COTORSION DIMENSIONS OVER GROUP RINGS

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Abstract. Let $\Gamma$ be a group, $\Gamma'$ a subgroup of $\Gamma$ with finite index and $M$ be a $\Gamma$-module. We show that $M$ is cotorsion if and only if it is cotorsion as a $\Gamma'$-module. Using this result, we prove that the global cotorsion dimensions of rings $\mathbb{Z}\Gamma$ and $\mathbb{Z}\Gamma'$ are equal.

1. Introduction

Harrison [11], Nunke [13] and Fuchs [9], independently, introduced the notion of cotorsion abelian groups. An abelian group is said to be cotorsion, if every extension of it by a torsion-free group splits. This notion was extended to modules over integral domains by Matlis [12] and Warfield [15] in two different ways. Finally in [8], Enochs has defined cotorsion modules over arbitrary associative rings as the modules $C$ for which $\text{Ext}^1_R(F, C) = 0$ for all flat modules $F$. Actually, Enochs’s definition generalizes the definitions of Harrison and Warfield and agrees with that of Fuchs but differs from that of Matlis.

In [6], Ding and Mao defined a homological dimension, the cotorsion dimension, $\text{cd}_R M$, for any $R$-module $M$. It is defined as the least non-negative integer $n$ satisfying $\text{Ext}^{n+1}_R(F, M) = 0$ for all flat $R$-modules $F$. They also defined the global cotorsion dimension of a ring $R$, denoted by $\text{Cot.D}(R)$, as the supremum of the cotorsion dimensions of all $R$-modules.

Recall that a ring $R$ is perfect, if every $R$-module has a projective cover. Bass [2] has proved that perfect rings are precisely those whose

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every flat module is projective. The global cotorsion dimension of rings measures how far away a ring is from being perfect. This exactly means that, for a ring $R$ and a positive integer $n$, $\text{Cot}.D(R) \leq n$ if and only if every flat $R$-module $F$ has projective dimension less than or equal to $n$; see [6, Theorem 7.2.5]. In particular, $R$ is perfect if and only if $\text{Cot}.D(R) = 0$. Ding and Mao have proved that the global cotorsion dimension gives an upper bound on the global dimension of rings. More precisely, they showed that for any ring $R$, we have the inequality

$$D(R) \leq w.D(R) + \text{Cot}.D(R).$$

The purpose of this note is to study the cotorsion modules over group rings. The main result of this paper asserts that a given $\mathbb{Z}\Gamma$-module $M$ is cotorsion if and only if it is cotorsion over $\mathbb{Z}\Gamma'$, where $\Gamma'$ is a finite index subgroup of $\Gamma$. This result enables us to deduce that if $M$ is a $\mathbb{Z}\Gamma$-module, then $\text{cd}_\Gamma M = \text{cd}_{\Gamma'} M$, where $\text{cd}_\Gamma M$ (resp., $\text{cd}_{\Gamma'} M$) denotes the cotorsion dimension of $M$ as a $\mathbb{Z}\Gamma$-module (resp., $\mathbb{Z}\Gamma'$-module).

Let $\Gamma$ be an abelian multiplicative group, and $R$ be a ring with identity. It is shown by Woods in [16] that, the group ring $R\Gamma$ is perfect if and only if $R$ is perfect and $\Gamma$ is a finite group. Several decades later, Bennis and Mahdou [3] extended the result of Woods and proved:

$$\text{Cot}.D(R) \leq \text{Cot}.D(R\Gamma) \leq \text{Cot}.D(R) + \text{pd}_{R\Gamma}(R).$$

Furthermore, if $\text{pd}_{R\Gamma} R$ is finite and $\Gamma$ is a finite group, then the equality $\text{Cot}.D(R) = \text{Cot}.D(R\Gamma)$ holds.

