ON GRADED INJECTIVE DIMENSION

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ABSTRACT: There are remarkable relations between the graded homological dimensions and the ordinary homological dimensions. In this paper, we study the injective dimension of a complex of graded modules and derive its some properties. In particular, we define the *dualizing complex for a graded ring and investigate its consequences.

1. INTRODUCTION

Let $R$ be a Noetherian $\mathbb{Z}$-graded ring. In [5] and [6], Fossum and Fossum-Foxby have studied the graded homological dimension of graded modules and compare them with classical homological dimensions. They showed that for a graded $R$-module $M$, one has

\[ ^*\text{id}_R M \leq \text{id}_R M \leq ^*\text{id}_R M + 1, \]

where $\text{id}_R M$ (resp., $^*\text{id}_R M$) denotes the injective dimension of $M$ in the category of $R$-modules (resp., category of graded $R$-modules). It is natural to ask how these inequalities hold for the injective dimension of a complex of graded modules and homogeneous homomorphisms. Section 2 of this paper is devoted to review some hyper-homological algebra for the derived category of the graded ring $R$. In Section 3, we define the *injective dimension of complexes of graded modules and homogeneous homomorphisms, and derive its some properties. Among other results, we prove the generalization of the dual version of Auslander-Buchbaum equality, which implies the known inequality

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let $X$ be a complex of $R$-modules and $R$-homomorphisms. The supremum and the infimum of a complex $X$, denoted by $\text{sup}(X)$ and $\text{inf}(X)$, are defined by the supremum and the infimum of the set \{ $i \in \mathbb{Z}$ | $H_i(X) \neq 0$ \}. For an integer $m$, $\Sigma^m X$ denotes the complex $X$ shifted $m$ degrees to the left; it is given by

$$(\Sigma^m X)_{\ell} = X_{\ell-m} \text{ and } \partial^m_{\ell} = (-1)^m \partial^X_{\ell-m},$$

for $\ell \in \mathbb{Z}$.

The symbol $\mathcal{D}(R)$ denotes the derived category of $R$-complexes. The full subcategories $\mathcal{D}_{\geq}(R)$, $\mathcal{D}_{\leq}(R)$, $\mathcal{D}_{=}(R)$ and $\mathcal{D}_{0}(R)$ of $\mathcal{D}(R)$ consist of $R$-complexes $X$ while $H_{\ell}(X) = 0$, for respectively $\ell \gg 0$, $\ell \ll 0$, $|\ell| \gg 0$ and $\ell \neq 0$. Homology isomorphisms are marked by the sign $\simeq$.

The right derived functor of the homomorphism functor of $R$-complexes and the left derived functor of the tensor product of $R$-complexes are denoted by $R \text{Hom}_R(\cdot, \cdot)$ and $- \otimes^L_R -$, respectively.

Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ and $N = \bigoplus_{n \in \mathbb{Z}} N_n$ be two graded $R$-modules. The $^*$ hom functor is defined by $^* \text{Hom}_R(M, N) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_i(M, N)$, such that $\text{Hom}_i(M, N)$ is a $\mathbb{Z}$-submodule of $\text{Hom}_R(M, N)$ consisting of all $\varphi : M \to N$ such that $\varphi(M_n) \subseteq N_{n+i}$ for all $n \in \mathbb{Z}$. In general, $^* \text{Hom}_R(M, N) \neq \text{Hom}_R(M, N)$, but equality holds if $M$ is finitely generated, see [6, Lemma 4.2]. Also, the tensor product $M \otimes_R N$ of $M$ and $N$ is a graded module with $(M \otimes_R N)_n$ generated (as a $\mathbb{Z}$-module) by elements $m \otimes n$ with $m \in M_i$ and $n \in N_j$ where $i + j = n$.

