hence, by Theorem 3.3, we have $H_n(\bigvee_{i\in\{1,\cdots,k\}}\mathbb{S}^n) = \mathbb{Z}^{(k)}$ and since $\bigvee_{i\in\{1,\cdots,k\}}\mathbb{S}^n$ is a Moore space of degree n, by Proposition 3.6, the capacity of $\bigvee_{i\in\{1,\cdots,k\}}\mathbb{S}^n$ is equal to $k+1$. □

Recall that a group G is called Hopfian if every epimorphism $f : G \longrightarrow G$ is an automorphism (equivalently, $N = \{1\}$ is the only normal subgroup of G for which $G/N \cong G$). It is easy to see that if G is a Hopfian group and $H \cong G$, then H is also Hopfian. Moreover, if G is an abelian Hopfian group and K is a direct summand of G, then K is also Hopfian [19].

Proposition 3.9. *Let $X = \bigvee_{\alpha\in I} M(A_\alpha, n_\alpha)$, where all n_α are distinct, $n_\alpha \geq 2$ and also all A_α are abelian Hopfian groups. Then, every topological space homotopy dominated by X has the homotopy type of $\bigvee_{\alpha\in I} M(B_\alpha, n_\alpha)$, where B_α is a direct summand of A_α for each $\alpha \in I$.*

Proof. Suppose that the space Y is homotopy dominated by X with a domination map $g : X \longrightarrow Y$ and a converse map $f : Y \longrightarrow X$. From $g \circ f \simeq id_Y$, we have

$$H_n(g) \circ H_n(f) = id_{H_n(Y)}, \tag{3.1}$$

for all $n \geq 1$. Let $h_n^X : \pi_n(X) \longrightarrow H_n(X)$ denotes the n-th Hurewicz map. By [1, Proposition 2.6.15], h_n^X is split surjective for all $n > 1$. So there exists a homomorphism $\phi_n^X : H_n(X) \longrightarrow \pi_n(X)$ such that

$$h_n^X \circ \phi_n^X = id_{H_n(X)}, \tag{3.2}$$

for all $n > 1$. Define homomorphism $\psi_n^Y : H_n(Y) \longrightarrow \pi_n(Y)$ by

$$\psi_n^Y = \pi_n(g) \circ \phi_n^X \circ H_n(f), \tag{3.3}$$

for all $n > 1$. Then by Eqs. (3.1), (3.2), (3.3) and the fact that the Hurewicz map is natural, we conclude that

$$h_n^y \circ \psi_n^Y = h_n^Y \circ (\pi_n(g) \circ \phi_n^X \circ H_n(f))$$

$$
\begin{aligned}
&= (h_n^Y \circ \pi_n(g)) \circ \phi_n^X \circ H_n(f) \\
&= (H_n(g) \circ h_n^X) \circ \phi_n^X \circ H_n(f) \\
&= H_n(g) \circ (h_n^X \circ \phi_n^X) \circ H_n(f) \\
&= H_n(g) \circ H_n(f) \\
&= id_{H_n(Y)},
\end{aligned}
$$

for all $n > 1$. This shows that h_n^Y is split surjective for all $n > 1$. So by [1, Proposition 2.6.15], Y has the homotopy type of a one point union of Moore spaces, say $\bigvee_{\alpha\in I} M(B_\alpha, n_\alpha)$. On the other hand, by the distinctness condition of n_α's, we have

$$\tilde{H}_{n_\alpha}(Y) \cong \tilde{H}_{n_\alpha}\Big(\bigvee_{\alpha\in I} M(B_\alpha, n_\alpha)\Big) \cong \bigoplus_{\alpha\in I} \tilde{H}_{n_\alpha}(M(B_\alpha, n_\alpha)) \cong B_\alpha$$

which implies that B_α is a direct summand of $\tilde{H}_{n_\alpha}(X) = A_\alpha$, for each $\alpha \in I$. Thus the proof is complete. □

The following corollary is a consequence of Proposition 3.9.