In this paper, we consider the global cotorsion dimension of the integral group ring of a group $\Gamma$, $\mathbb{Z}\Gamma$, and denote it by $\text{Cot}.D(\Gamma)$. We prove that $\text{Cot}.D(\Gamma) = \text{Cot}.D(\Gamma')$, where $\Gamma'$ is a finite index subgroup of $\Gamma$. Also, it is shown that there is a tight connection between the global cotorsion dimension of $\mathbb{Z}\Gamma$, the supremum of flat length of injective $\mathbb{Z}\Gamma$-modules, $\text{slf}\Gamma$, and the supremum of injective length of flat $\mathbb{Z}\Gamma$-modules, $\text{sflf}\Gamma$.

Throughout the paper, $\Gamma$ is a group and $\mathbb{Z}\Gamma$ is its integral group ring. By a $\Gamma$-module, we mean a $\mathbb{Z}\Gamma$-module. We follow this abbreviation in all of our notations. For example, for a $\Gamma$-module $M$, projective dimension of $M$ over $\mathbb{Z}\Gamma$ is denoted by $\text{pd}_\Gamma M$. The tensor product and Hom functor over $\mathbb{Z}\Gamma$ denoted by $\otimes_\Gamma -$ and $\text{Hom}_\Gamma(-,-)$, respectively. We also denote the tensor product and Hom functor over $\mathbb{Z}$ by $-\otimes -$ and $\text{Hom}(-,-)$, respectively.
2. Results and proofs

Let $\Gamma$ be a group. Following [8], a $\Gamma$-module $C$ is called cotorsion if $\text{Ext}^1_\Gamma(F, C) = 0$ for any flat $\Gamma$-module $F$. The class of cotorsion modules contains all pure-injective (and hence all injective) modules, and is closed under finite direct sums and direct summands.

**Lemma 2.1.** Let $\Gamma'$ be a subgroup of $\Gamma$ and let $M$ be a cotorsion $\Gamma'$-module. Then $M$ is a cotorsion $\Gamma'$-module.

**Proof.** Suppose that $F$ is a flat $\Gamma'$-module. Then $\mathbb{Z}\Gamma \otimes_{\Gamma'} F$ is a flat $\Gamma$-module. So $\text{Ext}^1_{\Gamma'}(\mathbb{Z}\Gamma \otimes_{\Gamma'} F, M) = 0$. Hence $M$ is a cotorsion $\Gamma'$-module. □

**Theorem 2.2.** Let $\Gamma'$ be a finite index subgroup of $\Gamma$ and let $M$ be a $\Gamma$-module. Then $M$ is cotorsion if and only if it is cotorsion as a $\Gamma'$-module.

**Proof.** According to the Lemma 2.1, we only need to show the ‘if’ part. So, assume that $M$ is a cotorsion $\Gamma'$-module. We must show that $M$ is cotorsion as a $\Gamma$-module. To this end, consider an arbitrary flat $\Gamma$-module $F$. Due to Lazard’s Theorem, there is a direct system $\{P_i\}_{i \in I}$ of finitely generated projective $\Gamma$-modules such that $F \cong \lim\limits_{\rightarrow} P_i$. Take for any $i$, a projective $\Gamma'$-module $P'_i$ such that $P_i \cong \mathbb{Z}\Gamma \otimes_{\Gamma'} P'_i$ as $\Gamma$-modules. Letting $F' \cong \lim\limits_{\rightarrow} P'_i$, one infers that $F'$ is a flat $\Gamma'$-module and $F \cong \mathbb{Z}\Gamma \otimes_{\Gamma'} F'$. Hence, we may have the following isomorphisms:

$$\text{Ext}^1_{\Gamma'}(F, M) \cong \text{Ext}^1_{\Gamma'}(\mathbb{Z}\Gamma \otimes_{\Gamma'} F', M) \cong \text{Ext}^1_{\Gamma'}(F', \text{Hom}_\Gamma(\mathbb{Z}\Gamma, M)) \cong \text{Ext}^1_{\Gamma'}(F', M),$$

in which, the second isomorphism obtains by adjointness of $\text{Hom}$ and $\otimes$. Since, by the assumption $M$ is a cotorsion $\Gamma'$-module, $\text{Ext}^1_{\Gamma'}(F', M) = 0$, implying that $\text{Ext}^1_{\Gamma'}(F, M) = 0$, as desired. □