Let $\{ M_\alpha \}_{\alpha \in I}$ be a family of graded $R$-modules. Then, $\bigoplus_\alpha M_\alpha$ becomes a graded $R$-module with $(\bigoplus_\alpha M_\alpha)_n = \bigoplus_\alpha (M_\alpha)_n$, for all $n \in \mathbb{Z}$, see [6, Page 289]. Recall that the direct products exist in the category of graded modules. Then the direct product is denoted by $^* \prod_\alpha M_\alpha$ and $(^* \prod_\alpha M_\alpha)_n = \prod_\alpha (M_\alpha)_n$ for all $n \in \mathbb{Z}$, see [6, Page 289]. In this case, there are the following bijections [6, Page 289]

$$^* \text{Hom}_R(\bigoplus_\alpha M_\alpha, -) \overset{\cong}{\longrightarrow} ^* \prod_\alpha ^* \text{Hom}_R(M_\alpha, -),$$

$$^* \text{Hom}_R(-, ^* \prod_\alpha M_\alpha) \overset{\cong}{\longrightarrow} ^* \prod_\alpha ^* \text{Hom}_R(-, M_\alpha).$$
Likewise, direct limits exist in the category of graded modules with
for all \( n \in \mathbb{Z} \), see [6, Page 289].

The symbol \( \mathcal{C}(R) \) denotes the category of complexes of graded \( R \)
modules and homogeneous differentials. Remember that the category
of graded modules is an abelian category. The derived category of
\( \mathcal{C}(R) \) will be denoted by \( \mathcal{D}(R) \), (see [10]). Analogously we have
\( \mathcal{C}_{\subseteq}(R) \) and \( \mathcal{C}_{\triangle}(R) \) and \( \mathcal{C}_{0}(R) \) (resp. \( \mathcal{D}_{\subseteq}(R) \), \( \mathcal{D}_{\triangle}(R) \) and \( \mathcal{D}_{0}(R) \)) which are the full subcategories of \( \mathcal{C}(R) \) (resp. \( \mathcal{D}(R) \)).

For \( R \)-complexes \( X \) and \( Y \) of graded modules, with homogeneous
differentials \( \partial^X \) and \( \partial^Y \) the homomorphism complex \( \text{Hom}_R(X,Y) \) is
defined as:

\[
\text{Hom}_R(X,Y)_\ell = \prod_{p \in \mathbb{Z}} \text{Hom}_R(X_p,Y_{p+\ell})
\]

and when \( \psi = (\psi_p)_{p \in \mathbb{Z}} \) belongs to \( \text{Hom}_R(X,Y)_\ell \), then the family
\( \partial^\ell \text{Hom}_R(X,Y)(\psi) \) in \( \text{Hom}_R(X,Y)_{\ell-1} \) has \( p \)-th component

\[
\partial^\ell \text{Hom}_R(X,Y)(\psi)_p = \partial^Y p+\ell \psi_p - (-1)^p \psi_{p-1} \partial^X p.
\]

When \( X \in \mathcal{C}_{\subseteq}(R) \) and \( Y \in \mathcal{C}_{\triangle}(R) \) all the products

\[
\prod_{p \in \mathbb{Z}} \text{Hom}_R(X_p,Y_{p+\ell})
\]

are finite. Note that for each \( p \in \mathbb{Z} \), \( X_p \) is finitely generated \( R \)-module,
thus \( \text{Hom}_R(X_p,Y_{p+\ell}) = \text{Hom}_R(X_p,Y_{p+\ell}) \), see [6, Lemma 4.2]. Therefore

\[
\prod_{p \in \mathbb{Z}} \text{Hom}_R(X_p,Y_{p+\ell}) = \text{Hom}_R(X_p,Y_{p+\ell}),
\]

for every \( \ell \in \mathbb{Z} \). Therefore \( \text{Hom}_R(X,Y) = \text{Hom}_R(X,Y) \).

Also the tensor product complex \( X \otimes_R Y \) is defined as:

\[
(X \otimes_R Y)_\ell = \bigoplus_{p \in \mathbb{Z}} (X_p \otimes_R Y_{\ell-p})
\]

and the \( \ell \)-th differential \( \partial^X \otimes_R Y \) is given on a generator \( x_p \otimes y_{\ell-p} \) in
\( (X \otimes_R Y)_\ell \), where \( x_p \) and \( y_{\ell-p} \) are homogeneous elements, by

\[
\partial^X \otimes_R Y(x_p \otimes y_{\ell-p}) = \partial^X p(x_p) \otimes y_{\ell-p} + (-1)^p x_p \otimes \partial^Y_{\ell-p}(y_{\ell-p}).
\]

If \( X \) and \( Y \) are \( R \)-complexes of graded modules, then \( \text{Hom}_R(X,-) \),
\( \text{Hom}_R(-,Y) \), and \( X \otimes_R - \) are graded functors on \( \mathcal{C}(R) \).