Corollary 3.10. *Let $X = \bigvee_{\alpha\in I} M(A_\alpha, n_\alpha)$, where n_α's are distinct, $n_\alpha \geq 2$ and A_α's are abelian Hopfian groups. Then*

$$C(X) = \prod_{\alpha\in I} C(M(A_\alpha, n_\alpha)).$$

Remark 3.11. Note that we can not omit the distinctness condition of n_α's in Corollary 3.10. For example,

$$C(\mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^3) = 6 \neq 8 = C(\mathbb{S}^2) \times C(\mathbb{S}^2) \times C(\mathbb{S}^3).$$

To overcome such difficulty, one can condiser $M(\bigoplus_{\alpha\in I} A_\alpha, n)$ instead of $\bigvee_{\alpha\in I} M(A_\alpha, n)$. To see this, consider the above example. Then $\mathbb{S}^2 \vee \mathbb{S}^2 = M(\mathbb{Z} \oplus \mathbb{Z}, 2)$ and $\mathbb{S}^3 = M(\mathbb{Z}, 3)$ which follows that

$$C(\mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^3) = C(\mathbb{S}^2 \vee \mathbb{S}^2) \times C(\mathbb{S}^3) = 3 \times 2 = 6.$$

Now, we are in a position to compute the capacity of wedge sum of finitely many spheres of the same or different dimensions.

Corollary 3.12. *The capacity of $\bigvee_{n\in I}(\vee_{i_n}\mathbb{S}^n)$ is equal to $\prod_{n\in I}(i_n+1)$, where $\vee_{i_n}\mathbb{S}^n$ denotes the wedge sum of i_n copies of $\mathbb{S}^n$, I is a finite subset of $\mathbb{N} \setminus \{1\}$ and $i_n \in \mathbb{N}$.*

Proof. It can be concluded from Corollaries 3.8 and 3.10. □

By the above corollary, the capacity of $\mathbb{S}^m \vee \mathbb{S}^n$ $(m, n \geq 2, m \neq n)$ is equal to 4.

4. The Capacity of Eilenberg-MacLane Spaces

In this section, we intend to compute the capacity of Eilenberg-MacLane spaces. Note that some of the results concerning the capacity of Eilenberg-MacLane spaces are similar to the results of previous section for Moore spaces, but their proofs are different. Also note that there exist Moore spaces which are not Eilenberg-MacLane spaces and vice versa. For example, $\mathbb{S}^n$ $(n \geq 2)$ is an example of a Moore space which is not an Eilenberg-MacLane space. Also, k-dimensional torus $\mathbb{T}^k$ $(k \geq 1)$ is an Eilenberg-MacLane space which is not a Moore space.

Recall that a space X having just one nontrivial homotopy group $\pi_n(X) \cong G$ is called an Eilenberg-MacLane space and is denoted by $K(G, n)$. The full subcategory of the category $hTop$ consisting of spaces $K(G, n)$ with $G \in \mathbf{Gp}$ is denoted by $\mathbf{K}^n$ (see [9]).

Theorem 4.1. [1]. *The n-th homotopy group functor $\pi_n : \mathbf{K}^n \longrightarrow \mathbf{Ab}$ is an equivalence of categories for $n \geq 2$. Moreover, the functor $\pi_1 : \mathbf{K}^1 \longrightarrow \mathbf{Gp}$ is also an equivalence of categoreis.*

Theorem 4.2. [9]. *The homotopy type of a CW complex $K(G, n)$ is uniquely determined by G and n.*

Lemma 4.3. *Let G be a group. Then the space X is homotopy dominated by Eilenberg-MacLane space $K(G, n)$ if and only if X has the homotopy type of $K(H, n)$, where H is an r-image of G.*

Proof. Suppose that X is homotopy dominated by $K(G, n)$. Then $\pi_i(X)$ is an r-image of $\pi_i(K(G, n))$, for each $i \geq 1$. Hence $\pi_i(X) = 0$ for each $i \neq n$ and $\pi_n(X) = H$, where H is an r-image of G. Therefore X has the homotopy type of an Eilenberg-MacLane space of the form $K(H, n)$.

Conversely, suppose that $\bar{f} : H \longrightarrow G$ and $\bar{g} : G \longrightarrow H$ are homomorphisms such that $\bar{g} \circ \bar{f} = id_H$. By Theorem 4.1, there exist maps $f : K(H, n) \longrightarrow K(G, n)$ and $g : K(G, n) \longrightarrow K(H, n)$ such that $\pi_n([f]) = \bar{f}$ and $\pi_n([g]) = \bar{g}$. Since $\bar{g} \circ \bar{f} = id_H$, we must have $g \circ f \simeq id_{K(H,n)}$. Hence $K(H, n)$ is homotopy dominated by $K(G, n)$. □

Now, similar to Moore spaces, we have the following result concerning the capacity of Eilenberg-MacLane spaces.