**Definition 2.3.** Let $M$ be a nonzero $\Gamma$-module. The cotorsion dimension of $M$, denoted by $\text{cd}_\Gamma M$, is defined to be the least non-negative integer $n$ such that $\text{Ext}^{n+1}_\Gamma(F, M) = 0$, for every flat $\Gamma$-module $F$. If no such $n$ exists, set $\text{cd}_\Gamma M = \infty$.

**Remark 2.4.** Suppose that $\Gamma$ is a group. It is easy to show that, for any $\Gamma$-module $M$ and integer $n \geq 0$, $\text{cd}_\Gamma M \leq n$ if and only if there exists an exact sequence

$$0 \longrightarrow M \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^n \longrightarrow 0,$$
where each $C^i$ is a cotorsion $\Gamma$-module, $i = 1, 2, \ldots, n$; see [6, Corollary 19.2.1].

**Corollary 2.5.** Let $\Gamma'$ be a subgroup of $\Gamma$ with finite index and let $M$ be a $\Gamma$-module. Then $\text{cd}_{\Gamma'}M = \text{cd}_{\Gamma}M$.

**Proof.** The inequality $\text{cd}_{\Gamma'}M \leq \text{cd}_{\Gamma}M$ follows immediately from Theorem 2.2. For the reverse inequality, we may assume that $\text{cd}_{\Gamma'}M = n < \infty$. If $F$ is an arbitrary flat $\Gamma$-module, then by a similar argument as that in the proof of Theorem 2.2, one may obtain the isomorphism $\text{Ext}^{n+1}_{\Gamma'}(F, M) \cong \text{Ext}^{n+1}_{\Gamma'}(F', M)$, in which $F'$ is a flat $\Gamma'$-module. By assumption $\text{Ext}^{n+1}_{\Gamma'}(F', M) = 0$. Therefore, $\text{Ext}^{n+1}_{\Gamma'}(F, M) = 0$. This implies the inequality, and the proof is complete. \hfill \Box

**Definition 2.6.** Assume that $R$ is an associative ring with identity. The left (resp., right) global cotorsion dimension of $R$, denoted by $l:\text{Cot}:D(R)$ (resp., $r:\text{Cot}:D(R)$) is defined as the supremum of the cotorsion dimensions of left (resp., right) $R$-modules. If $R = \mathbb{Z}$, where $\Gamma$ is a group, then $R$ is isomorphic with the opposite ring $R^{op}$ and so the distinction between left and right module is redundant. In this case, we drop the superfluous letters $l$ and $r$ and we write $\text{Cot}:D()$ instead of $\text{Cot}:D(\mathbb{Z})$.

**Lemma 2.7.** Let $\Gamma'$ be an arbitrary subgroup of $\Gamma$ and $C$ be a cotorsion $\Gamma'$-module. Then $\text{Hom}_{\Gamma'}(\mathbb{Z}, C)$ is a cotorsion $\Gamma$-module.

**Proof.** Suppose that $F$ is an arbitrary flat $\Gamma$-module. Using the adjointness of Hom and $\otimes$, we have the following isomorphisms:

$\text{Ext}^1_{\Gamma'}(F, \text{Hom}_{\Gamma'}(\mathbb{Z}, C)) \cong \text{Ext}^1_{\Gamma'}(\mathbb{Z} \otimes_{\Gamma} F, M) \cong \text{Ext}^1_{\Gamma'}(F, C)$.

Since $F$ is flat over $\Gamma'$, hence $\text{Ext}^1_{\Gamma'}(F, C) = 0$ and then

$\text{Ext}^1_{\Gamma'}(F, \text{Hom}_{\Gamma'}(\mathbb{Z}, C)) = 0$.