Note that any object of \( \mathcal{C}_{\subseteq}(R) \) has an injective resolution by [10,
Page 47], and any object of \( \mathcal{C}_{\triangle}(R) \) has a projective resolution by
[10, Page 48]. The right derived functor of the \( \text{Hom} \) functor in the
category of graded complexes is denoted by $R^*\text{Hom}_R(-,-)$ and set $^\ast\text{Ext}^i_R(-,-) = H_{-i}(R^*\text{Hom}_R(-,-))$. It is easily seen that if $R$ is a Noetherian $\mathbb{Z}$-graded ring and $X \in ^\ast\mathcal{C}_f(R)$ and $Y \in ^\ast\mathcal{C}_-(R)$ then $R^*\text{Hom}_R(X,Y) = R\text{Hom}_R(X,Y)$. Also the left derived functor of $- \otimes R -$ in the category of graded complexes is denoted by $- \otimes L^\ast R -$. Since $^\ast$projective graded $R$-modules coincide with projective $R$-modules by [6, Proposition 3.1] we easily see that $- \otimes L^\ast R -$ coincides with the ordinary left derived functor of $- \otimes R -$ in the category of complexes. So we use $- \otimes L^\ast R -$ instead of $- \otimes L^\ast R -$.

We recall the definition of the depth and width of complexes. Let $\mathfrak{a}$ be an ideal in a ring $R$ and $X$ a complex of graded $R$-modules. The $\mathfrak{a}$-depth and $\mathfrak{a}$-width of $X$ over $R$ are defined respectively by

$$\text{depth}(\mathfrak{a},X) := - \sup R^*\text{Hom}_R(R/\mathfrak{a},X),$$

$$\text{width}(\mathfrak{a},X) := \inf(R/\mathfrak{a} \otimes L^\ast R X).$$

For a local ring $(R,\mathfrak{m})$ set $\text{depth}_R X := \text{depth}(\mathfrak{m},X); \text{width}_R X := \text{width}(\mathfrak{m},X)$. Let $(R,\mathfrak{m})$ be a $^\ast$local graded ring and $X$ be a complex of graded $R$-modules. By [2, Proposition 1.5.15(c)], $- \otimes R \mathfrak{m}$ is a faithfully exact functor on the category of graded $R$-modules. Then we have

$$\text{width}(\mathfrak{m},X) = \inf\{i|H_i(R/\mathfrak{m} \otimes L^\ast R X) \neq 0\} = \inf\{i|H_i(R/\mathfrak{m} \otimes L^\ast R X) \otimes R \mathfrak{m} \neq 0\} = \inf\{i|H_i(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m} \otimes L^\ast R_\mathfrak{m} X_\mathfrak{m}) \neq 0\} = \text{width}(R_\mathfrak{m},X_\mathfrak{m}) = \text{width}_{R_\mathfrak{m}} X_\mathfrak{m}.$$  

Likewise we have $\text{depth}(\mathfrak{m},X) = \text{depth}_{R_\mathfrak{m}} X_\mathfrak{m}$.

3. *injective dimension

The injective dimension of a complex $X$, denoted by $\text{id}_R X$, is defined and studied in [1]. A graded module $J$ is called $^\ast$injective if it is an injective object in the category of graded modules. In other words, the functor $^\ast\text{Hom}_R(-,J)$ is exact in this category. A long exact sequence of $^\ast$injective modules is called $^\ast$injective resolution. The injective dimension of a graded module $M$ in the category of graded modules, is denoted by $^\ast\text{id}_R M$ (cf. [6, 2]). In this section we study the $^\ast$injective dimension of homologically left bounded complexes of graded modules.