Proposition 4.4. *There exists a one-to-one correspondence between the set of all homotopy types of spaces homotopy dominated by Eilenberg-MacLane space $K(G, n)$ and the set of all isomorphism classes of r-images of G.*

Proof. By Lemma 4.3, every space homotopy dominated by $K(G,n)$ has the form $K(H,n)$, where H is an r-image of G. Also, if H is an r-image of G, then $K(H,n)$ is homotopy dominated by $K(G,n)$. Now, By Theorem 4.2, it is obvious that $H \longmapsto K(H,n)$ is a one-to-one correspondence between the set of all isomorphism classes of r-images H of G and the set of all homotopy types of spaces homotopy dominated by $K(G,n)$. □

Note that by a result of Kolodziejczyk [15], the capacity of $K(G,n)$ is finite, for $n \geq 2$. Also, when G is abelian, by another result of Kolodziejczyk [13, Theorem 2], the capacity of $K(G,1)$ is also finite. By Corollary 2.6 and Proposition 4.4, we have the following corollary.

Corollary 4.5. *Let G be an abelian group. Then the capacity of $K(G,n)$ $(n \geq 1)$ is finite if and only if G has finitely many direct summands up to isomorphism.*

In the following, we compute the capacity of $K(G,n)$, when G is a finitely generated abelian group.

Proposition 4.6. *Let X be an Eilenberg-MacLane space $K(G,n)$ $(n \geq 1)$, where G is a finitely generated abelian group of the form*

$$\mathbb{Z}_{p_1^{\alpha_1}}^{(k_1)} \oplus \mathbb{Z}_{p_2^{\alpha_2}}^{(k_2)} \oplus \cdots \oplus \mathbb{Z}_{p_n^{\alpha_n}}^{(k_n)},$$

where for $i \neq j$, $p_i^{\alpha_i} \neq p_j^{\alpha_j}$, p_i's are prime numbers, α_i's are non-negative integers, $\mathbb{Z}_{p_i^{\alpha_i}}^{(k_i)}$ is the direct sum of k_i copies of $\mathbb{Z}_{p_i^{\alpha_i}}$ and $\mathbb{Z}_1 = \mathbb{Z}$. Then the capacity of X is equal to

$$(k_1 + 1) \times \cdots \times (k_n + 1).$$

Proof. This is a consequence of Proposition 2.7 and Proposition 4.4. □

As an example, the capacity of n-dimensional torus $\mathbb{T}^n$ is equal to $n+1$. Note that $\mathbb{T}^n$ is not a Moore space, so its capacity can not be computed by the results of the previous section.

Example 4.7. $K(\mathbb{Q},1)$ is an infinite CW-complex of capacity 2. Indeed, $\mathbb{Q}$ is not finitely generated abelian group and so by [20, Corollary 7.37], $K(\mathbb{Q},1)$ is an infinite CW-complex. Also, by Corollary 4.5 and the fact that $\mathbb{Q}$ has only two r-images up to isomorphism, the capacity of $K(\mathbb{Q},1)$ is 2.

Let A be an abelian group. A set $\{a_1, \cdots, a_k\}$ of nonzero elements of A is called linearly independent if $\sum_{i=1}^{k} n_i a_i = 0$ $(n_i \in \mathbb{Z})$ implies $n_1 a_1 = \cdots = n_k a_k = 0$. More explicitly, this means $n_i = 0$ if $o(a_i) = \infty$ and $o(a_i)|n_i$ if $o(a_i)$ is finite. By the rank $r(A)$ of A is meant the cardinal number of a maximal independent set containing only elements of infinite and prime power orders (see [8]). One can easily see that the rank of the additive group of rational numbers $\mathbb{Q}$ is 1. Accordingly, $\mathbb{Q}$ is a simple example of a finire rank torsion free abelian group which is not finitely generated.

From Proposition 4.6, the capacity of $K(G, n)$ for a finite rank torsion free abelian group G is finite. From the next corollary we can conclude that the capacity $K(G, n)$ is also finite when G is an infinitely generated finite rank torsion free abelian group.

Corollary 4.8. *The capacity of $K(G, n)$ for finite rank torsion free abelian group G is finite.*

Proof. It can be concluded from Corollary 4.5 and the fact that G has only finitely many direct summands, up to isomorphism (see [6]). □

Remark 4.9. By the definition of AKS $\mathbb{Z}$-module (for more details, see [7]), an abelian group is *AKS* $\mathbb{Z}$-module if and only if it has finitely many direct summands up to isomorphism. Hence, we can rewrite Corollary 4.5 for any abelian group G as follows:

" $K(G, n)$ has finite capacity if and only if G is an AKS $\mathbb{Z}$-module"

As an example, Artinian $\mathbb{Z}$-modules satisfy the definition of AKS $\mathbb{Z}$-module.