The proof is now finished. \hfill \Box

**Lemma 2.8.** Let $\Gamma'$ be a subgroup of $\Gamma$ with finite index. Then for any $\Gamma$-module $M$,

$\text{cd}_{\Gamma}M = \text{cd}_{\Gamma'}M = \text{cd}_{\Gamma}\text{Hom}_{\Gamma'}(\mathbb{Z} \Gamma, M) = \text{cd}_{\Gamma'}\text{Hom}_{\Gamma'}(\mathbb{Z} \Gamma, M)$.

**Proof.** The first and third equalities follows from Theorem 2.2. So, it is enough to show the second equality. To this end, first we show that $\text{cd}_{\Gamma}\text{Hom}_{\Gamma'}(\mathbb{Z} \Gamma, M) \leq \text{cd}_{\Gamma'}M$. If $\text{cd}_{\Gamma}M = \infty$, then there is no thing to prove. So assume that $\text{cd}_{\Gamma}M = n < \infty$. By Remark 2.4, there exists an exact sequence of $\Gamma$-modules;

$0 \longrightarrow M \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots \longrightarrow C^n \longrightarrow 0,$
where each $C^i$ is a cotorsion, and so cotorsion as a $\Gamma'$-module. Since $
abla \Gamma$ is a free $\nabla \Gamma'$-module, applying the functor $\text{Hom}_{\nabla \Gamma'}(\nabla \Gamma, -)$ to this sequence, gives rise to the following exact sequence of $\Gamma$-modules

$$0 \to \text{Hom}_{\nabla \Gamma'}(\nabla \Gamma, M) \to \text{Hom}_{\nabla \Gamma'}(\nabla \Gamma, C^0) \to \cdots \to \text{Hom}_{\nabla \Gamma'}(\nabla \Gamma, C^n) \to 0.$$ 

By Lemma 2.7, $\text{Hom}_{\nabla \Gamma'}(\nabla \Gamma, C^i)$'s are cotorsion $\Gamma$-modules. This means that $\text{cd}_{\nabla \Gamma'} \text{Hom}_{\nabla \Gamma'}(\nabla \Gamma, M) \leq n$.

For the converse inequality, consider the exact sequence of $\Gamma$-modules

$$0 \to M \to \text{Hom}_{\nabla \Gamma'}(\nabla \Gamma, M) \to K \to 0,$$

which splits over $\Gamma'$. So, $M$ is isomorphic to a direct summand of $\text{Hom}_{\nabla \Gamma'}(\nabla \Gamma, M)$ over $\Gamma'$. Hence $\text{cd}_{\nabla \Gamma'} M \leq \text{cd}_{\nabla \Gamma'} \text{Hom}_{\nabla \Gamma'}(\nabla \Gamma, M)$. In particular, by Theorem 2.2, $\text{cd}_{\nabla \Gamma} M \leq \text{cd}_{\nabla \Gamma} \text{Hom}_{\nabla \Gamma'}(\nabla \Gamma, M)$. This implies that the second equality. The proof is now finished. □

**Remark 2.9.** Related to the problem of extending the Farrell-Tate cohomology, two homological invariants were assigned to a group $\Gamma$ by Gedrich and Gruenberg, spli$\Gamma$, the supremum of the projective length of the injective $\Gamma$-modules, and silp$\Gamma$, the supremum of the injective lengths of the projective $\Gamma$-modules [10]. They studied these invariants and showed that for any group $\Gamma$, silp$\Gamma \leq$ spli$\Gamma$ and if spli$\Gamma$ is finite, then silp$\Gamma =$ spli$\Gamma$. These invariants then have been considered by several authors; see [4, 5]. For a long time, it was not known if the finiteness of silp$\Gamma$ implies the finiteness of spli$\Gamma$. In 2010, it is proved that by Emmanouil [7] that, for any group $\Gamma$, silp$\Gamma = $ spli$\Gamma$. While proving his interesting result, Emmanouil applied two new invariants silf$\Gamma$, the supremum of the injective length of the flat $\Gamma$-modules, and sli$\Gamma$, the supremum of the flat length of the injective $\Gamma$-modules. For any group $\Gamma$, let sclf$\Gamma$ denote the supremum of the cotorsion length of the flat $\Gamma$-modules. Note that since injective modules are cotorsion, one has the inequality sclf$\Gamma \leq$ silf$\Gamma$.