Let $n \in \mathbb{Z}$. A homologically left bounded complex of graded modules $X$, is said to have $^\ast$injective dimension at most $n$, denoted by $^\ast\text{id}_R X \leq n$, if there exists an $^\ast$injective resolution $X \to I$, such that $I_i = 0$ for
i < −n. If * id_R X ≤ n holds, but * id_R X ≤ n − 1 does not, we write * id_R X = n. If * id_R X ≤ n for all n ∈ Z we write * id_R X = −∞. If * id_R X ≤ n for no n ∈ Z we write * id_R X = ∞. The following theorem inspired by [1, Theorem 2.4.I and Corollary 2.5.I].

**Theorem 3.1.** For X ∈ *D_≤(R) and n ∈ Z the following are equivalent:

1. * id_R X ≤ n.
2. n ≥ − sup U − inf(R* Hom_R(U, X)) for all U ∈ *D_≤(R) and H(U) ≠ 0.
3. n ≥ − inf X and * Ext_R^{n+1}(R/J, X) = 0 for every homogeneous ideal J of R.
4. n ≥ − inf X and for any (resp. some) *injective resolution I of X, the graded R-module Ker(∂_−n : I_−n → I_−n−1) is *injective.

Moreover the following holds:

* id_R X = sup{ j ∈ Z | * Ext_R^j(R/J, X) ≠ 0 for some homogeneous ideal J}

Proof. (1) ⇒ (2) Let t := sup U and I be an *injective resolution of X, such that, for all i < −n, I_i = 0. Then we have

* Ext_R^t(U, X) ≅ H_−t(* Hom_R(U, I)).

Since * Hom_R(U, I)_−t = 0 for −t < −n − t, the assertion follows.

(2) ⇒ (3) It is trivial that * Ext_R^{n+1}(R/J, X) = 0 for every homogeneous ideal J of R. For the second assertion let U = R in (2). So that Ext_R^i(R, X) = * Ext_R^i(R, X) = 0 for i > n. Now by [1, Lemma 1.9(b)], we have H_−i(X) = 0 for −i < −n. This means that n ≥ − inf X.

(3) ⇒ (4) By hypothesis of (4) H_i(I) = 0 for i < −n. Thus the complex

\[ \cdots \to 0 \to 0 \to I_−n \to I_−n−1 \to \cdots \to I_i \to I_i−1 \to \cdots \]

gives an *injective resolution of Ker ∂_−n. In particular

* Ext_R^i(R/J, Ker ∂_−n) = H_−n−i * Hom_R(R/J, I) = * Ext_R^{n+1}(R/J, X) = 0

for every homogeneous ideal J of R. Thus Ker ∂_−n is *injective by [6, Corollary 4.3].

(4) ⇒ (1) Let I be any *injective resolution of X. By assumption, the module Ker ∂_−n is *injective. Thus * id_R X < −n by definition.

The last equalities are easy consequences of (1), . . . , (4). □

For a local ring (R, m, k) and for an R-complex X and i ∈ Z the ith Bass number and Betti number of X are defined respectively by
\[ \mu_i^R(X) := \dim_k H_i(\text{Hom}_R(k, X)) \] and \[ \beta_i^R(X) := \dim_k H_i(k \otimes L^R X) \].

It is well-known that for \( X \in \mathcal{D}_\geq(R) \) one has (cf. \[1, \text{Proposition 5.3.I}\])

\[ \text{id}_R X = \sup \{ m \in \mathbb{Z} | \exists p \in \text{Spec}(R); \mu^n_{R_p}(X_p) \neq 0 \} \]

As a graded analogue we have:

**Proposition 3.2.** For \( X \in \mathcal{D}_\geq(R) \) we have the following equality

\[ \text{id}_R X = \sup \{ m \in \mathbb{Z} | \exists p \in \text{Spec}(R); \mu^n_{R_p}(X_p) \neq 0 \} \]