To compute the capacity of finite product of Eilenberg-MacLane spaces, we give the following proposition.

Proposition 4.10. *Let $X = \prod_{\alpha \in I} K(G_\alpha, n_\alpha)$, where n_α's are distinct, $n_\alpha \geq 1$ and G_α's are Hopfian groups. Then every topological space homotopy dominated by X has the homotopy type of $\prod_{\alpha \in I} K(H_\alpha, n_\alpha)$, where H_α is an r-image of G_α for each α.*

Proof. Suppose that the space Y is homotopy dominated by X with a domination map $g : X \longrightarrow Y$ and a converse map $f : Y \longrightarrow X$. From $g \circ f \simeq id_Y$, we have $\pi_{n_\alpha}(g) \circ \pi_{n_\alpha}(f) = id_{\pi_{n_\alpha}(Y)}$ for each $\alpha \in I$. This shows that $\pi_{n_\alpha}(Y)$ is an r-image of $\pi_{n_\alpha}(X) = G_\alpha$ for each $\alpha \in I$. Then by Lemma 4.3, $K(\pi_{n_\alpha}(Y), n_\alpha)$ is homotopy dominated by $K(G_\alpha, n_\alpha)$ with a domination map $d_\alpha : K(G_\alpha, n_\alpha) \longrightarrow K(\pi_{n_\alpha}(Y), n_\alpha)$ and a converse map $u_\alpha : K(\pi_{n_\alpha}(Y), n_\alpha) \longrightarrow K(G_\alpha, n_\alpha)$ so that

$$\pi_{n_\alpha}(d_\alpha) = \pi_{n_\alpha}(g) \quad \text{and} \quad \pi_{n_\alpha}(u_\alpha) = \pi_{n_\alpha}(f). \qquad (*)$$

Then $\prod_{\alpha \in I} K(\pi_{n_\alpha}(Y), n_\alpha)$ is a homotopy dominated by space $X = \prod_{\alpha \in I} K(G_\alpha, n_\alpha)$ with domination map

$$d = \prod_{\alpha \in I} d_\alpha : X \longrightarrow \prod_{\alpha \in I} K(\pi_{n_\alpha}(Y), n_\alpha)$$

and converse map

$$u = \prod_{\alpha \in I} u_\alpha : \prod_{\alpha \in I} K(\pi_{n_\alpha}(Y), n_\alpha) \longrightarrow X.$$

Now, consider the map $g \circ u : \prod_{\alpha \in I} K(\pi_{n_\alpha}(Y), n_\alpha) \longrightarrow Y$. By (*),

$$\begin{aligned} \pi_{n_\alpha}(g \circ u)\Big(\pi_{n_\alpha}(\prod_{\alpha \in I} K(\pi_{n_\alpha}(Y), n_\alpha))\Big) &= \pi_{n_\alpha}(g)(\pi_{n_\alpha}(u)(\pi_{n_\alpha}(Y)))) \\ &= \pi_{n_\alpha}(g)(\pi_{n_\alpha}(f)(\pi_{n_\alpha}(Y))) \\ &= \pi_{n_\alpha}(Y) \end{aligned}$$

for each $\alpha \in I$. This show that for each $\alpha \in I$, $\pi_{n_\alpha}(g \circ u)$ is an epimorphism between two isomorphic Hopfian groups which must be an isomorphism. Now by Theorem 2.1, the map $g \circ u$ is a homotopy equivalence. Thus Y has the homotopy type of $\prod_{\alpha \in I} K(\pi_{n_\alpha}(Y), n_\alpha)$. □

Corollary 4.11. *Let $\{K(G_\alpha, n_\alpha)\}_{\alpha \in I}$ be a family of Eilenberg-MacLane spaces, where n_α's are distinct, $n_\alpha \geq 1$ and G_α's are Hopfian groups. Then $C(\prod_{\alpha \in I} K(G_\alpha, n_\alpha)) = \prod_{\alpha \in I} C\big(K(G_\alpha, n_\alpha)\big)$.*

Proof. This is a direct result of Proposition 4.10. □

Remark 4.12. Note that we can not omit the distinctness condition of n_α's in Proposition 4.10. For example,

$$C(\mathbb{S}^1 \times \mathbb{S}^1 \times K(\mathbb{Z}, 2)) = 6 \neq 8 = C(\mathbb{S}^1) \times C(\mathbb{S}^1) \times C(K(\mathbb{Z}, 2)).$$