The following proposition obtains immediately from [6, Theorem 7.2.5], but here we provide a short proof for it.

**Proposition 2.10.** Let $\Gamma$ be a group. Then,

(i) $\text{CotD}(\Gamma) = \text{sclf}\Gamma$.
(ii) $\text{CotD}(\Gamma) \leq \text{silp}\Gamma$.

**Proof.** (i). It is clear that sclf$\Gamma \leq \text{CotD}(\Gamma)$. To prove the inverse inequality, we may assume that sclf$\Gamma$ is finite, say $n$. Let $M$ be an arbitrary $\Gamma$-module. Clearly we are done, if we can show that
\[ \text{Ext}_{\Gamma}^{n+1}(F, M) = 0 \] for any flat \( \Gamma \)-module \( F \). To do this, consider a short exact sequence of \( \Gamma \)-modules;

\[ 0 \longrightarrow K \longrightarrow G \longrightarrow M \longrightarrow 0, \]

in which \( G \rightarrow M \) is a flat cover of the \( \Gamma \)-module \( M \). So by [17, Lemma 2.1.1], \( K \) is cotorsion. Assume that \( F \) is any flat \( \Gamma \)-module. Apply the functor \( \text{Hom}_\Gamma(F, -) \) to this sequence, to get the exact sequence

\[ \text{Ext}_{\Gamma}^{n+1}(F, G) \longrightarrow \text{Ext}_{\Gamma}^{n+1}(F, M) \longrightarrow \text{Ext}_{\Gamma}^{n+2}(F, K). \]

The first term vanishes because \( \text{cd}_\Gamma G \leq n \), and the last term vanishes because \( K \) is cotorsion; see [1, 2.2]. Hence, \( \text{Ext}_{\Gamma}^{n+1}(F, M) = 0 \), as needed.

(ii). In view of part (i), \( \text{Cot}.D(\Gamma') \leq \text{silf} \Gamma \). On the other hand, by [1, Theorem 3.3], \( \text{silf} \Gamma = \text{silp} \Gamma \), implying that \( \text{Cot}.D(\Gamma) \leq \text{silp} \Gamma \). The proof is complete. \( \square \)

**Theorem 2.11.** Let \( \Gamma' \) be a subgroup of \( \Gamma \) with finite index. Then \( \text{Cot}.D(\Gamma') = \text{Cot}.D(\Gamma) \).

**Proof.** In view of Corollary 2.5, we only need to show that \( \text{Cot}.D(\Gamma') \leq \text{Cot}.D(\Gamma) \). If \( \text{Cot}.D(\Gamma) = \infty \), there is nothing to prove. So assume that \( \text{Cot}.D(\Gamma) = n < \infty \). Take an arbitrary \( \Gamma' \)-module \( M \). By the hypothesis, \( \Gamma' \)-module \( Z \Gamma \otimes_{\Gamma'} M \) has cotorsion dimension at most \( n \), and hence Corollary 2.5, yields that the inequality \( \text{cd}_{\Gamma'}(Z \Gamma \otimes_{\Gamma'} M) \leq n \), implying that \( \text{cd}_{\Gamma'} M \leq n \), since \( M \) is a direct summand of \( Z \Gamma \otimes_{\Gamma'} M \) as a \( \Gamma' \)-module. Consequently, \( \text{Cot}.D(\Gamma') \leq n \), as desired. \( \square \)