**Proof.** The argument is the same as proof of \[1, \text{Proposition 5.3.I}\] with some changes. Denote the supremum by \( i \). By Theorem 3.1, we have \( \text{id}_R X \geq i \). Hence the equality holds if \( i = \infty \). Thus assume that \( i \) is finite. By Theorem 3.1 we have to show that if \( \text{Ext}^j_R(M, X) \neq 0 \) for some finitely generated graded \( R \)-module \( M \), then \( j \leq i \); this implies that \( \text{id}_R X \leq i \). The elements of \( \text{Ass}(M) \) are homogeneous prime ideals. Thus there exists a filtration \( 0 = M_0 \subset M_1 \subset \cdots \subset M_i = M \) of graded submodules of \( M \) such that for each \( i \) we have \( M_i/M_{i-1} \cong R/p_i \) with \( p_i \in \text{Supp} M \) and \( p_i \) is homogeneous. From the long exact sequence of \( \text{Ext}^j_R(\cdot, X) \neq 0 \) the set

\[ \{ q \in \text{Spec}(R) | \text{there is an } h \geq j \text{ such that } \text{Ext}^h_R(R/q, X) \neq 0 \} \]

turns to be non empty. Let \( p \) be maximal in this set and for a homogeneous \( x \in R\setminus p \) consider the exact sequence

\[ 0 \rightarrow R/p \xrightarrow{\cdot x} R/p \rightarrow R/(p + Rx) \rightarrow 0. \]

It induces an exact sequence

\[ \text{Ext}^h_R(R/(p + Rx), X) \rightarrow \text{Ext}^h_R(R/p, X) \xrightarrow{\cdot x} \text{Ext}^h_R(R/p, X) \]

\[ \rightarrow \text{Ext}^{h+1}_R(R/(p + Rx), X) \]

in which the left-hand term is trivial because of the maximality of \( p \). Thus \( \text{Ext}^h_R(R/p, X) \xrightarrow{\cdot x} \text{Ext}^h_R(R/p, X) \) is injective for all homogeneous elements \( x \in R\setminus p \), hence so is the homogeneous localization homomorphism \( \text{Ext}^h_R(R/p, X) \rightarrow \text{Ext}^h_R(R/p, X)_{(p)} \). Thus the free \( R_{(p)}/pR_{(p)} \)-module \( \text{Ext}^h_R(R/p, X)_{(p)} \) is nonzero. Consequently

\[ \text{Ext}^h_R(R/p, X)_{(p)} \cong \text{Ext}^h_{R_p}(R_p/pR_p, X_p) \]

is nonzero. This implies that \( j \leq h \leq i \). \( \square \)

**Remark 3.3.** (1) A graded module is called \( \ast \)-projective if it is a projective object in the category of graded modules. By \[6, \text{Proposition 3.1}\] the \( \ast \)-projective graded \( R \)-modules coincide with projective \( R \)-modules. The projective dimension of a graded module \( M \) in the category of

\[ \mu_i^R(X) := \dim_k H_i(\text{Hom}_R(k, X)) \] and \[ \beta_i^R(X) := \dim_k H_i(k \otimes L^R X) \].

It is well-known that for \( X \in \mathcal{D}_\geq(R) \) one has (cf. \[1, \text{Proposition 5.3.I}\])

\[ \text{id}_R X = \sup \{ m \in \mathbb{Z} | \exists p \in \text{Spec}(R); \mu^n_{R_p}(X_p) \neq 0 \} \]
graded modules, is denoted by $\pd_X^* M$ (cf. [6]). Let $n \in \mathbb{Z}$. A homologically right bounded complex of graded modules $X$, is said to have *projective dimension at most $n$, denoted by $\pd_X^* X \leq n$, if there exists a *projective resolution $P \to X$, such that $P_i = 0$ for $i > n$. If $\pd_X^* X \leq n$ holds, but $\pd_X^* X \leq n-1$ does not, we write $\pd_X^* X = n$. If $\pd_X^* X \leq n$ for all $n \in \mathbb{Z}$ we write $\pd_X^* X = -\infty$. If $\pd_X^* X \leq n$ for no $n \in \mathbb{Z}$ we write $\pd_X^* X = \infty$.