To overcome such difficulty, one can condiser $K(\prod_{\alpha \in I} G_\alpha, n)$ instead of $\prod_{\alpha \in I} K(G_\alpha, n)$. To see this, consider the above example. Then $\mathbb{S}^1 \times \mathbb{S}^1 = K(\mathbb{Z} \times \mathbb{Z}, 1)$ which follows that

$$C(\mathbb{S}^1 \times \mathbb{S}^1 \times K(\mathbb{Z}, 2)) = C(\mathbb{S}^1 \times \mathbb{S}^1) \times C(K(\mathbb{Z}, 2)) = 3 \times 2 = 6.$$

Let X be a topological space. The set of all maps $f : X \longrightarrow X$ satisfying the condition $f^2 = f$, constitute a subset of X^X which is denoted by $\mathcal{R}(X)$ (see [2]). Also, the set of all homotopy classes of maps $f : X \longrightarrow X$ with $f^2 \simeq f$ which are called homotopy idempotents of X, is denoted by $hI(X)$. Similarly, for a group G, the set of all homomorphisms $f : G \longrightarrow G$ with $f^2 = f$, is denoted by $\mathcal{R}(G)$.

Remark 4.13. By [10], we know that $|hI(X)|$ is an upper bound for the capacity of topological space X. Here we show that $|hI(X)|$ is not a good upper bound for the capacity of X. To see this, let X be an Eilenberg-MacLane space $K(G, 1)$. By Theorem 4.1, the correspondence $f \longmapsto f_*$ induces a one-to-one correspondence between $[X, X]$ and $Hom(\pi_1(X), \pi_1(X))$. So the number of homotopy classes of maps

$f : X \longrightarrow X$ with $f^2 \simeq f$ is equal to the number of homomorphisms $g : \pi_1(X) \longrightarrow \pi_1(X)$ with $g^2 = g$. Hence $|hI(X)| = |\mathcal{R}(\pi_1(X))|$. Now, suppose that X is the torus $\mathbb{T}^2$. Then $|hI(\mathbb{T}^2)| = |\mathcal{R}(\pi_1(\mathbb{T}^2))|$. Since $\pi_1(\mathbb{T}^2) \cong \mathbb{Z} \times \mathbb{Z}$, we have $Hom(\pi_1(\mathbb{T}^2), \pi_1(\mathbb{T}^2)) \cong M_2(\mathbb{Z})$. Therefore $|\mathcal{R}(\pi_1(\mathbb{T}^2))|$ equals to the number of idempotent matrices in $M_2(\mathbb{Z})$. But $M_2(\mathbb{Z})$ has infinite number idempotents such as $\begin{pmatrix} 1 & n \\ 0 & 0 \end{pmatrix}$ for $n \in \mathbb{Z}$. Hence $hI(\mathbb{T}^2)$ is infinite, while $C(\mathbb{T}^2) = 3$.

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ON THE CAPACITY OF EILENBERG–MACLANE AND MOORE SPACES

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درباره ظرفیت فضاهای آیلنبرگ–مک‌لین و مور

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کارول بورسوک در سال ۱۹۷۹، در کنفرانس توپولوژیکی در مسکو، مفهوم ظرفیت یک فضای متریک فشرده را معرفی کرد و چند سوال در ارتباط با خواص ظرفیت فضاهای متریک فشرده مطرح نمود. در این مقاله، ما به سه سوال از این مسائل پاسخ‌های جزئی مثبت در بعضی از حالات خواهیم داد. در حقیقت، با توصیف فضاهای تحت تسلط هموتوپی توسط فضاهای مور و آیلنبرگ–مک‌لین، ظرفیت‌های یک فضای مور $M(A,n)$ و یک فضای آیلنبرگ–مک‌لین $K(G,n)$ قابل محاسبه هستند. همچنین، ما ظرفیت الحاق تعداد متناهی فضای مور از درجه‌های متفاوت و ظرفیت حاصلضرب تعداد متناهی فضای آیلنبرگ–مک‌لین از انواع هموتوپی متفاوت را محاسبه خواهیم نمود. در حالت خاص، ظرفیت الحاق تعداد متناهی کره از ابعاد یکسان و یا متفاوت را محاسبه می‌کنیم.

کلمات کلیدی: تسلط هموتوپیکی، فضای آیلنبرگ–مک‌لین، فضای مور.