**Corollary 2.12.** If \( \Gamma \) is a finite group, then \( \text{Cot}.D(Z) = \text{Cot}.D(\Gamma) \).

**Remark 2.13.** Recall that a ring \( R \) is called (left) perfect if every (left) \( R \)-module has a projective cover. In [2], Bass proved that perfect rings are those rings such that every flat module is projective. This rings were characterized in term of the vanishing of \( \text{Cot}.D(R) \) by Ding and Mao as: \( R \) is a perfect ring if and only if \( \text{Cot}.D(R) = 0 \); see [6, Corollary 19.2.9]. It is clear that \( Z \) is not perfect. Hence, if \( \Gamma \) is a finite group, then the previous corollary implies that \( Z \Gamma \) is not a perfect ring.

**Corollary 2.14.** If \( \Gamma \) is a finite group, then \( \text{Cot}.D(\Gamma) = 1 \).

**Proof.** By Proposition 2.10, we have \( \text{Cot}.D(\Gamma) \leq \text{silp} \Gamma \). Since \( \Gamma \) is finite, by [7, Theorem 4.6] \( \text{silp} \Gamma = 1 \). So \( \text{Cot}.D(\Gamma) \leq 1 \). On the other hand, by the above Remark, \( \text{Cot}.D(\Gamma) \neq 0 \). Hence \( \text{Cot}.D(\Gamma) = 1 \). \( \square \)

**Theorem 2.15.** For any group \( \Gamma \), \( \text{silf} \Gamma \leq \text{Cot}.D(\Gamma) + \text{sfl}\Gamma \).
Proof. Assume that \( \text{Cot.D}(\Gamma) = n \) and \( s\text{fli}\Gamma = m \) are both finite. By [1, Theorem 3.3], in conjunction with Remark 2.9, it is enough for us to show that \( \text{pd}_I(I) \leq n + m \), for all injective \( \Gamma \)-modules \( I \). Since \( s\text{fli}\Gamma = m \), \( \text{fd}_I(I) \leq m \). Thus, there exists an exact sequence of \( \Gamma \)-modules;

\[
0 \longrightarrow F_m \longrightarrow F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I \longrightarrow 0,
\]
in which \( F_i \), for any \( i \), is flat. Take short exact sequences

\[
0 \longrightarrow L_i \longrightarrow F_i \longrightarrow L_{i-1} \longrightarrow 0,
\]
where, \( L_i = \ker(F_i \longrightarrow F_{i-1}) \), \( i = 0, 1, 2, \ldots, m - 1 \), \( F_1 = I \) and \( F_m = L_{m-1} \). By [6, Theorem 19.2.5] together with [14, Lemma 9.26], we have \( \text{pd}_I(L_{m-2}) \leq 1 + n \). So \( \text{pd}_I(I) \leq m + n \). This means that \( \text{split} \Gamma \leq m + n \). Therefore, \( \text{silf} \Gamma \leq m + n \), as required. \( \square \)

Remark 2.16. Let \( \Gamma \) be a finite group. Then by [1, Corollary 3.9] \( \text{silf} \Gamma = \text{sfli} \Gamma = 1 \). Also by Corollary 2.14, \( \text{Cot.D}(\Gamma) = 1 \). So in this case, the inequality is strict.

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References


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COTORSION DIMENSIONS OVER GROUP RINGS

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بعدهای همتایی روی حلقه گروه‌ها

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فرض کنید $\Gamma$ یک گروه و $\Gamma'$ زیرگروهی از $\Gamma$ با اندیس متناهی باشد. فرض کنید $M$ یک $\Gamma$-مدول باشد. نشان می‌دهیم که لازم است $\Gamma$-مدول $\pi(M)$ همتایی باشد. با استفاده از این نتیجه، ثابت می‌شود که $\Gamma$-مدول $\pi(M)$ همتایی جامع حلقه‌های $Z\Gamma'$ و $Z\Gamma$ هم مساوی هستند.

کلمات کلیدی: بعدهای همتایی، همتایی جامع، حلقه کامل