(2) For $X \in \mathcal{D}_\Delta(R)$ by the same method as in [1, Theorem 2.4,P and Corollary 2.5.P] we have

$\pd_X^* X = \sup\{j \in \mathbb{Z} | \Ext^j_R(X, N) \neq 0 \text{ for some graded } R\text{-module } N\}$

$= \sup\{\inf(U) - \inf(\Hom_R(X, U)) | U \neq 0 \text{ in } \mathcal{D}_\Delta(R)\}.$

(3) It is easy to see that for $X \in \mathcal{D}_\Delta(R)$, we have $\pd_X^* X \leq \pd_X^* X$.

(4) The notions of *flat module and *flat dimension are obtained by replacing ‘projective’ by ‘flat’ in (1). By [6, Proposition 3.2] the *flat graded $R$-modules coincide with flat $R$-modules. Therefore for a homologically right bounded complex of graded modules $X$, we have $\fd_X^* X \leq \pd_X^* X$.

The proof of the following proposition is easy so we omit it (see [2, Theorem 1.5.9]). Let $J$ be an ideal of the graded ring $R$. Then the graded ideal $J^*$ denotes the ideal generated by all homogeneous elements of $J$. It is well-known that if $p$ is a prime ideal of $R$, then $p^*$ is a homogeneous prime ideal of $R$ by [2, Lemma 1.5.6].

**Proposition 3.4.** Let $X \in \mathcal{D}_\Delta(R)$ and $p$ be a non-homogeneous prime ideal in $R$. Then $\mu_i^p(X_p) = \mu_i^{R_p}(X_p)$ and $\beta_i^{R_p}(X_p) = \beta_i^{R^{R_p}}(X_p)$ for any integer $i \geq 0$.

**Corollary 3.5.** Let $X \in \mathcal{D}_\Delta(R)$ and $p$ be a non-homogeneous prime ideal in $R$. Then

$\depth X_p = \depth X_{p^*} + 1.$

**Proof.** Using Proposition 3.4, we can assume that both $\depth X_p$ and $\depth X_{p^*}$ are finite. So the equality follows from the fact that over a local ring $(R, m, k)$ we have $\depth_R X = \inf\{i \in \mathbb{Z} | \mu_i^R(X) \neq 0\}$. □

Foxby defined the small support of a homologically right bounded complex $X$ over a Noetherian ring $R$, denoted by $\text{supp}_R X$, as the set of prime ideal of $R$ such that $R_{(p)}/p R_{(p)} \otimes_R^L X$ is non-trivial complex (See [7]). It is well known that;

$\text{supp}_R X = \{p \in \text{Spec } R | \exists m \in \mathbb{Z} : \beta_m^{R_p}(X_p) \neq 0\}.$

Let $^*\text{supp}_R X$ be a subset of $\text{supp}_R X$ consisting of homogeneous prime ideals of $\text{supp}_R X$. Then from Proposition 3.4 we see that $p \in \text{supp}_R X$
if and only if \( p^* \in \text{supp}_R X \). Also using [12, Lemma 2.3] for \( p \in \text{supp}_R X \) we have \( \text{depth } X_p < \infty \) if and only if \( \text{width}_{R_p} X_p < \infty \). Therefore by corollary 3.5 we get:

\[
\text{width}_{R_p} X_p < \infty \iff \text{width}_{R_{p^*}} X_{p^*} < \infty.
\]

**Proposition 3.6.** Let \( X \in \mathcal{D}(R) \) and \( p \) is a non-homogeneous prime ideal in \( R \). Then

\[
\text{width}_{R_p} X_p = \text{width}_{R_{p^*}} X_{p^*}.
\]

**Proof.** We can assume that both \( \text{width } X_p \) and \( \text{width } X_{p^*} \) are finite numbers, and the argument would be dual to the proof of [2, Theorem 1.5.9]. □

The ungraded version of the following theorem was proved for modules by Chouinard [3, Corollary 3.1], and extended to complexes by Yassemi [12, Theorem 2.10].

**Theorem 3.7.** Let \( X \in \mathcal{D}(R) \). If \( \ast \text{id}_R X < \infty \), then

\[
\ast \text{id}_R X = \sup \{ \text{depth } R_p - \text{width } X_p \mid p \in \ast \text{Spec}(R) \}.
\]

**Proof.** We have the following computations

\[
\ast \text{id}_R X = \sup \{ m \in \mathbb{Z} \mid \exists p \in \ast \text{Spec}(R) : \mu^m_{R_p}(X_p) \neq 0 \}
= \sup \{ m \in \mathbb{Z} \mid \exists p \in \ast \text{Spec}(R) : H_m(R \text{Hom}_{R_p}(\kappa(p), X_p)) \neq 0 \}
= \sup \{ - \inf R \text{Hom}_{R_p}(\kappa(p), X_p) \mid p \in \ast \text{Spec}(R) \}
= \sup \{ \text{depth } R_p - \text{width } R_p \text{X}_p \mid p \in \ast \text{Spec}(R) \}.
\]

The first equality holds by Proposition 3.2, and the last one holds by [12, Lemma 2.6(a)], since \( \text{id}_R X < \infty \) by Propositions 3.2 and 3.4. □

The following corollary was already known for graded modules in [6, Corollary 4.12].

**Corollary 3.8.** For every \( X \in \mathcal{D}(R) \), we have

\[
\ast \text{id}_R X \leq \text{id}_R X \leq \ast \text{id}_R X + 1.
\]

**Proof.** First of all note that by Proposition 3.4, \( \text{id}_R X < \infty \) if and only if \( \ast \text{id}_R X < \infty \). The first inequality is clear by Theorem 3.7 and [12, Theorem 2.10]. For the second one let \( p \in \text{Spec } R \) be such that \( \text{id}_R X = \text{depth } R_p - \text{width } R_p M_p \) by [12, Theorem 2.10]. By Corollary 3.5 and Proposition 3.6 we have

\[
\text{depth } R_p - \text{width } R_p M_p \leq \text{depth } R_{p^*} - \text{width } R_{p^*} M_{p^*} + 1 \leq \ast \text{id}_R X + 1,
\]

where the second inequality holds by Theorem 3.7. □
Here we define the *dualizing complex for a graded ring and prove some related results.

**Definition 3.9.** A *dualizing complex $D$ for a graded ring $R$ is a homologically finite and bounded complex of graded $R$-modules, such that $^*_\text{id}_RD < \infty$ and the homothety morphism $\psi : R \to R^*\text{Hom}_R(D, D)$ is invertible in $^*D(R)$.

**Corollary 3.10.** Any *dualizing complex for $R$ is a dualizing complex for $R$.

The proof of the following lemma is the same as [10, Chapter V, Proposition 3.4].

**Lemma 3.11.** Let $(R, m, k)$ be a *local ring and $D$ is a *dualizing complex of $R$. Then there exists an integer $t$ such that $H^i(R^*\text{Hom}_R(k, D)) = 0$ for $i \neq t$ and $H^t(R^*\text{Hom}_R(k, D)) \cong k$.

Assume that $(R, m)$ is a *local ring. A *dualizing complex $D$ is said to be normalized *dualizing complex if $t = 0$ in the lemma. It is easy to see that a suitable shift of any *dualizing complex is a normalized one. Also using [10, Chapter V, Proposition 3.4] we see that if $D$ is a normalized *dualizing complex for $(R, m)$, then $D_m$ is a normalized dualizing complex for $R_m$.

**Lemma 3.12.** Let $(R, m, k)$ be a *local ring and that $D$ is a normalized *dualizing complex for $R$. Then there exists a natural functorial isomorphism from the category of graded modules of finite length to itself

$$\phi : H^0(R^*\text{Hom}_R(-, D)) \to *\text{Hom}_R(-, *E_R(k)),$$

where $*E_R(k)$ is the *injective envelope of $k$ over $R$.

**Proof.** Since $D$ is a normalized *dualizing complex for $R$, the functor $T := H^0(R^*\text{Hom}_R(-, D))$ is an additive contravariant exact functor from the category of graded modules of finite length to itself. Let $M$ be a graded $R$-module and $m \in M$ is a homogeneous element of degree $\alpha$. Then $\epsilon_m : R(-\alpha) \to M$ is a homogeneous morphism which sends 1 into $m$. Thus we have a homogeneous morphism $\psi(M) : T(M) \to *\text{Hom}_R(M, T(R))$ which sends a homogeneous element $x \in T(M)$ to a morphism $f_x \in *\text{Hom}_R(M, T(R))$ such that $f_x(m) = T(\epsilon_m)(x)$ for every homogeneous element $m \in M$. It is easy to see that it is functorial on $M$. Thus there exists a natural functorial morphism $\psi : T \to *\text{Hom}_R(-, T(R))$. Note that if $M$ is a finite graded $R$-module, using a finite presentation of $M$, there is an isomorphism $*\lim \text{Hom}_R(M, T(R/m^n)) \cong *\text{Hom}_R(M, *\lim T(R/m^n))$. Therefore
by the same method of [9, Lemma 4.4 and Propositions 4.5], there is a functorial isomorphism

$$\phi: H^0(R^\ast \text{Hom}_R(-, D)) \to \ast \text{Hom}_R(-, \ast \lim T(R/m^n)),$$

from the category of graded modules of finite length to itself. Using the technique of proof of [9, Proposition 4.7] in conjunction with [6, Corollary 4.3], we see that $\ast \lim T(R/m^n)$ is an *injective $R$-module. Since $D$ is a normalized *dualizing complex for $R$ we have

$$\ast \text{Hom}_R(k, \ast \lim T(R/m^n)) \cong H^0(R^\ast \text{Hom}_R(k, D)) \cong k.$$ 

Particularly we can embed $k$ to $\ast \lim T(R/m^n)$. In order to show that $\ast \lim T(R/m^n)$ is an *essential extension of $k$, let $Q$ be a graded submodule of $\ast \lim T(R/m^n)$ such that $k \cap Q = 0$. Then $\ast \text{Hom}_R(k, Q)$ can be embed in

$$\ast \text{Hom}_R(k, \ast \lim T(R/m^n)) \cong k.$$ 

Therefore $\ast \text{Hom}_R(k, Q) = 0$. On the other hand for each $n \in \mathbb{N}$ the set $V(m)$ includes $\text{Ass}(T(R/m^n))$. Now by [11, Proposition 2.1], the fact that each prime ideal of $\text{Ass}(\ast \lim T(R/m^n))$ is the annihilator of a homogeneous element [2, Lemma 1.5.6], and the definition of $\ast \lim$, we have

$$\text{Ass}(\ast \lim T(R/m^n)) \subseteq \bigcup_{n \in \mathbb{N}} \text{Ass}(T(R/m^n)) \subseteq V(m).$$ 

Consequently $Q$ has support in $V(m)$, so that $Q = 0$. Therefore $\ast \lim T(R/m^n) \cong \ast E_R(k)$. \qed

Let $\mathfrak{a}$ be an ideal of $R$. The right derived local cohomology functor with support in $\mathfrak{a}$ is denoted by $R\Gamma_\mathfrak{a}(-)$. Its right adjoint, $L\Lambda^\mathfrak{a}(-)$, is the left derived local homology functor with support in $\mathfrak{a}$ (see [8] for detail).

Finally, we have the following proposition, its proof uses Lemma 3.12 and the argument is similar to [10, Chapter V, Proposition 6.1].

**Proposition 3.13.** Let $(R, \mathfrak{m}, k)$ be a *local ring and that $D$ be a normalized *dualizing complex for $R$. Then $R\Gamma_\mathfrak{m}(D) \simeq \ast E_R(k)$.

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رابطه‌های قابل ذکری مابین بعدهای همولوژیکی و بعدهای همولوژیکی مدرج وجود دارد. در این مقاله، بعد ازکتیو همبافت از مدول‌های مدرج مورد مطالعه قرار گرفت و خواص آن بررسی شد. به ویژه، همبافت دوگان‌ساز مدرج، برای یک حلقه مدرج را تعریف کرد و نتایج مربوطه را تعمیم می‌دهد.

کلمات کلیدی: حلقه‌های مدرج، مدول‌های مدرج، بعد ازکتیو.